BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Bull. Int. Math. Virtual Inst., **13**(2)(2023), 365–373
DOI: 10.7251/BIMVI2302365M

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

ON COMPLEX FUZZY IDEALS OF SEMIGROUPS

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Abstract. In this paper the concepts of complex fuzzy subsemigroup, complex fuzzy bi-ideal and complex fuzzy ideal in semigroups are introduced and some important characterizations have been obtained.

1. Introduction

After the introduction of the classical notion of fuzzy sets by Zadeh[**7**] in 1965, many scientists used fuzzy sets in different fields of science. The use of fuzzy sets in algebraic structures was done by Rosenfeld[**6**] in 1971. He defined fuzzy subgroups and discussed their important properties. A semigroup is an algebraic structure consisting of a non-empty set S together with an associative binary operation[**4**]. The formal study of semigroups began in the early 20th century. Semigroups are important in many areas of mathematics, for example, coding and language theory, automata theory, combinatorics and mathematical analysis. The idea of fuzzy ideals, fuzzy bi-ideals in semigroup was given by Nobuaki Kuroki[**5**]. With the passage of time researchers introduced many extensions of fuzzy sets, for example, intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets, bipolar fuzzy sets etc. Various operations of complex fuzzy sets were introduced by Zhang et al[**8**] in 2009. In 2018, Al-Tahan and Davvaz[**2**] introduced the concept of complex H_v subgroups and discussed various characteristics of these groups.

In this paper we have introduced the concept of complex fuzzy semigroups and defined new ideas like π-fuzzy subsemigroup, π-fuzzy bi-ideal and π-fuzzy ideal. We also introduced the concept of complex fuzzy subsemigroup, complex fuzzy bi-ideal

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2010 *Mathematics Subject Classification.* Primary 08A72; Secondary 20M12.

Key words and phrases. Semigroup, π-fuzzy subsemigroup, π-fuzzy bi-ideal, π-fuzzy ideal, Complex fuzzy subsemigroup, Complex fuzzy bi-ideal, Complex fuzzy left ideal, Complex fuzzy right ideal, Complex fuzzy ideal.

Communicated by Dusko Bogdanic.

and complex fuzzy ideal and investigated some of their important characteristics. Lastly we studied some properties of complex fuzzy subsemigroup, complex fuzzy bi-ideal and complex fuzzy ideal under semigroup homomorphism.

2. Preliminaries

DEFINITION 2.1. [**5**] *By a subsemigroup of* S *we mean a non-empty subset* A *of* S *such that* $A^2 \subseteq A$.

DEFINITION 2.2. [**5**] *A subsemigroup* A *of* S *is called a bi-ideal of* S *if* $ASA \subseteq A$.

DEFINITION 2.3. [**5**]*A left(right) ideal of a semigroup* S *is a non-empty subset* A *of* S *such that* $SA \subseteq A(AS \subseteq A)$. *If* A *is both a left and a right ideal of a semigroup* S, *then we say that* A *is an ideal of* S.

DEFINITION 2.4. [**5**] *Let* X *be a non-empty set. A fuzzy set of* X *is a function* $\mu : X \to [0,1]$ *and the complement of* μ, *denoted by* $\overline{\mu}$ *is a fuzzy set in* X *given by* $\overline{\mu}(x) = 1 - \mu(x)$ *for all* $x \in X$.

DEFINITION 2.5. [**5**] *A fuzzy subset of a semigroup* S *is called a fuzzy subsemigroup of* S *if* $\mu_A(xy) \geqslant \min\{\mu_A(x), \mu_A(y)\} \forall x, y \in S$.

DEFINITION 2.6. [**5**] *A fuzzy subsemigroup of a semigroup* S *is called a fuzzy bi-ideal of* S *if* $\mu_A(xyz) \geqslant \min\{\mu_A(x), \mu_A(z)\} \forall x, y, z \in S$.

DEFINITION 2.7. [**5**] *A non-empty fuzzy subset* μ_A *of a semigroup* S *is called a fuzzy left ideal of* S *if* $\mu_A(xy) \geqslant \mu_A(y) \forall x, y \in S$.

DEFINITION 2.8. [**5**] *A non-empty fuzzy subset* μ_A *of a semigroup* S *is called a fuzzy right ideal of* S *if* $\mu_A(xy) \geqslant \mu_A(x) \forall x, y \in S$.

DEFINITION 2.9. [**5**] *If* A *is both a left and a right ideal of a semigroup* S, *then we say that* A *is an ideal of* S.

DEFINITION 2.10. [**1**] *A complex fuzzy set, defined on a universe of discourse* U, *is characterized by a membership function* μ_A *that assigns any element a complex-valued grade of membership in A. By definition, the values* $\mu_A(x)$ *may receive all lie within the unit circle in the complex plane, and are thus of the form* $r_A(x)e^{i\omega_A(x)}$, *where* $i = \sqrt{-1}, r_A(x)$ *and* $\omega_A(x)$ *are both real-valued, and* $r_A(x) \in [0,1], \omega_A(x) \in [0, 2\pi]$. *The complex fuzzy set may be represented as the set of ordered pairs* $A = \{(x, \mu_A(x)) : x \in U\}$.

DEFINITION 2.11. [**1**] *Let* $A = \{(x, \mu_A(x)) : \mu_A(x) = r_A(x)e^{i\omega_A(x)}, x \in U\}$ *and* $B = \{(x, \mu_B(x)) : \mu_B(x) = r_B(x)e^{i\omega_B(x)}, x \in U\}$ *be two complex fuzzy sets of the same universe* U. *Then*

1. $\mu_{A\cap B}(x) = r_{A\cap B}(x)e^{i\omega_{A\cap B}(x)} = \min\{r_A(x), r_B(x)\}e^{i\min\{\omega_A(x), \omega_B(x)\}}$.

2. $\mu_{A^c}(x) = (1 - r_A(x))e^{i(2\pi - \omega_A(x))}$, *where* A^c *denotes the complement of* A.

DEFINITION 2.12. [**1**] *Let* $A = \{(x, \mu_A(x)) : x \in U\}$ *be a fuzzy set. Then the set* $A_\pi = \{(x, \gamma_{A_\pi}(x)) : \gamma_{A_\pi}(x) = 2\pi\mu_A(x), x \in U\}$ *is said to be a* π*-fuzzy set.*

3. Complex fuzzy subsemigroup and complex fuzzy bi-ideal

In this section we introduce the concepts of complex fuzzy subsemigroup and complex fuzzy bi-ideal and complex fuzzy ideal of a semigroup and investigate some of its important properties. Throughout this paper S stands for semigroup.

Definition 3.1. *Let S be a semigroup and A_π be a π-fuzzy set. Then A_π is said to be a π-fuzzy subsemigroup of S if $\gamma_{A_\pi}(xy) \geqslant \min\{\gamma_{A_\pi}(x), \gamma_{A_\pi}(y)\} \forall x, y \in S$.*

Definition 3.2. *Let S be a semigroup and A_π be a π-fuzzy set. Then A_π is said to be a π-fuzzy bi-ideal of S if $\gamma_{A_\pi}(xyz) \geqslant \min\{\gamma_{A_\pi}(x), \gamma_{A_\pi}(z)\} \forall x, y, z \in S$.*

From 3.1 and 3.2 following proposition follows easily.

Proposition 3.1. *A π-fuzzy set A_π a π-fuzzy subsemigroup(π-fuzzy bi-ideal) of S if and only if A is a fuzzy subsemigroup(fuzzy bi-ideal) of S.*

Definition 3.3. [**1**] *$A = \{(x, \mu_A(x)) : x \in U\}$ and $B = \{(x, \mu_B(x)) : x \in U\}$ be two complex fuzzy subsets of U, with membership functions $\mu_A(x) = r_A(x)e^{i\omega_A(x)}$ and $\mu_B(x) = r_B(x)e^{i\omega_B(x)}$ respectively. Then*

1. *A complex fuzzy subset A is said to be a homogeneous complex fuzzy subset if for all $x, y \in U$, $r_A(x) \leqslant r_A(y)$ if and only if $\omega_A(x) \leqslant \omega_A(y)$.*

2. *A complex fuzzy subset A is said to be a homogeneous complex fuzzy subset with B, if for all $x, y \in U$, $r_A(x) \leqslant r_B(y)$ if and only if $\omega_A(x) \leqslant \omega_B(y)$.*

Proposition 3.2. [**1**] *A complex fuzzy subset A of S is a homogeneous complex fuzzy subset of S if and only if A^c is a homogeneous complex fuzzy subset of S.*

Theorem 3.1. *Let S be a semigroup and $A = \{(x, \mu_A(x)) : x \in S\}$ be a homogeneous complex fuzzy subset with membership function $\mu_A(x) = r_A(x)e^{i\omega_A(x)}$. Then A is a complex fuzzy subsemigroup of S if and only if:* (1). *The fuzzy set $\overline{A} = \{(x, r_A(x)) : x \in S, r_A(x) \in [0, 1]\}$ is a fuzzy subsemigroup of S.* (2). *The π-fuzzy set $\underline{A} = \{(x, \omega_A(x)) : x \in S, \omega_A(x) \in [0, 2\pi]\}$ is a π-fuzzy subsemigroup of S.*

Proof. Let A be a complex fuzzy subsemigroup of S and $x, y \in S$. Then we have

$$\begin{aligned} r_A(xy)e^{i\omega_A(xy)} &= \mu_A(xy) \\ &\geqslant \min\{\mu_A(x), \mu_A(y)\} \\ &= \min\{r_A(x)e^{i\omega_A(x)}, r_A(y)e^{i\omega_A(y)}\} \\ &= \min\{\mu_A(x), \mu_A(y)\}e^{i\min\{\omega_A(x), \omega_A(y)\}} \end{aligned}$$

which implies $r_A(xy) \geqslant \min\{r_A(x), r_A(y)\}$ and $\omega_A(xy) \geqslant \min\{\omega_A(x), \omega_A(y)\}$. Hence $\overline{A}$ is a fuzzy subsemigroup and $\underline{A}$ is a π-fuzzy subsemigroup of S.

Conversely, let $\overline{A}$ be a fuzzy subsemigroup and $\underline{A}$ be a π-fuzzy subsemigroup of S. Let $x, y \in S$. Then

$$\begin{aligned}
\mu_A(xy) &= r_A(xy)e^{i\omega_A(xy)} \\
&= \min\{\mu_A(x), \mu_A(y)\}e^{i\min\{\omega_A(x),\omega_A(y)\}} \\
&= \min\{r_A(x)e^{i\omega_A(x)}, r_A(y)e^{i\omega_A(y)}\} \\
&\geqslant \min\{\mu_A(x), \mu_A(y)\}
\end{aligned}$$

Hence A is a complex fuzzy subsemigroup of S. □

By routine calculation we can prove the following theorem.

THEOREM 3.2. *Let S be a semigroup and $A = \{(x, \mu_A(x)) : x \in S\}$ be a homogeneous complex fuzzy subset with membership function $\mu_A(x) = r_A(x)e^{i\omega_A(x)}$. Then A is a complex fuzzy bi-ideal of S if and only if:* (1). *The fuzzy set $\overline{A} = \{(x, r_A(x)) : x \in S, r_A(x) \in [0,1]\}$ is a fuzzy bi-ideal of S.* (2). *The π-fuzzy set $\underline{A} = \{(x, \omega_A(x)) : x \in S, \omega_A(x) \in [0, 2\pi]\}$ is a π-fuzzy bi-ideal of S.*

PROPOSITION 3.3. *Let $\{A_i : i \in I\}$ be a family of complex fuzzy subsemigroups (complex fuzzy bi-ideals) of a semigroup S such that A_j is homogeneous with A_k for all $j, k \in I$. Then $\bigcap_{i\in I} A_i$ is complex fuzzy subsemigroup(complex fuzzy bi-ideal) of S.*

PROOF. For all $i \in I$ we have $r_{A_i}(x)$ is a fuzzy subsemigroup and $\omega_{A_i}(x)$ is a π-fuzzy subsemigoup of S. A_j is homogeneous with A_k for all $j, k \in I$.

$$\begin{aligned}
\mu_{\cap_{i\in I}} A_i(xy) &= r_{\cap_{i\in I}} A_i(xy)e^{\omega_{\cap_i} A_i(xy)} \\
&= \min_{i\in I}\{r_{A_i}(xy)\}e^{i\min_{i\in I}\{\omega_{A_i}(xy)\}} \\
&\geqslant \min_{i\in I}\{\min\{r_{A_i}(x), r_{A_i}(y)\}\}e^{i\min_{i\in I}\{\min\{\omega_{A_i}(x),\omega_{A_i}(y)\}\}} \\
&= \min\{\min_{i\in I}\{r_{A_i}(x)\}, \min_{i\in I}\{r_{A_i}(y)\}\}e^{i\min\{\min_{i\in I}\{\omega_{A_i}(x)\},\min_{i\in I}\{\omega_{A_i}(y)\}\}} \\
&= \min\{\min_{i\in I}\{r_{A_i}(x)\}e^{i\min_{i\in I}\{\omega_{A_i}(x)\}}, \min_{i\in I}\{r_{A_i}(y)\}e^{i\min_{i\in I}\{\omega_{A_i}(y)\}}\} \\
&= \min\{\mu_{\cap_{i\in I}} A_i(x), \mu_{\cap_{i\in I}} A_i(y)\}
\end{aligned}$$

Hence $\bigcap_{i\in I} A_i$ is complex fuzzy subsemigroup of S. Similarly we can prove the other case also.

□

THEOREM 3.3. *A complex fuzzy set A is a complex fuzzy subsemigroup of S if and only if A^c is a complex fuzzy subsemigroup of S.*

PROOF. Let A be a complex fuzzy subsemigroup of S. Let $x, y \in S$. Then

$$\begin{aligned}
\mu_{A^c}(xy) &= (1 - r_A(xy))e^{i(2\pi-\omega_A(xy))} \\
&\geqslant (1 - \min\{r_A(x), r_A(y)\})e^{i(2\pi-\min\{\omega_A(x),\omega_A(y)\})} \\
&= \max\{(1 - r_A(x)), (1 - r_A(y))\}e^{i\max\{(2\pi-\omega_A(x)),(2\pi-\omega_A(y))\}} \\
&\geqslant \min\{(1 - r_A(x)), (1 - r_A(y))\}e^{i\min\{(2\pi-\omega_A(x)),(2\pi-\omega_A(y))\}} \\
&\geqslant \min\{(1 - r_A(x))e^{i(2\pi-\omega_A(x))}, (1 - r_A(y))e^{(2\pi-\omega_A(y))}\} \\
&\geqslant \min\{\mu_{A^c}(x), \mu_{A^c}(y)\}
\end{aligned}$$

Hence A^c is a complex fuzzy subsemigroup of S.

Conversely, let $x, y \in S$. Then

$$\begin{aligned}
\mu_A(xy) &= r_A(xy))e^{i\omega_A(xy))} \\
&= (1-(1-r_A(xy)))e^{i(2\pi-(2\pi-\omega_A(xy)))} \\
&\geqslant (1-\min\{r_A(x), r_A(y)\})e^{i(2\pi-\min\{\omega_A(x),\omega_A(y)\})} \\
&= (1-\min\{(1-r_A(x)),(1-r_A(y))\})e^{i(2\pi-\min\{(2\pi-\omega_A(x)),(2\pi-\omega_A(y))\})} \\
&= \max\{r_A(x), r_A(y)\}e^{i\max\{\omega_A(x),\omega_A(y)\}} \\
&\geqslant \min\{r_A(x), r_A(y)\}e^{i\min\{\omega_A(x),\omega_A(y)\}} \\
&= \min\{r_A(x)e^{i\omega_A(x)}, r_A(y)e^{i\omega_A(y)}\} \\
&= \min\{\mu_A(x), \mu_A(y)\}
\end{aligned}$$

Hence A is a complex fuzzy subsemigroup of S. □

By similar arguments we can prove the following theorem.

Theorem 3.4. *A complex fuzzy set A is a complex fuzzy bi-ideal of S if and only if A^c is a complex fuzzy bi-ideal of S.*

Definition 3.4. [**1**] *Let $A = \{(x, \mu_A(x)) : \mu_A(x) = r_A(x)e^{i\omega_A(x)}, x \in S\}$ be a complex fuzzy subset of S. For $\alpha \in [0,1]$ and $\beta \in [0, 2\pi]$, the set $A_{(\alpha,\beta)} = \{x \in S : r_A(x) \geqslant \alpha, \omega_A(x) \geqslant \beta\}$ is called a level subset of the complex fuzzy subset A. In particular if $\beta = 0$, then we get the level subset $A_\alpha = \{x \in S : r_A(x) \geqslant \alpha\}$ and if $\alpha = 0$, then we get the level subset $A_\beta = \{x \in S : \omega_A(x) \geqslant \beta\}$.*

Proposition 3.4. *If $A = \{(x, \mu_A(x)) : \mu_A(x) = r_A(x)e^{i\omega_A(x)}, x \in S\}$ is a complex fuzzy subsemigroup(complex fuzzy bi-ideal) of S then the level subset $A_{(\alpha,\beta)}$ is subsemigroup(resp. bi-ideal) of S.*

Proof. Let $A = \{(x, \mu_A(x)) : \mu_A(x) = r_A(x)e^{i\omega_A(x)}, x \in S\}$ is a complex fuzzy subsemigroup of S. Let $x, y \in A_{(\alpha,\beta)}$. Then $r_A(x) \geqslant \alpha$ and $\omega_A(x) \geqslant \beta$, also $r_A(y) \geqslant \alpha$ and $\omega_A(y) \geqslant \beta$. Now

$$\begin{aligned}
r_A(xy)e^{i\omega_A(xy)} &= \mu_A(xy) \\
&\geqslant \min\{\mu_A(x), \mu_A(y)\} \\
&= \min\{r_A(x)e^{i\omega_A(x)}, r_A(y)e^{i\omega_A(y)}\} \\
&= \min\{\mu_A(x), \mu_A(y)\}e^{i\min\{\omega_A(x),\omega_A(y)\}}
\end{aligned}$$

This implies

$$\begin{aligned}
r_A(xy) &\geqslant \min\{r_A(x), r_A(y)\} \\
&\geqslant \min\{\alpha, \alpha\} \\
&= \alpha
\end{aligned}$$

and

$$\begin{aligned}
\omega_A(xy) &\geqslant \min\{\omega_A(x), \omega_A(y)\} \\
&\geqslant \min\{\beta, \beta\} \\
&= \beta
\end{aligned}$$

So $xy \in A_{(\alpha,\beta)}$. Therefore $A^2_{(\alpha,\beta)} \subseteq A_{(\alpha,\beta)}$. Hence $A_{(\alpha,\beta)}$ is a subsemigroup of S. Similarly by routine verification we can prove other case also. □

By routine calculation we can prove the following corollary.

COROLLARY 3.1. *If $A = \{(x, \mu_A(x)) : \mu_A(x) = r_A(x)e^{i\omega_A(x)}, x \in S\}$ is a complex fuzzy subsemigroup(complex fuzzy bi-ideal) of S then the level subsets $A_\alpha = \{x \in S : r_A(x) \geqslant \alpha\}$ and $A_\beta = \{x \in S : \omega_A(x) \geqslant \beta\}$ are two subsemigroups(resp. bi-ideals) of S.*

DEFINITION 3.5. [**5**] *A mapping Φ from a semigroup S to another semigroup T is called a homomorphism if $\Phi(xy) = \Phi(x)\Phi(y) \forall x, y \in S$.*

THEOREM 3.5. *Let $f : S \to T$ be an homomorphism of semigroups and $B = \{(x, \mu_B(x)) : \mu_B(x) = r_B(x)e^{i\omega_B(x)}, x \in T\}$ is a complex fuzzy subsemigroup(complex fuzzy bi-ideal) of T. Then the pre-image $f^{-1}(B)$ is a complex fuzzy subsemigroup(resp. complex fuzzy bi-ideal) of S, where $f^{-1}(B)(x) := \{(x, f^{-1}(\mu_B)(x)) : f^{-1}(\mu_B)(x) = f^{-1}(r_B)(x)e^{if^{-1}(\omega_B)(x)}\} = \{(x, (\mu_B)(f(x)) : (\mu_B)(f(x)) = r_B(f(x))e^{i\omega_B(f(x))}\}$.*

PROOF. Let B be a complex fuzzy subsemigroup of T and $x, y \in T$ and $f : S \to T$ be an homomorphism of semigroups. Then

$$\begin{aligned}
f^{-1}(\mu_B)(xy) &= f^{-1}(r_B)(xy)e^{if^{-1}(\omega_B(xy))} \\
&= r_B(f(xy))e^{i(\omega_B(f(xy))} \\
&= r_B(f(x)f(y))e^{i(\omega_B(f(x)f(y))} \\
&\geqslant \min\{r_B(f(x)), r_B(f(y))\}e^{i\min\{\omega_B(f(x)),\omega_B(f(y))\}} \\
&= \min\{r_B(f(x))e^{i\omega_B(f(x))}, r_B(f(y))e^{i\omega_B(f(y))}\} \\
&= \min\{f^{-1}(\mu_B)(x), f^{-1}(\mu_B)(y)\}
\end{aligned}$$

Hence the pre-image $f^{-1}(B)$ of B under f is a complex fuzzy subsemigroup of S. Similarly we can prove the other case also. □

4. Complex fuzzy ideal

DEFINITION 4.1. *Let S be a semigroup and A_π be a π-fuzzy set. Then A_π is said to be a π-fuzzy left ideal of S if $\gamma_{A_\pi}(xy) \geqslant \gamma_{A_\pi}(y) \forall x, y \in S$.*

DEFINITION 4.2. *Let S be a semigroup and A_π be a π-fuzzy set. Then A_π is said to be a π-fuzzy right ideal of S if $\gamma_{A_\pi}(xy) \geqslant \gamma_{A_\pi}(x) \forall x, y \in S$.*

DEFINITION 4.3. *If A_π is both a π-fuzzy left and π-fuzzy right ideal of S, then we say that A_π is a π-fuzzy ideal of S.*

From 4.1 and 4.2 following proposition follows easily.

PROPOSITION 4.1. *A π-fuzzy set A_π is a π-fuzzy left ideal(right ideal, ideal) of S if and only if A is a fuzzy left ideal(resp. fuzzy right ideal, fuzzy ideal) of S.*

THEOREM 4.1. *Let S be a semigroup and $A = \{(x, \mu_A(x)) : x \in S\}$ be a homogeneous complex fuzzy subset with membership function $\mu_A(x) = r_A(x)e^{i\omega_A(x)}$. Then A is a complex fuzzy left ideal(fuzzy right ideal, fuzzy ideal) of S if and only if:*

(1). *The fuzzy set* $\overline{A} = \{(x, r_A(x)) : x \in S, r_A(x) \in [0, 1]\}$ *is a fuzzy left ideal(fuzzy right ideal, fuzzy ideal) of* S. (2). *The* π*-fuzzy set* $\underline{A} = \{(x, \omega_A(x)) : x \in S, \omega_A(x) \in [0, 2\pi]\}$ *is a* π*-fuzzy left ideal(resp. fuzzy right ideal, fuzzy ideal) of* S.

PROOF. Let A be a complex fuzzy left ideal of S and $x, y \in S$. Then we have

$$\begin{aligned} r_A(xy)e^{i\omega_A(xy)} &= \mu_A(xy) \\ &\geqslant \mu_A(y) \\ &= r_A(y)e^{i\omega_A(y)} \end{aligned}$$

which implies $r_A(xy) \geqslant r_A(y)$ and $\omega_A(xy) \geqslant \omega_A(y)$. Hence $\overline{A}$ is a fuzzy left ideal and $\underline{A}$ is a π-fuzzy left ideal of S.

Conversely, let $\overline{A}$ be a fuzzy left ideal and $\underline{A}$ be a π-fuzzy left ideal of S. Let $x, y \in S$. Then

$$\begin{aligned} \mu_A(xy) &= r_A(xy)e^{i\omega_A(xy)} \\ &\geqslant r_A(y)e^{i\omega_A(y)} \\ &= \mu_A(y) \end{aligned}$$

Hence A is a complex fuzzy left ideal of S. Similarly we can prove the other cases also. □

PROPOSITION 4.2. *Let* $\{A_i : i \in I\}$ *be a family of complex fuzzy left ideals(complex fuzzy right ideals, complex fuzzy ideals) of a semigroup* S *such that* A_j *is homogeneous with* A_k *for all* $j, k \in I$. *Then* $\bigcap_{i \in I} A_i$ *is complex fuzzy left ideal(resp. complex fuzzy right ideal, complex fuzzy ideal) of* S.

PROOF. For all $i \in I$ we have $r_{A_i}(x)$ is a fuzzy left ideal and $\omega_{A_i}(x)$ is a π-fuzzy left ideal of S. A_j is homogeneous with A_k for all $j, k \in I$.

$$\begin{aligned} \mu_{\cap_{i \in I}} A_i(xy) &= r_{\cap_{i \in I}} A_i(xy)e^{\omega_{\cap_i} A_i(xy)} \\ &= \min_{i \in I}\{r_{A_i}(xy)\}e^{i \min_{i \in I}\{\omega_{A_i}(xy)\}} \\ &\geqslant \min_{i \in I} r_{A_i}(y)e^{i \min_{i \in I} \omega_{A_i}(y)} \\ &= \mu_{\cap_{i \in I}} A_i(y) \end{aligned}$$

Hence $\bigcap_{i \in I} A_i$ is complex fuzzy left ideal of S. Similarly we can prove the other cases also. □

THEOREM 4.2. *A complex fuzzy set* A *is a complex fuzzy left ideal(complex fuzzy right ideal, complex fuzzy ideal) of* S *if and only if* A^c *is a complex fuzzy left ideal(resp. complex fuzzy right ideal, complex fuzzy ideal) of* S.

PROOF. Let A be a complex fuzzy left ideal of S. Let $x, y \in S$. Then

$$\begin{aligned} \mu_{A^c}(xy) &= (1 - r_A(xy))e^{i(2\pi - \omega_A(xy))} \\ &\geqslant (1 - r_A(y))e^{i(2\pi - \omega_A(y))} \\ &= \mu_{A^c}(y) \end{aligned}$$

Hence A^c is a complex fuzzy left ideal of S.

Conversely, let $x, y \in S$. Then

$$\begin{array}{rcl} \mu_A(xy) & = & r_A(xy))e^{i\omega_A(xy))} \\ & = & (1-(1-r_A(xy)))e^{i(2\pi-(2\pi-\omega_A(xy)))} \\ & \geqslant & (1-(1-r_A(y)))e^{i(2\pi-(2\pi-\omega_A(y)))} \\ & = & r_A(y)e^{i\omega_A(y)} \\ & = & \mu_A(y) \end{array}$$

Hence A is a complex fuzzy left ideal of S. Similarly we can prove the other cases also. □

PROPOSITION 4.3. *If $A = \{(x, \mu_A(x)) : \mu_A(x) = r_A(x)e^{i\omega_A(x)}, x \in S\}$ is a complex fuzzy left ideal(complex fuzzy right ideal, complex fuzzy ideal) of S then the level subset $A_{(\alpha,\beta)}$ is a left ideal(resp. right ideal, ideal) of S.*

PROOF. Let $A = \{(x, \mu_A(x)) : \mu_A(x) = r_A(x)e^{i\omega_A(x)}, x \in S\}$ is a complex fuzzy left ideal of S. Let $y \in A_{(\alpha,\beta)}$ and $x \in S$. Then $r_A(y) \geqslant \alpha$ and $\omega_A(y) \geqslant \beta$.

$$\begin{array}{rcl} r_A(xy)e^{i\omega_A(xy)} & = & \mu_A(xy) \\ & \geqslant & \mu_A(y) \\ & = & r_A(y)e^{i\omega_A(y)} \end{array}$$

This implies

$$\begin{array}{rcl} r_A(xy) & \geqslant & r_A(y) \\ & = & \alpha \end{array}$$

and

$$\begin{array}{rcl} \omega_A(xy) & \geqslant & \omega_A(y) \\ & = & \beta \end{array}$$

So $xy \in A_{(\alpha,\beta)}$. Therefore $SA_{(\alpha,\beta)} \subseteq A_{(\alpha,\beta)}$. Hence $A_{(\alpha,\beta)}$ is a left ideal of S. Similarly by routine verification we can prove other cases also. □

By routine calculation we can prove the following corollary.

COROLLARY 4.1. *If $A = \{(x, \mu_A(x)) : \mu_A(x) = r_A(x)e^{i\omega_A(x)}, x \in S\}$ is a complex fuzzy left ideal(complex fuzzy right ideal, complex fuzzy ideal) of S then the level subsets $A_\alpha = \{x \in S : r_A(x) \geqslant \alpha\}$ and $A_\beta = \{x \in S : \omega_A(x) \geqslant \beta\}$ are two left ideals(resp. right ideals, ideals) of S.*

THEOREM 4.3. *Let $f : S \to T$ be an homomorphism of semigroups and $B = \{(x, \mu_B(x)) : \mu_B(x) = r_B(x)e^{i\omega_B(x)}, x \in T\}$ is a complex fuzzy left ideal(complex fuzzy right ideal, complex fuzzy ideal) of T. Then the pre-image $f^{-1}(B)$ of B under f is a complex fuzzy left ideal(resp. complex fuzzy right ideal, complex fuzzy ideal) of S, where $f^{-1}(B)(x) := \{(x, f^{-1}(\mu_B)(x)) : f^{-1}(\mu_B)(x) = f^{-1}(r_B)(x)e^{if^{-1}(\omega_B)(x)}\} = \{(x, (\mu_B)(f(x)) : (\mu_B)(f(x)) = r_B(f(x))e^{i\omega_B(f(x))}\}$.*

PROOF. Let B be a complex fuzzy left ideal of T and $x, y \in T$ and $f : S \to T$ be an homomorphism of semigroups. Then

$$\begin{aligned} f^{-1}(\mu_B)(xy) &= f^{-1}(r_B)(xy)e^{if^{-1}(\omega_B)(xy)} \\ &= r_B(f(xy))e^{i(\omega_B)(f(xy))} \\ &= r_B(f(x)f(y))e^{i(\omega_B)(f(x)f(y))} \\ &\geqslant r_B(f(y))e^{i\omega_B(f(y))} \\ &= f^{-1}(\mu_B)(y) \end{aligned}$$

Hence the pre-image $f^{-1}(B)$ of B under f is a complex fuzzy left ideal of S. Similarly we can prove the other cases also. □

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Received by editors 23.11.2022; Revised version 5.8.2023; Available online 25.9.2023.

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