BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Bull. Int. Math. Virtual Inst., **13**(2)(2023), 375–381
DOI: 10.7251/BIMVI2302375E

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

SOLVABLE MULTIGROUP AND ITS PROPERTIES

Paul Augustine Ejegwa, Johnson Mobolaji Agbetayo, John Abah Agba, and Idris Muhammad Adamu

Abstract. The concept of multigroups is the study of groups in multiset setting. Though several notions in group theory have been investigated in multigroup, some concepts are yet to be applied to multisets. In this present work, the notion of solvable multigroups is introduced and certain results are established. Solvable series for a multigroup is defined in such a way that the family of the submultigroups of the considered multigroup has identical support. Among several results, it is established that there exists an equivalent condition between the solvability of a multigroup and its support.

1. Introduction

One of the limitation of set theory is the omission of repeated elements in a collection, which is acceptable in real-life problems. The term multiset was introduced as an extensional set where repetition of elements is allowed [**25**]. Multiset was first suggested by N. G. de Bruijn (cf. [**8**]) in a communication to D. E. Knuth. Due to the appropriateness of the concept, it has been researched and applied in diverse areas [**6, 7, 21, 24, 28, 30**]. In a way to show the application of multiset in group theory, Nazmul et al. [**26**] introduced multigroup in multiset framework and presented a number of results. An elaborate work on multigroup theory has been carried out [**10**]. Numerous results in multigroup theory have been discussed [**2, 11**]. The notion of cuts of multigroups was introduced as a bridge between multigroup and group, and extended to homomorphism [**12**]. The ideas of strongly invariant

2010 *Mathematics Subject Classification.* Primary 03E72; Secondary 06D72, 11E57, 19A22.
Key words and phrases. Multiset, Multigroup, Normality in multigroup, Quotient multigroup, Solvable multigroup.
Communicated by Dusko Bogdanic.

submultigroups, normal submultigroups, characteristic submultigroups and Frattini submultigroups have been investigated in multigroup setting with some results [**1, 13, 23, 27**].

The notions of order in multigroup, comultisets, cyclic multigroups and factor multigroups have been established with results [**3, 4, 5, 14**]. Some group's analogous concepts like direct product, actions, etc. have been studied in multigroup context [**15, 16, 17, 22, 18, 19**]. Some algebraic structures have been studied based on multiset approach [**9, 20, 29, 31**].

Although numerous group's theoretic notions have been investigated in multigroup setting, solvable group is yet to be studied via multiset approach. The fact that concepts like comultiset, normal submultigroups and factor multigroups have been established, it is then needful to discuss solvability in multigroup setting. To this end, this paper seeks to discuss solvable multigroup and characterize its properties. The rest of the paper are delineated as follows; Section 2 presents preliminaries on multisets and multigroups, Section 3 presents the concept of solvable multigroups and discusses certain of its properties, and Section 4 summarizes and gives recommendations for future studies.

2. Preliminaries

We denote a non-empty set as X and a group as G throughout the paper.

Definition 2.1. [**24**] A multiset M over X is a pair $\langle X, C_M\rangle$, where

$$C_M : X \to \mathcal{N} = \{0, 1, 2, ...\}$$

is a function, such that for $x \in X$ implies $M(x) = C_M(x) > 0$ and $C_M(x)$ denoted the number of times an object x occur in M. If $C_M(x) = 0$, then $x \notin X$. The set X is called the ground or generic set of the class of all multisets containing objects from X.

Definition 2.2. [**30**] Let M and N be multisets of X. Then

(i) $M = N \iff C_M(x) = C_N(x)$ for all $x \in X$,

(ii) $M \subseteq N \iff C_M(x) \leqslant C_N(x)$ for all $x \in X$,

(iii) $M \cap N \implies C_{M\cap N}(x) = \min\Big(C_M(x), C_N(x)\Big)$ for all $x \in X$,

(iv) $M \cup N \implies C_{M\cup N}(x) = \max\Big(C_M(x), C_N(x)\Big)$ for all $x \in X$,

(v) $M \oplus N \implies C_{M\oplus N}(x) = C_M(x) \oplus C_N(x)$ for all $x \in X$.

Definition 2.3. [**26**] A multiset M of G is a multigroup of G if (i) $C_M(xy) \geqslant \min\Big(C_M(x), C_M(y)\Big)$, (ii) $C_M(x^{-1}) = C_M(x)$ for all $x, y \in G$. Since $C_M(e) = C_M(xx^{-1}) \geqslant \min\Big(C_M(x), C_M(x)\Big) = C_M(x)$ for all $x \in G$, where e is the identity element of G, then $C_M(e)$ is the upper bound of M, which we call the tip of M.

Definition 2.4. [**26**] Let M be a multigroup of G. Then, the support of M is the set $\text{supp}(M) = \{x \in G \mid C_M(x) \geqslant 0\}$.

Proposition 2.1. [**26**] *The support of a multigroup M of G is a subgroup of G.*

Definition 2.5. [**10**] Let M and N be multigroups of G. Then, the product $M \circ N$ is defined to be a multiset of G as follows:

$$C_{M\circ N}(x) = \begin{cases} \bigvee_{x=yz} \min\Big(C_M(y), C_N(z)\Big), & \text{if } \exists\, y, z \in X \text{ such that } x = yz \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.6. [**17**] A multigroup M of G is said to be commutative if $C_M(xy) = C_M(yx)$ for all $x, y \in G$. Certainly, if G is a commutative group, then a multigroup M of G is commutative.

Definition 2.7. [**17**] Let M and N be multigroups of G. We say M is a submultigroup of N if $M \subseteq N$. Again, M is a proper submultigroup of N if $M \subseteq N$ and $M \neq N$.

Definition 2.8. [**13**] Let M be a submultigroup of a multigroup N of G. We say M is normal in N if $C_M(xy) = C_M(yx) \iff C_M(y) = C_M(x^{-1}yx)$ for all $x, y \in G$, and we write $M \lhd N$. In fact, every normal submultigroup is self-normal and commutative.

Definition 2.9. [**14**] Suppose M is a submultigroup of a multigroup N of G. Then, the submultiset yM of N for $y \in G$ defined by $C_{yM}(x) = C_M(y^{-1}x)$ for all $x \in G$ is called the left comultiset of M. Similarly, the submultiset My of N for $y \in G$ defined by $C_{My}(x) = C_M(xy^{-1})$ for all $x \in G$ is called the right comultiset of M.

Definition 2.10. [**14**] Let N be a multigroup of a group G and M be a normal submultigroup of N. Then the set of right/left comultisets of M with the property $C_{xM\circ yM}(z) = C_{xyM}(z)$ for all $x, y, z \in X$ form a multigroup called factor or quotient multigroup of N determined by M, denoted as N/M.

3. Main results

Before the establishment of solvable multigroup, we first present some results as follows:

Theorem 3.1. *Let M be a multigroup of G. Then supp(M) is abelian iff M is commutative.*

Proof. Let $x, y \in \text{supp}(M)$. If supp(M) is abelian then $xy = yx$, and so $C_M(xy) = C_M(yx)$ for all $x, y \in G$.

Conversely, if M is commutative then $C_M(xy) = C_M(yx)$ for all $x, y \in G$. Thus supp(M) is an abelian group because $C_M(xy) > 0 < C_M(yx) \implies xy = yx$ for all $x, y \in \text{supp}(M)$. □

Theorem 3.2. *(i) Every commutative multigroup is self-normal. (ii) If M and N are multigroups of G such that $M \lhd N$, then M is self-normal.*

Proof. Suppose M is a commutative multigroup of G. Then

$$C_M(xy) = C_M(yx) \text{ for all } x, y \in G,$$

and so $C_M(y) = C_M(x^{-1}yx)$. Hence $M \lhd M$, which proves (i). Again, if $M \lhd N$ then $C_M(x) \leqslant C_N(x)$ for all $x \in G$ and $C_M(x) > 0 < C_M(y) \Longrightarrow C_M(xy) = C_M(yx)$ for all $x, y \in G$. Thus (ii) holds from (i). $\square$

THEOREM 3.3. *Let M, N and O be multigroups of G such that (i) N/M and O/M are both in the canonical form, (ii) $N/M \lhd O/M$, (iii) $(O/M)/(N/M)$ is commutative. Then $N \lhd O$ and O/N is commutative.*

PROOF. Let H, H_1 and H_2 be the supports of M, N and O, respectively, and let H' be the zone of M, O. Then N/M and O/M are both multigroups of H'/H. If $x \in H_1$, then $C_N(x) = C_{N/M}(xH) \leqslant C_{O/M}(xH) = C_O(x)$, and $N \subseteq O$. Thus $C_O(x) > 0 < C_O(y) \Longrightarrow C_{O/M}(xH) = C_O(x) > 0 < C_O(y) = C_{O/M}(yH) \Longrightarrow C_{N/M}(xyH) = C_{N/M}(yxH) \Longrightarrow C_N(xy) = C_N(yx)$ for all $x, y \in G$, and so $N \lhd O$.

Again, we have $\mathrm{supp}(O/N) = H_2/H_1 \approx (H_2/H)/(H_1/H) = \mathrm{supp}(O/M)/\mathrm{supp}(N/M)$, which is abelian. By Theorem 3.1, O/N is commutative. $\square$

Now, we define the notion of solvable multigroup as follows:

DEFINITION 3.1. If M is a multigroup of G, then there exists a chain of successive submultigroups of M:

$$M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M, \tag{3.1}$$

such that $\mathrm{supp}(M_0) = \mathrm{supp}(M_1) = \cdots = \mathrm{supp}(M_n) = \mathrm{supp}(M)$.

Thus (3.1) can be rewritten as

$$C_{M_0}(x) \leqslant C_{M_1}(x) \leqslant \cdots \leqslant C_{M_n}(x) = C_M(x) \text{ for all } x \in G.$$

Albeit, if M is a trivial multigroup, we have $M_0 = M$.

DEFINITION 3.2. A multigroup M of G is solvable/soluble if there exists a chain of successive submultigroups

$$M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M,$$

where $M_i \lhd M_{i+1}$ and M_{i+1}/M_i is commutative for all $0 \leqslant i \leqslant n-1$.

Thus, such a finite chain of successive submultigroups of M is a solvable/soluble series for M denoted by M_i. Without contradiction, the solvable series for M can be written as

$$M_0 \lhd M_1 \lhd \cdots \lhd M_n = M.$$

THEOREM 3.4. *A multigroup M of a group G is solvable iff supp(M) is a solvable group.*

PROOF. Assume M is a solvable multigroup of G. Then there exists a solvable series for M as follows:

$$M_0 \lhd M_1 \lhd \cdots \lhd M_n = M.$$

Set $H = \mathrm{supp}(M)$, i.e. H is a subgroup of G. Then

$$\{e\} = H_0 \lhd H_1 \lhd \cdots \lhd H_n = H$$

is a solvable series for H since $\mathrm{supp}(M) = \mathrm{supp}(M_i)$. Thus H is a solvable group.

Conversely, let $H = \mathrm{supp}(M)$ be a solvable group (certainly, a subgroup of G). Then

$$\{e\} = H_0 \lhd H_1 \lhd \cdots \lhd H_n = H$$

is a solvable series for H. Consequently, we have

$$M_0 \lhd M_1 \lhd \cdots \lhd M_n = M,$$

which is a solvable series for M. Hence M is a solvable multigroup of G □

THEOREM 3.5. *Let M and N be multigroups of G with the same support H such that $M \subseteq N$ and M is self-normal. If M is solvable, then N is a solvable multigroup of G.*

PROOF. Let $M_0 \lhd M_1 \lhd \cdots \lhd M_n = M$ be a solvable series for M. Because M is self-normal and $\mathrm{supp}(M) = \mathrm{supp}(N) = H$, we get $M \lhd N$. Consequently, we have $\mathrm{supp}(N/M) = H/H = H$ and is abelian. Thus

$$M_0 \lhd M_1 \lhd \cdots \lhd M_n = M \lhd N$$

is a solvable series for N. Hence N is a solvable multigroup of G. □

THEOREM 3.6. *Let N be a solvable multigroup of G and let M be a self-normal submultigroup of N with $M \subseteq N_i$. Then M is solvable.*

PROOF. Let $N_0 \lhd N_1 \lhd \cdots \lhd N_n = N$ be a solvable series for N since N is solvable. Because $M \subseteq N_i$, we have

$$N_0 \cap M \subseteq N_1 \cap M \subseteq \cdots \subseteq N_n \cap M = M.$$

Clearly, $C_{N_1 \cap M}(x) > 0 < C_{N_1 \cap M}(y) \Longrightarrow C_{N_1}(x) > 0 < C_{N_1}(y)$ and $C_M(x) > 0 < C_M(y) \Longrightarrow C_{N_1}(xy) > 0 < C_{N_1}(yx)$ and $C_M(xy) > 0 < C_M(yx) \Longrightarrow C_{N_1 \cap M}(xy) = C_{N_1 \cap M}(yx)$ for all $x, y \in G$. Thus

$$N_0 \cap M \lhd N_1 \cap M \lhd \cdots \lhd N_n \cap M = M.$$

Again, let $H_i = \mathrm{supp}(N_i)$ and $H = \mathrm{supp}(M)$. Then we get a quotient $(H_2 \cap H)/(H_1 \cap H)$, which is abelian because H_2/H_1 is abelian. The same logic holds for the other quotients, and thus

$$N_0 \cap M \subseteq N_1 \cap M \subseteq \cdots \subseteq N_n \cap M = M$$

is the solvable series for M. Hence M is solvable. □

THEOREM 3.7. *Let M be a normal submultigroup of a multigroup N of G, and let N be self-normal. If M and N/M are solvable, then N is a solvable multigroup of G.*

PROOF. Assuming N'/M' is the canonical form of N/M. Then we have $M \subseteq M'$, $N \subseteq N'$, $\mathrm{supp}(M') = \mathrm{supp}(M) = H_1$ and $\mathrm{supp}(N') = \mathrm{supp}(N) = H_2$. Thus, there exists a solvable series

$$O_0 \lhd O_1 \lhd \cdots \lhd O_n = N'/M'.$$

Set $M' = M_m$ and $N' = N_n$. Assume there exist multigroups N_i of G such that $M' \lhd N_i \lhd N_{i+1}$ and $O_i = N_i/M'$ in the canonical form for $0 \leqslant i \leqslant n-1$. Thus

$$N_0/M' \lhd N_1/M' \lhd \cdots \lhd N_n/M' = N'/M'$$

is a solvable series for N'/M'. Since N_0/M' is a trivial submultigroup of N'/M', then it is meet to say that $N_0 = M'$. By Theorem (3.3), we have

$$M' = N_0 \lhd N_1 \lhd \cdots \lhd N_n = N', \tag{3.2}$$

where N_{i+1}/N_i is commutative for $0 \leqslant i \leqslant n-1$.

Next, M is self-normal by Theorem (3.2), and M' is solvable by Theorem (3.5). Thus, there exists a solvable series for M' as follows:

$$M_0 \lhd M_1 \lhd \cdots \lhd M_m = M'. \tag{3.3}$$

By juxtaposing (3.2) and (3.3), we have a solvable series for N'. Hence N is solvable by Theorem (3.6). $\square$

References

1. U. Adamu, M. A. Ibrahim, Strongly invariant submultigroups. *Theory Appl. Math. Computer Sci.*, **10**(2) (2020), 96 –103.
2. J. A. Awolola, A. M. Ibrahim, Some results on multigroups. *Quasi. Related Syst.*, **24**(2) (2016), 169–177.
3. J. A. Awolola, On multiset relations and factor multigroups. *South East Asian J. Math. Mathemat. Sci.*, **15**(3) (2019), 1–10.
4. J. A. Awolola, On cyclic multigroup family. *Ratio Mathematica*, **37** (2019), 61–68.
5. J. A. Awolola, P. A. Ejegwa, On some algebraic properties of order of an element of a multigroup. *Quasi. Related Syst.*, **25**(1) (2017), 21–26.
6. W. D. Blizard, Multiset theory. *Notre Dame J. logic*, **31** (1989), 36–65.
7. W. D. Blizard, The development of multiset theory. *Modern Logic*, **1**(4) (1991), 319–352.
8. N. G. DeBruijin, Denumerations of rooted trees and multisets. *Discrete Appl. Math.*, **6** (1983), 25–33.
9. S. Debnath, A. Debnath, Study of ring structure from multiset context. *Applied Sci.*, **21** (2019), 84–90.
10. P. A. Ejegwa, *A study of multigroup structure and its acting principles on multiset.* A PhD Thesis, Ahmadu Bello University, Zaria, Nigeria, 2018.
11. P. A. Ejegwa, A. M. Ibrahim, Some homomorphic properties of multigroups. *Bul. Acad. Stiinte Repub. Mold. Mat.*, **83**(1) (2017), 67–76.
12. P. A. Ejegwa, A. M. Ibrahim, Homomorphism of cuts of multigroups. *Gulf J. Math.*, **6**(1) (2018), 61–73.
13. P. A. Ejegwa, A. M. Ibrahim, Normal submultigroups and comultisets of a multigroup. *Quasi. Related Syst.*, **25**(2) (2017), 231–244.
14. P. A. Ejegwa, A. M. Ibrahim, On comultisets and factor multigroups. *Theory Appl. Math. Computer Sci.*, **7**(2) (2017), 124–140.
15. P. A. Ejegwa, A. M. Ibrahim, Direct product of multigroups and its generalization. *Int. J. Math. Combin.*, **4** (2017), 1–18.
16. P. A. Ejegwa, A. M. Ibrahim, Some group's analogous results in multigroup setting. *Ann. Fuzzy Math. Inform.*, **17**(3) (2019), 231–245.
17. P. A. Ejegwa, A. M. Ibrahim, Some properties of multigroups. *Palestine J. Math.*, **9**(1) (2020), 31–47.
18. P. A. Ejegwa, J. M. Agbetayo, Some results on commutators in multigroup framework. *J. Ramanujan Soc. Math. Mathemat. Sci.*, **7**(2) (2020), 67–82.

19. P. A. Ejegwa, A. M. Ibrahim, On divisible and pure multigroups and their properties. *Open J. Math. Sci.*, **4** (2020), 377–385.
20. P. A. Ejegwa, Concept of anti multigroups and its properties. *Earthline J. Math. Sci.*, **4**(1) (2020), 83–97.
21. K. P. Girish, S. J. John, Relations and functions in multiset context. *Inf. Sci.*, **179** (2009), 758–768.
22. A. M. Ibrahim, P. A. Ejegwa, Multigroup actions on multiset. *Ann. Fuzzy Math. Inform.*, **14**(5) (2017), 515–526.
23. A. M. Ibrahim, P. A. Ejegwa, Characteristic submultigroups of a multigroup. *Gulf J. Math.*, **5**(4) (2017), 1–8.
24. S. P. Jena, S. K. Ghosh, B. K. Tripathy, On the theory of bags and lists. *Inf. Sci.*, **132** (2001), 241–254.
25. D. Knuth, *The art of computer programming.* Semi Numerical Algorithms, Second Edition, volume 2, Addison-Wesley, Reading, Massachusetts, 1981.
26. Sk. Nazmul, P. Majumdar, S. K. Samanta, On multisets and multigroups. *Ann. Fuzzy Math. Inform.*, **6**(3) (2013), 643–656.
27. J. A. Otuwe, M. A. Ibrahim, Frattini submultigroups of multigroups. *Ratio Mathematica*, **39** (2020), 147–163.
28. D. Singh, A. M. Ibrahim, T. Yohanna, J. N. Singh, An overview of the applications of multisets. *Novi Sad J. Math.*, **37**(2), (2007) 73–92.
29. P. Suma, S. J. John, Multiset approach to algebraic structures. In: C. Jana, T. Senapati, M. Pal (Eds.), *Handbook of Research on Emerging Applications of Fuzzy Algebraic Structures* (pp. 78–90). IGI Global Publisher, Hershey, Pennsylvania 17033-1240, USA, 2020.
30. A. Syropoulos, *Mathematics of multisets.* Springer-Verlag Berlin Heidelberg, 347–358, 2001.
31. B. C. Tripathy, S. Debnath, D. Rakshit, On multiset group. *Proyecciones J. Math.*, **37**(3) (2018), 479–489.

Received by editors 10.10.2022; Revised version 15.7.2023; Available online 25.9.2023.

Paul Augustine Ejegwa, Department of Mathematics, Joseph Sarwuan Tarka University, Makurdi, Nigeria
Email address: `ejegwa.augustine@uam.edu.ng`

Johnson Mobolaji Agbetayo, Department of Mathematics, Joseph Sarwuan Tarka University, Makurdi, Nigeria
Email address: `agbetayojohnson@gmail.com`

John Abah Agba, Department of Mathematics, University of Calabar, Calabar, Nigeria
Email address: `maths4deworld@gmail.com`

Idris Muhammad Adamu, Department of Mathematics, Federal University Dutse, Dutse, Nigeria
Email address: `idreesmuhammadadam@gmail.com`