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QUASI-INTERIOR IDEALS AND THEIR PROPERTIES

Murali Krishna Rao Marapureddy

Abstract. In this paper, we introduce the notion of quasi-interior ideal as a generalization of quasi ideals, interior ideals, left(right) ideal, ideals of a $\Gamma-$semiring. We study the properties of quasi-interior ideals of $\Gamma-$semiring and characterize the quasi-interior simple $\Gamma-$semiring, regular $\Gamma-$semiring using quasi-interior ideal ideals of a $\Gamma-$semiring.

1. Introduction

Ideals play an important role in advance studies and uses of algebraic structures. Generalization of ideals in algebraic structures is necessary for further study of algebraic structures. Many mathematicians introduced various generalizations of concept of ideals in algebraic structures, proved important results and charecterizations of algebraic structures. The notion of a semiring was introduced by Vandiver[**19**] in 1934, but semirings had appeared in earlier studies on the theory of ideals of rings. Semiring is a generalization of a ring but also of a generalization of distributive lattice. Semirings are structually similar to semigroups than to rings. Semiring theory has many applications in other branches of mathematics.
In 1995, M.Murali Krishna Rao[**5, 6, 8**] introduced the notion of a $\Gamma-$semiring as a generalization of a $\Gamma-$ ring, a ternary semiring and a semiring. As a generalization of ring, the notion of a $\Gamma-$ring was introduced by Nobusawa in 1964[**15**]. In 1981, Sen introduced the notion of a $\Gamma-$semigroup as a generalization of semigroup[**16**]. The notion of a ternary algebraic system was introduced by Lehmer. Murali Krishna Rao and Venkateswarlu[**11, 12, 13**] studied regular $\Gamma-$incline, field $\Gamma-$semiring and derivations.

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Henriksen[**2**] and Shabir [**17**] et al. studied ideals in semirings. We know that the notion of an one sided ideal of any algebraic structure is a generalization of notion of an ideal. The quasi ideals are generalization of left ideal and right ideal whereas the bi-ideals are generalization of quasi ideals. In 1952, the concept of bi-ideals was introduced by Good and Hughes[**1**] for semigroups. The notion of bi-ideals in rings and semirings were introduced by Lajos and Szasz[**4**]. Bi-ideal is a special case of (m-n) ideal. Steinfeld[**18**] first introduced the notion of quasi ideals for semigroups and then for rings. Iseki[**3**] introduced the concept of quasi ideal for a semiring. Quasi ideals, bi-ideals in $\Gamma-$semirings studied by Jagtap and Pawar. Murali Krishna Rao[**7, 8, 10**] introduced the notion of left (right) bi-quasi ideal, the notion of bi-interior ideal and the notion of bi quasi-interior ideal of $\Gamma-$semiring as a generalization of ideal of $\Gamma-$semiring, studied their properties and characterized the left bi-quasi simple $\Gamma-$semiring and regular $\Gamma-$semiring. In this paper, we introduce the notion of quasi-interior ideals as a generalization of quasi ideal, interior ideal, left(right) ideal and ideal of a $\Gamma-$semiring and study the properties of quasi-interior ideals of a $\Gamma-$semiring.

2. Preliminaries

In this section, we will recall some of the fundamental concepts and definitions which are necessary for this paper.

Definition 2.1. [1] *A set S together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$ respectively will be called semiring provided*

(i) *addition is a commutative operation.*
(ii) *multiplication distributes over addition both from the left and from the right.*
(iii) *there exists $0 \in S$ such that $x + 0 = x$ and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in S$.*

Definition 2.2. [15] *Let $(M, +)$ and $(\Gamma, +)$ be commutative semigroups. Then we call M a $\Gamma-$semiring, if there exists a mapping $M \times \Gamma \times M \to M$ (the image of (x, α, y) will be denoted by $x\alpha y, x, y \in M, \alpha \in \Gamma$) such that it satisfying the following axioms for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$*

(i) $x\alpha(y + z) = x\alpha y + x\alpha z$
(ii) $(x + y)\alpha z = x\alpha z + y\alpha z$
(iii) $x(\alpha + \beta)y = x\alpha y + x\beta y$
(iv) $x\alpha(y\beta z) = (x\alpha y)\beta z$.

Every semiring R is a $\Gamma-$semiring with $\Gamma = R$ and ternary operation $x\gamma y$ defined as the usual semiring multiplication.

example 2.1. *Let S be a semiring and $M_{p,q}(S)$ denote the additive abelian semigroup of all $p \times q$ matrices with identity element whose entries are from S. Then $M_{p,q}(S)$ is a $\Gamma-$ semiring with $\Gamma = M_{p,q}(S)$ ternary operation is defined by $x\alpha z = x(\alpha^t)z$ as the usual matrix multiplication, where α^t denote the transpose of the matrix α; for all x, y and $\alpha \in M_{p,q}(S)$.*

EXAMPLE 2.2. *Let M be the set of all natural numbers. Then $(M,\ max,\ min)$ is a semiring. If $\Gamma = M$, then M is a $\Gamma-$semiring.*

EXAMPLE 2.3. *Let M be the additive semigroup of all $m \times n$ matrices over the set of non negative rational numbers and Γ be the additive semigroup of all $n \times m$ matrices over the set of non negative integers.Then with respect to usual matrix multiplication M is a $\Gamma-$semiring.*

DEFINITION 2.3. *A $\Gamma-$semiring M is said to be commutative $\Gamma-$semiring if $x\alpha y = y\alpha x$, for all $x, y \in M$ and $\alpha \in \Gamma$.*

DEFINITION 2.4. *A $\Gamma-$semiring M is said to have zero element if there exists an element $0 \in M$ such that $0 + x = x = x + 0$ and $0\alpha x = x\alpha 0 = 0$, for all $x \in M, \alpha \in \Gamma$.*

DEFINITION 2.5. *An element $a \in M$ is said to be idempotent if there exists $\alpha \in \Gamma$ such that $a = a\alpha a$.*

DEFINITION 2.6. *A non-empty subset A of a $\Gamma-$semiring M is called*

(i) *a $\Gamma-$subsemiring of M if $(A, +)$ is a subsemigroup of $(M, +)$ and $A\Gamma A \subseteq A$.*
(ii) *a quasi-ideal of M if A is a $\Gamma-$subsemiring of M and $A\Gamma M \cap M\Gamma A \subseteq A$.*
(iii) *a bi-ideal of M if A is a $\Gamma-$subsemiring of M and $A\Gamma M\Gamma A \subseteq A$.*
(iv) *an interior ideal of M if A is a $\Gamma-$subsemiring of M and $M\Gamma A\Gamma M \subseteq A$.*
(v) *a left (right) ideal of M if A is a $\Gamma-$subsemiring of M and $M\Gamma A \subseteq A (A\Gamma M \subseteq A)$.*
(vi) *an ideal if A is a $\Gamma-$subsemiring of $M, A\Gamma M \subseteq A$ and $M\Gamma A \subseteq A$.*
(vii) *a $k-$ideal if A is a $\Gamma-$subsemiring of $M, A\Gamma M \subseteq A, M\Gamma A \subseteq A$ and $x \in M,\ x + y \in A, y \in A$ then $x \in A$.*

DEFINITION 2.7. [20] *Let M be a $\Gamma-$semiring. A non-empty subset B of M is said to be bi-interior ideal of M if B is a $\Gamma-$subsemiring of M and $M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B$.*

DEFINITION 2.8. [22] *Let M be a $\Gamma-$semiring. A non-empty subset L of M is said to be left bi-quasi ideal (right bi-quasi ideal) of M if L is a subsemigroup of $(M, +)$ and $M\Gamma L \cap L\Gamma M\Gamma L \subseteq L$ $(L\Gamma M \cap L\Gamma M\Gamma L \subseteq L)$.*

DEFINITION 2.9. [24] *Let M be a $\Gamma-$semiring. A non-empty subset B of M is said to be bi-quasi interior ideal of M if B is a $\Gamma-$subsemiring of M and $B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B$.*

DEFINITION 2.10. [23] *Let M be a $\Gamma-$semiring. L is said to be bi-quasi ideal of M if it is both a left bi-quasi ideal and a right bi-quasi ideal of M.*

DEFINITION 2.11. [23] *A $\Gamma-$semiring M is called a left bi-quasi simple $\Gamma-$semiring if M has no left bi-quasi ideal other than M itself.*

Remark: A bi-quasi interior ideal of a $\Gamma-$semiring M need not be bi-ideal, quasi-ideal, interior ideal, bi-interior ideal, and bi-quasi ideals of a $\Gamma-$semiring M.

Definition 2.12. *Let M be a $\Gamma-$semiring. An element $1 \in M$ is said to be unity if for each $x \in M$ there exists $\alpha \in \Gamma$ such that $x\alpha 1 = 1\alpha x = x$.*

Definition 2.13. *In a $\Gamma-$semiring M with unity 1, an element $a \in M$ is said to be left invertible (right invertible) if there exist $b \in M, \alpha \in \Gamma$ such that $b\alpha a = 1 (a\alpha b = 1)$.*

Definition 2.14. *In a $\Gamma-$semiring M with unity 1, an element $a \in M$ is said to be invertible if there exist $b \in M, \alpha \in \Gamma$ such that $a\alpha b = b\alpha a = 1$.*

Definition 2.15. *A $\Gamma-$semiring M is called a division $\Gamma-$semiring if for each non-zero element of M has multiplication inverse.*

example 2.4.

(i) *Let Q be the set of all rational numbers, $M = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in Q \right\}$ be the additive semigroup of M matrices and $\Gamma = M$. The ternary operation $A\alpha B$ is defined as usual matrix multiplication of A, α, B, for all $A, \alpha, B \in M$.Then M is a $\Gamma-$semiring*

(a) *If $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid 0 \neq a, 0 \neq b \in Q \right\}$ then R is a quasi ideal of a Γ-semiring M and R is neither a left ideal nor a right ideal.*

(b) *If $S = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid 0 \neq a \in Q \right\}$ then S is a bi-ideal of $\Gamma-$semiring M.*

(ii) *If $M = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in Q \right\}$ and $\Gamma = M$ then M is a $\Gamma-$semiring with respect to usual addition of matrices and ternary operation is defined as usual matrix multiplication and $A = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid 0 \neq a, 0 \neq b \in Q \right\}$. Then A is not a bi-ideal of $\Gamma-$semiring M.*

example 2.5. *Let N be a the set of all natural numbers and $\Gamma = 4\mathcal{N}$ be additive abelian semigroups. Tennary operation is defined as $(x, \alpha, y) \to x + \alpha + y$, where $+$ is the usual addition of integers. Then N is a $\Gamma-$semiring. A subset $I = 2N$ of N is a bi-quasi-interior ideal of N but not bi-ideal,quasi-ideal, interior ideal, bi-interior ideal and bi-quasi ideal of $\Gamma-$semiring N.*

3. Quasi-interior ideals of $\Gamma-$semirings

In this section, we introduce the notion of quasi-interior ideal as a generalization of quasi-ideals and interior-ideals of a $\Gamma-$semiring and study the properties of quasi-interior ideals of a $\Gamma-$semiring. Throughout this paper M is a $\Gamma-$semiring with unity element.

Definition 3.1. *A non-empty subset B of a $\Gamma-$semiring M is said to be left quasi-interior ideal of M if B is a $\Gamma-$subsemiring of M and $M\Gamma B\Gamma M\Gamma B \subseteq B$.*

Definition 3.2. *A non-empty subset B of a $\Gamma-$semiring M is said to be right quasi-interior ideal of M if B is a $\Gamma-$subsemiring of M and $B\Gamma M\Gamma B\Gamma M \subseteq B$.*

DEFINITION 3.3. *A non-empty subset B of a $\Gamma-$semiring M is said to be quasi-interior ideal of M if B is a $\Gamma-$subsemiring of M and $B$is left and right quasi-interior ideal of M.*

Remark: A quasi-interior ideal of a $\Gamma-$semiring M need not be quasi-ideal, interior ideal, bi-interior ideal. and bi-quasi ideal of $\Gamma-$semiring M.

EXAMPLE 3.1. *Let N be a the set of all natural numbers and $\Gamma = \mathcal{N}$ be additive abelian semigroups. Tennary operation is defined as $(x, \alpha, y) \to x + \alpha + y$, where $+$ is the usual addition of integers. Then N is a $\Gamma-$semiring.Let I be the set of all odd natural numbers.Then I is a quasi-interior ideal of N but not quasi-ideal, interior ideal, bi-interior ideal and bi-quasi ideal of $\Gamma-$semiring N.*

In the following theorem, we mention some important properties and we omit the proofs since they are straight forward.

THEOREM 3.1. *Let M be a $\Gamma-$semring. Then the following hold.*

(1) *Every left ideal is a left quasi-interior ideal of M.*
(2) *Every right ideal is a right quasi-interior ideal of M.*
(3) *Every quasi ideal is a quasi-interior ideal of M.*
(4) *Every ideal is a quasi-interior ideal of M.*
(5) *The intersection of a right ideal and a left ideal of M is a quasi-interior ideal of M.*
(6) *If L is a left ideal and R is a right ideal of a $\Gamma-$semiring M then $B = R\Gamma L$ is a quasi-interior ideal of M.*
(7) *If B is a quasi-interior ideal and T is a $\Gamma-$subsemiring of M then $B \cap T$ is a quasi-interior ideal of ring M.*
(8) *Let M be a $\Gamma-$semiring and B be a $\Gamma-$subsemiring of M. If $M\Gamma M\Gamma M\Gamma B \subseteq B$ then B is a left quasi-interior ideal of M.*
(9) *Let M be a $\Gamma-$semiring and B be a $\Gamma-$subsemiring of M.If $M\Gamma M\Gamma M\Gamma B \subseteq B$ and$B\Gamma M\Gamma M\Gamma M \subseteq B$ then B is a quasi-interior ideal of M.*
(10) *The intersection of a right quasi-interior ideal and a left quasi-interior ideal of M is a quasi-interior ideal of M.*
(11) *If L is a left ideal and R is a right ideal of M then $B = R \cap L$ is a quasi-interior ideal of M.*

THEOREM 3.2. *If B be a left quasi-interior ideal of a $\Gamma-$semiring M, then B is a left bi-quasi ideal of M.*

PROOF. Suppose B is a left quasi-interior ideal of a $\Gamma-$semiring M. Then $M\Gamma B\Gamma M\Gamma B \subseteq B$. We have $B\Gamma M\Gamma B \subseteq M\Gamma B\Gamma M\Gamma B$ Therefore, $M\Gamma B \cap B\Gamma M\Gamma B \subseteq B\Gamma M\Gamma B \subseteq M\Gamma B\Gamma M\Gamma B \subseteq B$ Hence, B is a left bi-quasi ideal of M. □

COROLLARY 3.1. *If B be a right quasi-interior ideal of a $\Gamma-$semiring M, then B is a right bi-quasi ideal of M.*

COROLLARY 3.2. *If B be a quasi-interior ideal of a $\Gamma-$semiring M, then B is a bi-quasi ideal of M.*

THEOREM 3.3. *If B be a left quasi-interior ideal $\Gamma-$ semiring of M, then B is a bi-interior ideal of M.*

PROOF. Suppose B is a left quasi-interior ideal of a $\Gamma-$semiring M.Then $M\Gamma B\Gamma M\Gamma B \subseteq B$. We have $M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B\Gamma M\Gamma B \subseteq M\Gamma B\Gamma M\Gamma B \subseteq B$. Hence, B is a bi-interior ideal of M. □

COROLLARY 3.3. *If B be a right quasi-interior ideal of a $\Gamma-$ semiring M, then B is a bi–interior ideal of M.*

COROLLARY 3.4. *If B be a quasi-interior ideal of a $\Gamma-$ semiring M, then B is a bi-interior ideal of M.*

THEOREM 3.4. *Every left quasi-interior ideal of a $\Gamma-$semiring M is a bi-ideal of a $\Gamma-$semiring M.*

PROOF. Let B be a left quasi-interior ideal of a $\Gamma-$semiring M.
Then $B\Gamma M\Gamma B \subseteq M\Gamma B\Gamma M\Gamma B \subseteq B$. Therefore $B\Gamma M\Gamma B \subseteq B$.
Hence every left quasi-interior ideal of a $\Gamma-$semiring M is a bi-ideal of $\Gamma-$semiring M. □

COROLLARY 3.5. *Every right quasi-interior ideal of a $\Gamma-$semiring M is a bi-ideal of $\Gamma-$semiring M.*

COROLLARY 3.6. *Every quasi-interior ideal of a $\Gamma-$semiring M is a bi-ideal of $\Gamma-$semiring M.*

THEOREM 3.5. *Every left quasi-interior ideal of a $\Gamma-$semiring M is a bi-quasi interior ideal of $\Gamma-$semiring M.*

PROOF. Let B be a left quasi-interior ideal of $\Gamma-$semiring M.
Then $M\Gamma B\Gamma M\Gamma B \subseteq B$. Therefore $B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq M\Gamma B\Gamma M\Gamma B \subseteq B$. □

COROLLARY 3.7. *Every right quasi-interior ideal of a $\Gamma-$semiring M is a bi-quasi interior ideal of $\Gamma-$semiring M.*

COROLLARY 3.8. *Every quasi-interior ideal of a $\Gamma-$semiring M is a bi-quasi interior ideal of $\Gamma-$semiring M.*

THEOREM 3.6. *Every interior ideal of a $\Gamma-$semiring M is a left quasi-interior ideal of M.*

PROOF. Let I be an interior ideal of the $\Gamma-$semiring M.
Then $M\Gamma I\Gamma M\Gamma I \subseteq M\Gamma I\Gamma M \subseteq I$.
Hence, I is a left quasi-interior ideal of $\Gamma-$semiring M. □

THEOREM 3.7. *Let M be a $\Gamma-$semiring and B be a $\Gamma-$subsemiring of M. B is a quasi-interior ideal of M if and only if there exist left ideals L and R such that $R\Gamma L \subseteq B \subseteq R \cap L$.*

PROOF. Suppose B is a quasi-interior ideal of a $\Gamma-$semiring$M$.
Then $M\Gamma B\Gamma M\Gamma B \subseteq B$. Let $R = M\Gamma B$ and $L = M\Gamma B$.
Then L and R are left ideals of M.
Therefore $R\Gamma L \subseteq B \subseteq R \cap L$.
Conversely, suppose that there exist L and R are left ideals of M such that $R\Gamma L \subseteq B \subseteq R \cap L$.Then
$M\Gamma B\Gamma M\Gamma B \subseteq M\Gamma(R \cap L)\Gamma M\Gamma(R \cap L) \subseteq M\Gamma(R)\Gamma M\Gamma(L) \subseteq R\Gamma L \subseteq B$.
Hence, B is a left quasi-interior ideal of M. □

COROLLARY 3.9. *Let M be a $\Gamma-$semiring and B be a $\Gamma-$subsemiring of M. B is a right quasi-interior ideal of M if and only if there exist right ideals L and R such that $R\Gamma L \subseteq B \subseteq R \cap L$.*

THEOREM 3.8. *The intersection of a left quasi-interior ideal B of a $\Gamma-$semiring M and a right ideal A of M is always a left quasi-interior ideal of M.*

PROOF. Suppose $C = B \cap A$.

$$M\Gamma C\Gamma M\Gamma C \subseteq M\Gamma B\Gamma M\Gamma B \subseteq B$$

$M\Gamma C\Gamma M\Gamma C \subseteq M\Gamma A\Gamma M\Gamma A \subseteq A$. Since A is a left ideal of M.

Therefore, $M\Gamma C\Gamma M\Gamma C \subseteq B \cap A = C$.

Hence, the intersection of a left quasi-interior ideal B of a $\Gamma-$semiring M and a left ideal A of M is always a left quasi-interior ideal of M. □

COROLLARY 3.10. *The intersection of a right quasi-interior ideal B of a $\Gamma-$semiring M and a right ideal A of M is always a right quasi-interior ideal of M.*

COROLLARY 3.11. *The intersection of a quasi-interior ideal B of a $\Gamma-$semiring M and an ideal A of M is always a quasi-interior ideal of M.*

THEOREM 3.9. *Let A and C be left quasi-interior ideals of a $\Gamma-$semiring M, $B = A\Gamma C$ and B is additively subsemigroup of M. If $C\Gamma C = C$ then B is a left quasi-interior ideal of M.*

PROOF. Let A and C be left quasi-interior ideals of a $\Gamma-$semiring M and $B = A\Gamma C$.
Then $B\Gamma B = A\Gamma C\Gamma A\Gamma C = A\Gamma C\Gamma C\Gamma C\Gamma A\Gamma C \subseteq A\Gamma C\Gamma M\Gamma C\Gamma M\Gamma C \subseteq A\Gamma C = B$.
Therefore, $B = A\Gamma C$ is a $\Gamma-$subsemiring of M.

$$\begin{aligned} M\Gamma B\Gamma M\Gamma B &= M\Gamma A\Gamma C\Gamma M\Gamma A\Gamma C \\ &\subseteq M\Gamma A\Gamma M\Gamma A\Gamma C \subseteq A\Gamma C = B. \end{aligned}$$

Hence, B is a left quasi-interior ideal of M. □

COROLLARY 3.12. *Let A and C be quasi-interior ideals of a $\Gamma-$semiring M, $B = C\Gamma A$ and B is additively subsemigroup of M. If $C\Gamma C = C$ then B is a quasi-interior ideal of M.*

THEOREM 3.10. *Let A and C be $\Gamma-$subsemirings of a $\Gamma-$semiring M and $B = A\Gamma C$ and B is additively subsemigroup of M. If A is the left ideal of M then B is a quasi-interior ideal of M.*

Proof. Let A and C be $\Gamma-$subsemirings of M and $B = A\Gamma C$. Suppose A is the left ideal of M. Then $B\Gamma B = A\Gamma C\Gamma A\Gamma C \subseteq A\Gamma C = B$.

$$\begin{aligned} M\Gamma B\Gamma M\Gamma B &= M\Gamma A\Gamma C\Gamma M\Gamma A\Gamma C \\ &\subseteq A\Gamma C = B. \end{aligned}$$

Hence, B is a left quasi-interior ideal of M. □

Corollary 3.13. *Let A and C be $\Gamma-$subsemirings of a $\Gamma-$semiring M and $B = A\Gamma C$ and B is additively subsemigroup of M. If C is a right ideal then B is a right quasi-interior ideal of M.*

Theorem 3.11. *Let M be a $\Gamma-$semiring and T be a non-empty subset of M. If $\Gamma-$ subsemiring B of M containing $M\Gamma T\Gamma M\Gamma T$ and $B \subseteq T$ then B is a left quasi-interior ideal of $\Gamma-$semiring M.*

Proof. Let B be a $\Gamma-$ subsemiring of M containing $M\Gamma T\Gamma M\Gamma T$. Then

$$\begin{aligned} M\Gamma B\Gamma M\Gamma B &\subseteq M\Gamma T\Gamma M\Gamma T \\ &\subseteq B. \end{aligned}$$

Therefore, $B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B$.
Hence, B is a left quasi-interior ideal of M. □

Theorem 3.12. *If B is a left quasi-interior ideal of $\Gamma-$semiring M, $B\Gamma T$ is an additively subsemigroup of M and $T \subseteq B$ then $B\Gamma T$ is a left quasi-interior ideal of M.*

Proof. Suppose B is a left quasi-interior ideal of a $\Gamma-$semiring M, $B\Gamma T$ is an additively subsemigroup of M and $T \subseteq B$.Then $B\Gamma T\Gamma B\Gamma T \subseteq B\Gamma T$. Hence, $B\Gamma T$ is a $\Gamma-$subsemiring of M.

$$\text{We have } \; M\Gamma B\Gamma T\Gamma M\Gamma B\Gamma T \subseteq M\Gamma B\Gamma M\Gamma B\Gamma T \subseteq B\Gamma T.$$

Hence, $B\Gamma T$ is a left quasi-interior ideal of $\Gamma-$semiring M. □

Theorem 3.13. *Let B be a left quasi-interior ideal of a $\Gamma-$semiring M and I be interior ideal of M. Then $B \cap I$ is a left quasi-interior ideal of M.*

Proof. Suppose B is a bi-ideal of M and I is an interior ideal of M. Obviously $B \cap I$ is $\Gamma-$subsemiring of M. Then

$$M\Gamma(B \cap I)\Gamma M\Gamma(B\Gamma I) \subseteq M\Gamma B\Gamma M\Gamma B \subseteq B$$
$$M\Gamma(B \cap I)\Gamma M\Gamma(B\Gamma I) \subseteq M\Gamma I\Gamma M\Gamma I \subseteq I$$

Therefore, $M\Gamma(B \cap I)\Gamma M\Gamma(B\Gamma I) \subseteq B \cap I$.
Hence, $B \cap I$ is a left quasi-interior ideal of M. □

Theorem 3.14. *Let M be a $\Gamma-$semiring and T be a $\Gamma-$subsemiring of M. Then every $\Gamma-$subsemiring of T containing $M\Gamma T\Gamma M\Gamma T$ is a left quasi-interior ideal of M.*

PROOF. Let C be a $\Gamma-$subsemiring of T containing $M\Gamma T\Gamma M\Gamma T$. Then

$$C\Gamma M\Gamma C \subseteq M\Gamma T\Gamma M\Gamma T \subseteq C.$$

Hence, C is a left quasi-interior ideal of M . □

THEOREM 3.15. *The intersection of $\{B_\lambda \mid \lambda \in A\}$ left quasi-interior ideals of a $\Gamma-$semiring M is a left quasi-interior ideal of M.*

PROOF. Let $B = \bigcap_{\lambda \in A} B_\lambda$. Then B is a $\Gamma-$subsemiring of M.
Since B_λ is a left quasi-interior ideal of M, we have

$$\begin{aligned} &M\Gamma B_\lambda \Gamma M \Gamma B_\lambda \subseteq B_\lambda, \text{ for all } \lambda \in A \\ \Rightarrow &M\Gamma \cap B_\lambda \Gamma M \cap B_\lambda \subseteq \cap B_\lambda \\ \Rightarrow &M\Gamma B\Gamma M\Gamma B \subseteq B. \end{aligned}$$

Hence, B is a left quasi-interior ideal of M. □

THEOREM 3.16. *Let B be a left quasi-interior ideal of a $\Gamma-$semiring M ,$e \in B$ and e be $\beta-$idempotent. Then $e\Gamma B$ is a left quasi-interior ideal of M.*

PROOF. Let B be a left quasi-interior ideal of a $\Gamma-$semiring M. Suppose $x \in B \cap e\Gamma M$. Then, $x \in B$ and $x = e\alpha y, \alpha \in \Gamma, y \in M$.

$$\begin{aligned} x &= e\alpha y \\ &= e\beta e\alpha y \\ &= e\beta(e\alpha y) \\ &= e\beta x \in e\Gamma B. \end{aligned}$$

$$\begin{aligned} \text{Therefore } B \cap e\Gamma M &\subseteq e\Gamma B \\ e\Gamma B \subseteq B &\text{ and } e\Gamma B \subseteq e\Gamma M \\ \Rightarrow e\Gamma B &\subseteq B \cap e\Gamma M \\ \Rightarrow e\Gamma B &= B \cap e\Gamma M. \end{aligned}$$

Hence, $e\Gamma B$ is a left quasi-interior ideal of M. □

COROLLARY 3.14. *Let M be a $\Gamma-$semiring M and e be $\alpha-$idempotent. Then $e\Gamma M$ and $M\Gamma e$ are left quasi-interior ideal and right quasi-interior ideal of M respectively.*

THEOREM 3.17. *Let M be a $\Gamma-$semiring. If $M = M\Gamma a$, for all $a \in M$. Then every left quasi-interior ideal of M is a quasi ideal of M.*

Proof. Let B be a left quasi-interior ideal of a $\Gamma-$semiring M and $a \in B$. Then

$$\begin{aligned}
&\Rightarrow M\Gamma a \subseteq M\Gamma B,\\
&\Rightarrow M \subseteq M\Gamma B \subseteq M\\
&\Rightarrow M\Gamma B = M\\
&\Rightarrow B\Gamma M = B\Gamma M\Gamma B \subseteq M\Gamma B\Gamma M\Gamma B \subseteq B\\
&\Rightarrow M\Gamma B \cap B\Gamma M \subseteq M \cap B\Gamma M \subseteq B\Gamma M \subseteq B.
\end{aligned}$$

Therefore, B is a left quasi ideal of M. □

Theorem 3.18. *B is a left quasi-interior ideal of a $\Gamma-$semiring M if and only if B is a left ideal of some ideal of a $\Gamma-$semiring$M$.*

Proof. Suppose B is a left ideal of ideal R of a $\Gamma-$semiring M. Then $R\Gamma B \subseteq B, M\Gamma R \subseteq R$ and $M\Gamma B\Gamma M\Gamma B \subseteq M\Gamma R\Gamma M\Gamma B \subseteq R\Gamma M\Gamma B \subseteq R\Gamma B \subseteq B$. Therefore, B is a left quasi-interior ideal of a $\Gamma-$semiring M .
Conversely suppose that B is a left quasi-interior ideal of a $\Gamma-$semiring M. Then $M\Gamma B\Gamma M\Gamma B \subseteq B$. Therefore, B is a left ideal of an ideal $M\Gamma B\Gamma M$ of a $\Gamma-$semiring M. □

Corollary 3.15. *B is a right quasi-interior ideal of a $\Gamma-$semiring M if and only if B is a right ideal of some ideal of a $\Gamma-$semiring$M$.*

Corollary 3.16. *B is a quasi-interior ideal of a $\Gamma-$semiring M if and only if B is an ideal of some ideal of a $\Gamma-$semiring M.*

4. Left(Right) quasi-interior simple $\Gamma-$semiring

In this section, we introduce the notion of a left quasi-interior simple $\Gamma-$semiring and characterize the left quasi-interior simple $\Gamma-$semiring using left quasi-interior ideals of a $\Gamma-$semiring and study the properties of minimal left quasi-interior ideals of a $\Gamma-$semiring and the properties of left(right) quasi-interior ideals of a regular $\Gamma-$semiring.

Definition 4.1. *A $\Gamma-$semiring M is a left (right) simple $\Gamma-$semiring if M has no proper left (right) ideals of M.*

Definition 4.2. *A $\Gamma-$semiring M is said to be simple $\Gamma-$semiring if M has no proper ideals of M.*

Definition 4.3. *A $\Gamma-$semiring M is said to be left(right) quasi-interior simple $\Gamma-$semiring if M has no left(right) quasi-interior ideal other than M itself.*

Definition 4.4. *A $\Gamma-$semiring M is said to be quasi-interior simple $\Gamma-$semiring if M has no quasi-interior ideal other than M itself.*

Theorem 4.1. *If M is a division $\Gamma-$semiring then M is a left quasi-interior simple $\Gamma-$semiring.*

Proof. Let B be a proper left quasi-interior ideal of the division $\Gamma-$semiring M, $x \in M$ and $0 \neq a \in B$. Since M is a division $\Gamma-$semiring, there exist $b \in M, \alpha \in \Gamma$ such that $a\alpha b = 1$. Then there exists $\beta \in \Gamma$ such that $a\alpha b\beta x = x = x\beta a\alpha b$. Therefore $x \in B\Gamma M$ and $M \subseteq B\Gamma M$. We have $B\Gamma M \subseteq M$. Hence $M = B\Gamma M$. Similarly, we can prove $M\Gamma B = M$.

$$M = M\Gamma B = M\Gamma B\Gamma B = M\Gamma B\Gamma B\Gamma B$$
$$\subseteq M\Gamma B\Gamma M\Gamma B \subseteq B$$
$$M \subseteq B$$
$$\text{Therefore } M = B.$$

Hence, division $\Gamma-$semiring M has no proper left-quasi-interior ideals. □

Corollary 4.1. *If M is a division $\Gamma-$semiring then M is a right quasi-interior simple $\Gamma-$semiring.*

Corollary 4.2. *If M is a division $\Gamma-$semiring then M is a quasi-interior simple $\Gamma-$semiring.*

Theorem 4.2. *Let M be a simple $\Gamma-$semiring. Every left quasi-interior ideal of M is a left ideal of M.*

Proof. Let M be a simple $\Gamma-$semiringl and B be a left quasi-interior ideal of M.
Then $M\Gamma B\Gamma M\Gamma B \subseteq B$ and $M\Gamma B\Gamma M$ is an ideal of M.
Since M is a simple $\Gamma-$semiring, we have $M\Gamma B\Gamma M = M$. Therefore

$$M\Gamma B\Gamma M\Gamma B \subseteq B$$
$$\Rightarrow M\Gamma B \subseteq B.$$

□

Corollary 4.3. *Let M be a simple $\Gamma-$semiring. Every right quasi-interior ideal is a right ideal of M.*

Corollary 4.4. *Let M be a simple $\Gamma-$semiring. Every quasi-interior ideal is an ideal of M.*

Theorem 4.3. *Let M be a $\Gamma-$semiring. Then M is a left quasi-interior simple $\Gamma-$semiring if and only if $< a >= M$, for all $a \in M$ and where $< a >$ is the smallest left quasi-interior ideal generated by a.*

Proof. Suppose M is the left quasi-interior simple $\Gamma-$semiring, $a \in M$ and $B = M\Gamma a$.
Then B is a left ideal of M.
Therefore, B is a left quasi-interior ideal of M.
Therefore, $B = M$. Hence, $M\Gamma a = M$, for all $a \in M$.

$$M\Gamma a \subseteq < a > \subseteq M$$
$$\Rightarrow M \subseteq < a > \subseteq M.$$
$$\text{Therefore } M = < a > .$$

Suppose $< a >$ is the smallest left quasi-interior ideal of M generated by a, $< a >= M$, A is the left quasi-interior ideal and $a \in A$. Then

$$\begin{aligned} &< a > \subseteq A \subseteq M \\ \Rightarrow &M \subseteq A \subseteq M. \end{aligned}$$

Therefore, $A = M$. Hence M is a left quasi-interior ideal simple $\Gamma-$semiring. □

COROLLARY 4.5. *Let M be a $\Gamma-$semiring.Then M is a right quasi-interior simple $\Gamma-$semiring if and only if $< a >= M$, for all $a \in M$ and where $< a >$ is the smallest right quasi-interior ideal generated by a.*

COROLLARY 4.6. *Let M be a $\Gamma-$semiring.Then M is a quasi-interior simple $\Gamma-$semiring if and only if $< a >= M$, for all $a \in M$ and where $< a >$ is the smallest quasi-interior ideal generated by a.*

THEOREM 4.4. *Let M be a $\Gamma-$semiring. Then M is a left quasi-interior simple $\Gamma-$semriring if and only if $M\Gamma a\Gamma M\Gamma a = M$, for all $a \in M$.*

PROOF. Suppose M is the left-quasi interior simple $\Gamma-$semiring and $a \in M$.
Then $M\Gamma a\Gamma M\Gamma a$ is a quasi-interior ideal of M.
Hence, $M\Gamma a\Gamma M\Gamma a = M$, for all $a \in M$.
Conversely, suppose that $M\Gamma a\Gamma M\Gamma a = M$, for all $a \in M$.
Let B be a left quasi-interior ideal of the $\Gamma-$semiring M and $a \in B$.

$$\begin{aligned} M &= M\Gamma a\Gamma M\Gamma a \\ &\subseteq M\Gamma B\Gamma M\Gamma B \subseteq B \end{aligned}$$

Therefore $M = B$.

Hence, M is a left quasi-interior simple $\Gamma-$semiring. □

COROLLARY 4.7. *Let M be a $\Gamma-$semiring. Then M is a right quasi-interior simple $\Gamma-$semiring if and only if $a\Gamma M\Gamma a\Gamma M = M$, for all $a \in M$.*

COROLLARY 4.8. *Let M be a $\Gamma-$semiring. Then M is a quasi-interior simple $\Gamma-$semiring if and only if $a\Gamma M\Gamma a\Gamma M = M$ and $M\Gamma a\Gamma M\Gamma a = M$,for all $a \in M$.*

THEOREM 4.5. *If $\Gamma-$semiring M is a left simple $\Gamma-$semiring then every left quasi-interior ideal of M is a right ideal of M.*

PROOF. Let B be a left quasi-interior ideal of the left simple $\Gamma-$semiring M. Then $M\Gamma B$ is a left ideal of M and $M\Gamma B \subseteq M$. Therefore $M\Gamma B = M$. Then

$$\begin{aligned} &\Rightarrow B\Gamma M = B\Gamma M\Gamma B \\ &\Rightarrow \ \subseteq M\Gamma B\Gamma M\Gamma B \subseteq B \\ &\Rightarrow B\Gamma M \subseteq B. \end{aligned}$$

Hence, every left quasi-interior ideal is a right ideal of M.

□

COROLLARY 4.9. *If $\Gamma-$semiring M is right simple $\Gamma-$semiring then every right quasi-interior ideal of M is a left ideal of M.*

COROLLARY 4.10. *Every quasi-interior ideal of left and right simple $\Gamma-$semiring M is an ideal of M.*

THEOREM 4.6. *Let M be a $\Gamma-$semiring and B be a left quasi-interior ideal of M. Then B is a minimal left quasi-interior ideal of M if and only if B is a left quasi-interior simple $\Gamma-$subsemiring of M.*

PROOF. Let B be a minimal left quasi-interior ideal of a $\Gamma-$semiring M and C be a left quasi-interior ideal of B. Then $B\Gamma C\Gamma B\Gamma C \subseteq C$.
and $B\Gamma C\Gamma B\Gamma C$ is a left quasi-interior ideal of M.
Since C is a quasi-interior ideal of B,

$$\begin{aligned} &B\Gamma C\Gamma B\Gamma C = B \\ \Rightarrow &B = B\Gamma C\Gamma B\Gamma C \subseteq C \\ \Rightarrow &B = C. \end{aligned}$$

Conversely, suppose that B is the left quasi-interior simple $\Gamma-$subsemiring of M. Let C be a left quasi-interior ideal of M and $C \subseteq B$.

$\Rightarrow B\Gamma C\Gamma B\Gamma C \subseteq M\Gamma C\Gamma M\Gamma C \subseteq M\Gamma B\Gamma M\Gamma B \subseteq B$.Hence, C is a left quasi-interior of B.

$\Rightarrow B = C$.Since B is a left quasi-interior simple $\Gamma-$subsemiring of M.

Hence, B is a minimal left quasi-interior ideal of M. □

COROLLARY 4.11. *Let M be a $\Gamma-$semiring and B be a right quasi-interior ideal of M. Then B is a minimal right quasi-interior ideal of M if and only if B is a right quasi-interior simple $\Gamma-$subsemiring of M.*

COROLLARY 4.12. *Let M be a $\Gamma-$semiring and B be a quasi-interior ideal of M. Then B is a minimal quasi-interior ideal of M if and only if B is a quasi-interior simple $\Gamma-$subsemiring of M.*

THEOREM 4.7. *Let M be a $\Gamma-$semiring and $B = L\Gamma L$, where L is a minimal left ideal of M . Then B is a minimal left quasi-interior ideal of M.*

PROOF. Obviously $B = L\Gamma L$ is a left quasi-interior ideal of M. Let A be a left quasi-interior ideal of M such that $A \subseteq B$.
We have $M\Gamma A$ is a left ideal of M. Then

$$\begin{aligned} M\Gamma A &\subseteq M\Gamma B \\ &= M\Gamma L\Gamma L \\ &\subseteq L, \text{ since } L \text{ is a left ideal of } M. \\ \text{Therefore } M\Gamma A &= L, \text{ since } L \text{ is a minimal left ideal of } M. \\ \text{Hence } \ B &= M\Gamma A\Gamma M\Gamma A \\ &\subseteq A. \end{aligned}$$

Therefore $A = B$. Hence B is a minimal left quasi-interior ideal of M.

□

COROLLARY 4.13. *Let M be a $\Gamma-$semiring and $B = R\Gamma R$, where R is a minimal right ideal of M . Then B is a minimal right quasi-interior ideal of M.*

5. Left quasi-interior ideals of regular $\Gamma-$semiring

In this section, we characterize regular $\Gamma-$semiring using left quasi-interior ideals of a $\Gamma-$semiring.

THEOREM 5.1. *Let M be a regular $\Gamma-$semiring. Then every left quasi-interior ideal of a regular $\Gamma-$semiring M is a left ideal of M.*

PROOF. Let B be a left quasi-interior ideal of a regular $\Gamma-$semiring M. Then

$$\begin{aligned} & M\Gamma B\Gamma M\Gamma B \subseteq B \\ \Rightarrow & M\Gamma B \subseteq M\Gamma B\Gamma M\Gamma M\Gamma B, \text{ since } M \text{ is regular} \\ & \subseteq M\Gamma B\Gamma M\Gamma B \subseteq B. \end{aligned}$$

□

COROLLARY 5.1. *Let M be a regular $\Gamma-$semiring. Then every right quasi-interior ideal is a right ideal of M.*

COROLLARY 5.2. *Let M be a regular $\Gamma-$semiring. Then every quasi-interior ideal is an ideal of M.*

THEOREM 5.2. *M is a regular $\Gamma-$semiring if and only if $A\Gamma B = A\cap B$ for any right ideal A and left ideal B of a $\Gamma-$semiring M.*

THEOREM 5.3. *Let M be a $\Gamma-$semiring Then B is a quasi-interior ideal of a regular $\Gamma-$semiring M if and only if $B\Gamma M\Gamma B\Gamma M = B$ and $M\Gamma B\Gamma M\Gamma B = B$ for all quasi-interior ideals B of M.*

PROOF. Suppose M is a regular $\Gamma-$semiring, B is a quasi-interior ideal of M and $x \in B$. Then $M\Gamma B\Gamma M\Gamma B \subseteq B$ and there exist $y \in M,\ \alpha, \beta \in \Gamma$ such that $x = x\alpha y\beta x\alpha y\beta x \in M\Gamma B\Gamma M\Gamma B$. Therefore $x \in M\Gamma B\Gamma M\Gamma B$.
Hence $M\Gamma B\Gamma M\Gamma B = B$. Similarly we can prove $B\Gamma M\Gamma B\Gamma M = B$.
Conversely, suppose that $B\Gamma M\Gamma B\Gamma M = B$ and $M\Gamma B\Gamma M\Gamma B = B$ for all quasi-interior ideals B of M.
Let $B = R\cap L$ and $C = R\Gamma L$, where R is a right ideal and L is a left ideal of M. Then B and C are quasi-interior ideals of M.
Therefore $(R\cap L)\Gamma M\Gamma(R\cap L)\Gamma M = R\cap L$

$$\begin{aligned} R\cap L &= (R\cap L)\Gamma M\Gamma(R\cap L)\Gamma M \\ &\subseteq R\Gamma M\Gamma L\Gamma M \\ &\subseteq R\Gamma L\Gamma M \\ R\cap L &= (R\cap L)\Gamma M\Gamma(R\cap L)\Gamma M \\ &\subseteq R\Gamma L\Gamma M\Gamma R\Gamma L\Gamma M \\ &\subseteq R\Gamma L \\ &\subseteq R\cap L \text{ (since } R\Gamma L \subseteq L \text{ and } R\Gamma L \subseteq R). \end{aligned}$$

Therefore, $R\cap L = R\Gamma L$. Hence M is a regular $\Gamma-$semiring. □

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Murali Krishna Rao Marapureddy, Department of Mathematics, Sankethika Institute of Tech. and Management,, Andhra Pradesh, India.

Email address: mmarapureddy@gmail.com