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ON THE BANACH PRINCIPLE IN b-METRIC SPACES

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Abstract. In 1989, Bakhtin proved a version of the Banach contraction principle in the context of b-metric spaces. More precisely, let $(X, d; s)$ be a complete b-metric space with parameter $s \geqslant 1$ and let T a contractive map on X, that is a self-map T of X satisfying

$$d(Tx, Ty) \leqslant \lambda d(x, y),\ \forall x, y \in X, \qquad (B_\lambda)$$

with some $\lambda \in [0, 1)$. Bakhtin proved that if $\lambda \in [0, \frac{1}{s})$, then T has a unique fixed point. In the two last decades many research papers were produced by many authors in the setting of b-metric spaces. In 2021, the author published a note on the Banach contraction principle in b-metric spaces in *Divilgaciones Matématicas* dealing with some complements to Bakhtin results. The aim of this paper is to provide other complements. In particular, we discuss the remaining case where $\lambda \in [\frac{1}{s}, 1)$. We give an evaluation of the order of convergence for the iterative Picard process and a posteriori error estimate for this process and we estimate the diameter of the T-orbits. We investigate well-posedness of the fixed problem of a map T satisfying B_λ when X is T-orbitally complete and we establish two shadowing properties for these maps without requiring completeness of X.

1. Introduction

We recall the following definition (see [**9**], [**15**] and [**16**]).

Definition 1.1. *Let X be a non-empty set and let $d : X \times X \to [0, +\infty)$ be a function. Then d is said to be a b-metric on the set X, if the following conditions are satisfied:*

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(i) $d(x,y) \geqslant 0$ *and* $d(x,y)=0$ *if and only if* $x=y$*;*
(ii) $d(x,y)=d(y,x)$*;*
(iii) there exists a real number $s \geqslant 1$ *such that:*

$$d(x,y) \leqslant s\,[d(x,u)+d(u,y)], \text{ for all } x,y,u \in X.$$

The triplet $(X,d;s)$ *is said to be a b-metric space with parameter* s*.*
The inequality (iii) is called the s*-triangle inequality.*

Throughout this paper, $(X,d;s)$ will be a b-metric space with parameter $s \geqslant 1$. Let T be a contraction on X. That is a selfmap of X satisfying

$$d(Tx,Ty) \leqslant \lambda d(x,y), \ \forall x,y \in X, \qquad (B_\lambda)$$

with some $\lambda \in [0,1)$.

In 1989, Bakhtin established in [**9**] the following result, which may be considered as an extension the classical Banach contraction principle to the context of b-metric spaces.

THEOREM 1.1. *(*[**9**]*) Let* $(X,d;s)$ *be a complete b-metric space with parameter* s *and* $f : X \to X$ *a mapping such that, for some* $\lambda > 0$*,*

$$d(f(x),f(y)) \leqslant \lambda\, d(x,y), \quad \text{for all } x,y \in X. \tag{1.1}$$

If $0 < \lambda < 1/s$*, then* f *has a unique fixed point* z *and, for every* $x \in X$*, the sequence* $\big(f^n(x)\big)_{n\in\mathbb{N}}$ *converges to* z *as* $n \to \infty$*. Furthermore, the following evaluation of the order of convergence holds*

$$d(x_n,z) \leqslant \frac{sd(x_0,x_1)}{1-\lambda s} \cdot \lambda^n\,, \quad \text{for all} \quad n \in \mathbb{N}. \tag{1.2}$$

Among the first fixed point results known in the context of b-metric spaces, we find the above result, the articles [**15**], [**16**] published in 1993 and 1998 respectively, by S. Czerwik and the article [**10**] published by V. Berinde in 1993.

During the last decade, a very intensive research work was conducted in b-metric spaces and in their generalizations.

The survey [**14**] of S. Cobzaş contains a large view on the evolution and recent developments of the theory of b-metric spaces and fixed point theory therein.

The survey [**21**] of E. Karapinar contains a short survey on some recent fixed point results obtained in the context of b-metric spaces.

The reader is invited to consult the articles listed in the references of this work and the references therein.

We notice that the result of Bakhtin [**9**] is concerned only with contraction maps with Lipschitz constant $\lambda \in [\frac{1}{s},1)$. So it is natural to raise the following question: What happens when $\lambda \in [\frac{1}{s},1]$?

One of the aims of this paper is to propose an answer to that question and to provide some complements to Theorem (1.1).

A general result focusing on necessary and sufficient conditions for existence of fixed points of Banach contraction was established in distance spaces by Berinde et al. in [**12**]. To recall its statement, we need to review some terminology from [**12**].

Definition 1.2. *A distance space is a pair (X, ρ) consisting of a set X and a non-negative real-valued function ρ on the set $X \times X$ satisfying the following condition:*

(D) $\rho(x, y) + \rho(y, x) = 0$ if and only if $x = y$.

A distance space (X, ρ) is said to be a symmetric space if it satisfies the following condition:

(S)$\rho(x, y) = \rho(y, x)$, for all $x, y \in X$.

Definition 1.3. *A CF-distance space is a distance space (X, ρ) satisfying the following conditions:*

(CF1) $\rho(x, y) = 0$ if and only if $x = y$;

(CF2) $\rho(x, y) = \rho(y, x)$, for all $x, y \in X$;

(CF3) for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that for all $x, y, z \in X$ the inequalities $\rho(x, y) \leqslant \delta$ and $\rho(y, z) \leqslant \delta$ imply $\rho(x, z) \leqslant \varepsilon$.

The notion of CF-metric (or of distance with the Fréchet-Chittenden condition) was introduced by M. Fréchet and studied by E. W. Chittenden (see [**11**]).

Definition 1.4. *An F-distance space is a distance space (X, ρ) satisfying the following conditions:*

(F1) $\rho(x, y) = 0$ if and only if $x = y$;

(F2) for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x, y, z \in X$ the inequalities: $\rho(x, y) \leqslant \delta$ and $\rho(y, z) \leqslant \delta$ imply $\rho(x, z) \leqslant \varepsilon$ and $\rho(z, x) \leqslant \varepsilon$.

It is easy to see that a b-metric space is CF-distance space and then it is an F-distance space. (see [**12**]).

The following theorem proved by V. Berinde and M. Choban in [**12**] gives necessary and sufficient conditions for the existence of fixed points for contractions in any arbitrary F-distance space.

Theorem 1.2 (Theorem 6.3. in [**12**]). *Let $(X; \rho)$ be an F-distance space, $0 < r < 1$, $f : X \to X$ be a mapping and*

$$\rho(f(x), f(y)) \leqslant r\rho(x, y), \quad \text{for all } x, y \in X.$$

The following assertions are equivalent:

1. There exists a point $a \in X$ such that $f(a) = a$.

2. If x_0 is a fixed point of the space X and $x_{n+1} = f(x_n)$, for each $n \geqslant 0$, then the sequence $(x_n)_{n \geqslant 0}$ is convergent.

3. If x_0 is a fixed point of the space X and $x_{n+1} = f(x_n)$, for each $n \geqslant 0$, then the sequence $(x_n)_{n \geqslant 0}$ contains a convergent sub-sequence.

4. For some fixed point $x_0 \in X$ the sequence $(x_n)_{n \geqslant 0}$, where $x_{n+1} = f(x_n)$, for each $n \geqslant 0$, contains a convergent sub-sequence.

As mentioned above, the aim of this paper is to give some complements to Bakhtin's result. This is done in Section three by Theorem 3.1, where we provide some new estimates concerning the Picard sequence starting from any arbitrary point in X. In section two, we recall some definitions. In Section four, we investigate Well-posedness for a selfmap T of a b-metric space $(X, d; s)$ satisfying the contractive condition B_λ, when X is T-orbitally complete. The section five is devoted to study two shadowing properties of such maps without requiring any kind of completeness on the b-metric space $(X, d; s)$.

We point out that this paper is a natural continuation to the work published by the author in [**1**].

2. Recall of some definitions

In all this paper, we denote the set of non-negative integers by $\mathbb{N}_0$. As usual, $\mathbb{N}$ designates the set of positive integers. A sequence $\{x_n\}$ $(n \in \mathbb{N}_0)$ of elements of a set X will be also denoted by $(x_n)_{n\in\mathbb{N}_0}$ and its range set will be denoted by $\{x_n : n \in \mathbb{N}_0\}$.

Before going further, we recall some definitions which will be useful to formulate the results of this paper. Most of them are known in the context of b-metric spaces, or already known in the usual metric spaces but are easy to be extended to the framework of b-metric spaces. So we list them without references.

Definition 2.1. *Let $(X, d; s)$ be a b-metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then, the following are defined as follows:*

(i) The sequence $\{x_n\}$ is said to be a Cauchy sequence, if for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that: for all $n \geqslant N$ and all $p \in \mathbb{N}$, we have $d(x_n, x_{n+p}) < \epsilon$.

(ii) The sequence $\{x_n\}$ is said to be convergent to x, if for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that: for all $n \geqslant N$, we have $d(x_n, x) < \epsilon$. In this case, we write: $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(iii) $(X, d; s)$ is said to be complete b-metric space if every Cauchy sequence in X converges to some $x \in X$.

(iv) $(X, d; s)$ is said to be sequentially compact if every sequence in X has a convergent sub-sequence.

We observe that every converging sequence in a b-metric space is Cauchy sequence. In general, The converse is not true.

It is easy to see that the limit of a converging sequence (in a b-metric space) is unique.

Definition 2.2. *Let $(X, d; s)$ be a b-metric space and $\{x_n\}$ be a sequence in X. Let A be a non empty subset of X. Then A is said to be bounded if, $\delta(A)$ is finite, where $\delta(A) := \sup\{d(x, y) : (x, y) \in A \times A\}$.*

We observe that every Cauchy sequence in a b-metric space is bounded but, in general, the converse is not true.

Definition 2.3. *Let X be a non empty set and let T be a self-mapping of X. Then, for every $x \in X$, the set $O_T(x) := \{x, Tx, T^2x, T^3x, \dots\}$ is called the orbit of T at $x \in X$.*

Now, we define the concepts of T-orbitally completeness.

Definition 2.4. *Let $(X, d; s)$ be a b-metric space and let T be a self-mapping on X.*

X is said to be $T-$orbitally complete, if for any $x \in X$, every Cauchy sequence of the orbit $O_T(x) := \{x, Tx, T^2x, \dots\}$ is convergent in X.

In the case of metric spaces, the concept of orbitally completeness was first introduced in 1974 by Ćirić in [**13**].

3. Some complements to Bakhtin's result

Let $(X, d; s)$ be a b-metric space with parameter $s \geqslant 1$.

Before stating the main result of this section, we need to recall the following fondamental lemma:

LEMMA 3.1. *Let $(X, d; s)$ be a b-metric space with parameter $s \geqslant 1$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements from X having the property that there exists $\gamma \in [0, 1)$ such that*

$$d(x_{n+1}, x_n) \leqslant \gamma d(x_n, x_{n-1}), \;\; \forall n \in \mathbb{N}.$$

Then the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy.

Lemma 3.1 was established in 2017 by R. Miculescu and A. Mihail in their paper [**23**] and by T. Suzuki in his paper [**33**] by a different method.

In [**18**], T. M. Došenović, M. V. Pavlović and S. N. Radenović made a discussion concerning Lemma 3.1 above and showed that various known fixed point results in the context of b-metric spaces can be shortened by the use of this lemma.

Since $(X, d; s')$ is a b-metric space for every parameter $s' \geqslant s$, one may suppose, without loss of generality, that $s > 1$.

The following result provides some complements to the result of Bakhtin (see Theorem 1.1).

THEOREM 3.1. *Let $(X, d; s)$ be a b-metric space with parameter $s > 1$ and $T : X \to X$ a mapping such that*

$$d(T(x), T(y)) \leqslant \lambda\, d(x, y), \quad \text{for all } x, y \in X, \tag{3.1}$$

for some $\lambda \in [0, 1)$.

Suppose that X is $T-$orbitally complete. Then

(P1) *T has a unique fixed point z in X.*

(P2) *For every $x \in X$, the Picard sequence $(x_n)_{n\geqslant 0}$ defined by*

$$x_0 := x \quad \text{and} \quad x_n := T^n(x), \quad \text{for all integer} \quad n \geqslant 1,$$

converges to z as $n \to \infty$.

(P3) *If $0 \leqslant \lambda < \frac{1}{s}$ then the following evaluation of the order of convergence holds*

$$d(T^n(x), z) \leqslant \frac{s\, d(x, Tx)}{1 - \lambda s} \cdot \lambda^n, \quad \text{for all} \quad n \geqslant 0. \tag{3.2}$$

(P4) *If $\frac{1}{s} \leqslant \lambda < 1$, let p_0 is the smallest positive integer satisfying: $s\,\lambda^{p_0} < 1$. Then:*

(i) *the following evaluation of the order of convergence holds*

$$d(T^n(x), z) \leqslant \frac{s\, d(x, T^{p_0}x)}{1 - s\,\lambda^{p_0}} \cdot \lambda^n, \quad \text{for all} \quad n \geqslant 0. \tag{3.3}$$

(ii) *The following a posteriori error estimate holds:*

$$d(x_n, z) \leqslant \frac{s\lambda}{1 - s\,\lambda^{p_0}} \cdot d(x_{n-1}, T^{p_0-1}x_n)\,, \quad \textit{for all} \quad n \geqslant 1. \tag{3.4}$$

(iii) *The orbit $O_T(x)$ is bounded and we have the following estimate for its diameter:*

$$\delta(O_T(x)) \leqslant \frac{s}{1 - s\,\lambda^{p_0}} \cdot \max_{1 \leqslant j \leqslant p_0} d(x, T^j x). \tag{3.5}$$

(P5) *The rate of convergence of Picard iteration is given by*

$$d(x_n, z) \leqslant \lambda\, d(x_{n-1}, z), \quad \textit{for all integer} \quad n \geqslant 1. \tag{3.6}$$

Proof. (1) The case where $0 \leqslant \lambda < \frac{1}{s}$ was studied in Theorem 1.1 of Bakhtin. The property (P3) is given by Bakhtin's theorem.

So, we are led to suppose that $\frac{1}{s} \leqslant \lambda < 1$.

(2) For every $x_0 \in X$, we consider the sequence $x_n := T^n(x_0)$ (for all $n \in \mathbb{N}$). We set $\tau_n := d(x_n, x_{n+1})$ for all non-negative integer $n \geqslant 0$. By virtue of the condition (3.1), we have $\tau_n \leqslant \lambda\tau_{n-1}$ for every positive integer n. As $\lambda < 1$, Lemma 3.1, infers that the sequence $(x_n)_n$ is Cauchy. Since X is supposed to be orbitally complete, the sequence converges to a point (say) $z \in X$.

For evey nonnegative integer n, we have

$$d(T(x_n), T(z)) \leqslant \lambda\, d(x_n, z).$$

The above inequality, implies that the sub-sequence $(x_{n+1})_n$ converges to Tz. By uniqueness of the limit, we infer that $z = Tz$. Thus z is a fixed point of T.

It is clear, by the property (3.1), that z is the unique fixed point of T.

For the sequel, we let p_0 be the smallest positive integer satisfying $s\,\lambda^{p_0} < 1$. We observe that $p_0 \geqslant 2$.

(3) Let $x \in X$. Next we show the estimate (3.3). Indeed, for all positive integer n, we have

$$\begin{aligned} d(T^n x, z) &\leqslant s\left[d(T^n x, T^{n+p_0}x) + d(T^{n+p_0}x, T_0^p z)\right] \\ &\leqslant s\,\lambda^n d(x, T^{p_0}x) + s\lambda^{p_0} d(T^n x, z), \end{aligned}$$

from which we get the inequality (3.3).

(4) Next, we prove the estimate (3.4). Let $x \in X$ and let $(x_n)_{n\geqslant 0}$ be the Picard sequence associated to x. Then for every positive integer n, we have

$$\begin{aligned} d(x_n, z) = d(Tx_{n-1}, Tz) &\leqslant \lambda d(x_{n-1}, z) \\ &\leqslant s\lambda\left[d(x_{n-1}, T^{p_0-1}x_n) + d(T^{p_0-1}x_n, T^{p_0-1}z)\right] \\ &\leqslant s\,\lambda d(x_{n-1}, T^{p_0-1}x_n) + s\lambda^{p_0} d(x_n, z), \end{aligned}$$

from which we get the inequality (3.4).

(5) Next, we prove the estimate (3.5).

Let $u \in X \setminus \{z\}$, we know that $\{T^n u\}$ is Cauchy, therefore it is bounded. Since for every integers p, q with $1 \leqslant p < q$, we have

$$d(T^p u, T^q u) \leqslant \lambda^p\, d(u, T^{q-p}u) < d(u, T^{q-p}u).$$

Then the diameter $\delta(O_T(u))$ of the orbit $O_T(u)$ satisfies

$$\delta(O_T(u)) = \sup_{k\in\mathbb{N}} d(u, T^k u).$$

Let n be an integer such that $n \geqslant p_0 + 1$. Then by using the s-triangle inequality, we have

$$\begin{aligned} d(u, T^n u) &\leqslant s\, d(u, T^{p_0} u) + s\, d(T^{p_0} u, T^n u) \\ &\leqslant s\, d(u, T^{p_0} u) + s\, \lambda^{p_0} d(u, T^{n-p_0} u) \\ &\leqslant s\, d(u, T^{p_0} u) + s\, \lambda^{p_0} \delta(O_T(u)). \end{aligned}$$

If $1 \leqslant n \leqslant p_0$, then obviously, we have

$$d(u, T^n u) \leqslant s \max_{1\leqslant j\leqslant p_0} d(u, T^j u).$$

From the above analysis, we infer that

$$\delta(O_T(u)) \leqslant s \max_{1\leqslant j\leqslant p_0} d(u, T^j u) + s\, \lambda^{p_0} \delta(O_T(u)),$$

which implies that

$$\delta(O_T(u)) \leqslant \frac{s}{1 - s\, \lambda^{p_0}} \max_{1\leqslant j\leqslant p_0} d(u, T^j u).$$

For $u = z$ the above inequality is evident.

(6) The property (P5) is clear. This ends the proof. □

Next, we give an illustrative example.

EXAMPLE 3.1. Take $X = [0, 1]$. For $x, y \in [0, 1]$, we set $d(x, y) :=\mid x - y \mid^2$. It is easy to see that $([0, 1], d)$ is a b-metric space with parameter $s = 2$. We set $Tx = \frac{1}{\sqrt{8}} x^2$, for all $x \in [0, 1]$. Then it is easy to see that T satisfies the following

$$\begin{aligned} sd(Tx, Ty) &= 2\left|\frac{x^2}{\sqrt{8}} - \frac{y^2}{\sqrt{8}}\right|^2 \\ &= \frac{1}{4}(x + y)^2 |x - y|^2 \\ &= \frac{1}{4}(x + y)^2 d(x, y) \\ &< d(x, y), \quad \text{for all } x, y \in X, \text{ with } x \neq y. \end{aligned}$$

Therefore, we have

$$d(Tx, Ty) < \frac{1}{2} d(x, y), \quad \text{for all } x, y \in [0, 1] \text{ with } x \neq y. \tag{3.7}$$

Thus T is a b-contractive mapping of $[0, 1]$. By Theorem 1.1 or its corollary, T has a unique fixed point (which is zero).

We point out that this example can not be handled by Theorem 1.1 of Bakhtin, because we have here $s = 2$, $\lambda = \frac{1}{2}$ and $s\lambda = 1$.

Thus Theorem 3.1 brings a true extension to Bakhtin result.

4. Well posedness

We start this section by recalling the concept of well-posedness of a fixed point problem. In the metric case, this concept was introduced by F.S. De Blasi and J. Myjak [**17**]. After their work and the work du to S. Reich and A.J. Zaslavski [**28**], many authors have been interested by the study of well-posednes of fixed point problems (see [**22**], [**26**], [**30**], [**24**], [**27**], [**3**], [**4**],).

The following definition (see [**2**]) is a natural extension (to the case of b-metric spaces) of the concept of well-posedness known for metric spaces.

Definition 4.1. *Let $(X, d; s)$ be a b-metric space with constant $s \geqslant 1$ and $T : X \to X$ be a mapping. The fixed point problem of T is said to be well-posed if:*
(i) T has a unique fixed point z in X,
(ii) for any sequence $\{x_n\}$ of points in X such that $\lim_{n\to\infty} d(Tx_n, x_n) = 0$, *we have* $\lim_{n\to\infty} d(x_n, z) = 0$.

According to this definition, we investigate the well-posedness of contractions in a b-metric space $(X, d; s)$. More precisely, we propose the following.

Theorem 4.1. *Let $(X, d; s)$ be a b-metric space with parameter $s \geqslant 1$ and $T : X \to X$ a mapping such that*

$$d(T(x), T(y)) \leqslant \lambda\, d(x, y), \quad \text{for all } x, y \in X, \tag{4.1}$$

for some $\lambda \in [0, 1)$.
Suppose that X is $T-$orbitally complete.
Then, the fixed point problem for T is well-posed.

Proof. The existence and uniqueness of the fixed point (say) z of T has been ensured by Theorem 3.1.

Let $\{y_n\}$ be any arbitrary sequence of points in X such that

$$\lim_{n\to\infty} d(Ty_n, y_n) = 0.$$

We need to show that the sequence $\{y_n\}$ converges to the unique fixed point z of T. To this end, we start by observing the following.

$$\lim_{n\to+\infty} d(y_n, T^p y_n) = 0, \quad \forall p \in \mathbb{N}.$$

Indeed, for every positive integer $p \geqslant 2$, by using the s-relaxed triangle inequality of order p, we have

$$\begin{aligned} d(y_n, T^p y_n) &\leqslant sd(y_n, Ty_n) + s^2 d(Ty_n, T^2 y_n) + \ldots + s^p d(T^{p-1} y_n, T^p y_n) \\ &\leqslant s\, d(y_n, Ty_n) + s^2 \lambda d(y_n, Ty_n) + \ldots + s^p \lambda^{p-1} d(y_n, Ty_n) \\ &= s\left(1 + s\lambda + \ldots + (s\lambda)^{p-1}\right) d(y_n, Ty_n), \end{aligned}$$

from which, we deduce that $\lim_{n\to+\infty} d(y_n, T^p y_n) = 0$.

Let p be a positive integer such that $s\,\lambda^p < 1$. By using the s-triangle property and the contractive condition (4.1), for every non-negative integer n, we have:

$$\begin{aligned} d(y_n, z) &\leqslant s\,[d(y_n, T^p y_n) + d(T^p y_n, T^p z)] \\ &\leqslant s\,d(y_n, T^p y_n) + s\lambda^p d(y_n, z), \end{aligned}$$

from which, we obtain

$$d(y_n, z) \leqslant \frac{s}{1 - s\lambda^p} d(y_n, T^p y_n), \text{ for every non-negative integer } n.$$

The above inequality implies that $\lim_{n\to\infty} d(y_n, z) = 0$. This shows that the fixed point problem for T is well posed. This completes the proof. □

5. Shadowing properties

In dynamical systems theory, shadowing concepts were introduced to investigate the qualitative properties of dynamical systems [**8, 25**]. Various other kinds of shadowing property have been developed in the course of such studies so far. The limit shadowing property introduced in [**19**] is one of the variants of shadowing property, which concerns asymptotic shadowing of pseudo orbits for which one-step errors are converging to zero.

In metric fixed point theory, the concept of shadowing was introduced by I. Rus in [**29**].

The concepts of shadowing properties can be extended to b-metric spaces as follows.

Definition 5.1. *Let $(X, d; s)$ be a b-metric space with parameter $s \geqslant 1$. Let T be a self-map of X.*

For $\delta > 0$, a sequence $(x_n)_{n\geqslant 0}$ of points in X is said to be a δ-pseudo orbit of T if we have $d(x_{n+1}, T(x_n)) \leqslant \delta$ for all $n \geqslant 0$.

For given $\epsilon > 0$, a δ-pseudo orbit $(x_n)_{n\geqslant 0}$ is said to be ϵ-shadowed by $y \in X$ if $d(x_n, T^n(y)) \leqslant \epsilon$ for all $n \geqslant 0$.

We say that T has the shadowing property if for every $\epsilon > 0$, there is $\delta > 0$ such that every δ-pseudo orbit of T is ϵ-shadowed by some point of X.

A sequence $(x_n)_{n\geqslant 0}$ of points in X is said to be a limit pseudo orbit of T if we have $\lim_{n\to\infty} d(x_{n+1}, T(x_n)) = 0$.

We say that T has the limit shadowing property if for any limit pseudo orbit $(x_n)_{n\geqslant 0}$ of T, there is a point $z \in X$ such that $\lim_{n\to\infty} d(x_n, T^n(z)) = 0$. Such z is called a limit shadowing point of the sequence $(x_n)_{n\geqslant 0}$.

In [**32**], the following definition was given.

Definition 5.2. *(see Definition 15 in [**32**]) Let (M, d) be a metric space and $f : M \to M$ a self-mapping of M. We say that the fixed point problem of f has the limit shadowing property in M if, for any sequence $(x_n)_{n\geqslant 0}$ in M satisfying* $\lim_{n\to\infty} d(x_n, f(x_n)) = 0$, *it follows that there exists a point $z \in M$ such that* $\lim_{n\to\infty} d(f^n(z), x_n) = 0$.

As we see, this definition does not correspond to the one already existing in the theory of dynamical systems in metric spaces (see [**25**]). Moreover, the limit shadowing property, mostly concerns dynamical properties of the map f in M rather than its fixed point problem. So the definition (5.2) will not be used here.

To establish the main result of this section, we need to recall the following lemma.

LEMMA 5.1. *If α is a real number such that $0 \leqslant \alpha < 1$, and $(\varepsilon_n)_{n\geqslant 0}$ is a sequence of non-negative numbers such that $\lim_{n\to+\infty} \varepsilon_n = 0$, then for any sequence of non-negative numbers $(u_n)_{n\geqslant 0}$ satisfying:*

$$u_{n+1} \leqslant \alpha u_n + \varepsilon_n, \quad \forall n \geqslant 0,$$

we have

$$\lim_{n\to+\infty} u_n = 0.$$

The lemma 5.1 above was proved by V. Berinde in [**11**] by using a result of Harder and Hicks (see [**20**]).

Concerning the shadowing properties (according to Definition (5.1)) for contractions in b-metric spaces, we propose the following.

THEOREM 5.1. *Let $(X, d; s)$ be a b-metric space with parameter $s \geqslant 1$ and $T : X \to X$ a mapping such that*

$$d(T(x), T(y)) \leqslant \lambda\, d(x, y), \quad \textit{for all } x, y \in X, \tag{5.1}$$

for some $\lambda \in [0, \frac{1}{s})$. Then

(a) T has the limit shadowing property.

(b) T has the shadowing property.

PROOF. Let $(x_n)_{n\geqslant 0}$ be any sequence of points in X. For every integer $n \geqslant 1$, we set $\varepsilon_n := d(x_n, Tx_{n-1})$. For any point $y \in X$, we set $u_n := d(T^n(y), x_n)$, for all integer $n \geqslant 0$.

(a) By the s-triangle inequality, for all positive integer n, we have

$$\begin{aligned} d(x_n, T^n y) &\leqslant sd(x_n, Tx_{n-1}) + sd(Tx_{n-1}, T^n y) \\ &\leqslant s\varepsilon_n + s\lambda d(x_{n-1}, T^{n-1} y). \end{aligned}$$

This shows that

$$u_n \leqslant s\lambda u_{n-1} + s\varepsilon_n, \quad \forall n \geqslant 1. \tag{5.2}$$

By assumption, we have $0 \leqslant s\lambda < 1$. Then by using Lemma 5.1, we deduce that $\lim_{n\to\infty} u_n = 0$ if $\lim_{n\to\infty} d(x_n, Tx_{n-1}) = 0$. Thus we have proved that T has the limit shadowing property.

(b) For all point $y \in X$, we have the following inequality:

$$d(T^n(y), x_n) \leqslant s \sum_{j=0}^{n-1} (s\lambda)^{n-j} \varepsilon_j + (s\lambda)^n d(x_0, y). \tag{5.3}$$

Indeed, the inequality (5.3) is easily obtained by mathematical induction on $n \geqslant 1$ and by using the inequality (5.2).

Now let $\epsilon > 0$ given and consider any positive number δ satisfying $\delta \in (0, (\frac{1}{s} - \lambda)\epsilon)$. suppose that $(x_n)_{n\geqslant 0}$ is a δ-pseudo orbit of T. That is $\varepsilon_n \leqslant \delta$, for all integer $n \geqslant 0$.

Next we show that $(x_n)_{n\geqslant 0}$ is ϵ-shadowed by every $y \in X$ such that $d(x_0, y) \leqslant R(\epsilon, \delta)$, where $R(\epsilon, \delta) := \epsilon - \frac{s\delta}{1-s\lambda}$.

Indeed, for such point y, by virtue of (5.3), we have

$$\begin{aligned} d(x_n, T^n y) &\leqslant s\delta \sum_{j=0}^{n-1} (s\lambda)^j + (s\lambda)^n d(x_0, y) \\ &\leqslant s\delta \frac{1-(s\lambda)^n}{1-s\lambda} + d(x_0, y) \\ &\leqslant \frac{s\delta}{1-s\lambda} + R(\epsilon, \delta) = \epsilon. \end{aligned}$$

The inequality above is true for all positive integer n. But, it is evident for $n = 0$. Thus we have proved that $d(x_n, T^n y) \leqslant \epsilon$ for every integer $n \geqslant 0$. That is the δ-pseudo orbit of T is ϵ-shadowed by every y in the closed ball of X having x_0 as centre and $R(\epsilon, \delta)$ as radius. Hence T has the shadowing property. This ends the proof. □

Remark 5.1. We observe that the results (a) and (b) of Theorem (5.1) are proved without using fixed point theory for the self-map T and without requiring any kind of completeness on the b-metric space $(X, d; s)$.

Remark 5.2. When $s > 1$, we address the following question: Is it possible to extend the results of Theorem (5.1) for all values of $\lambda \in [0, 1)$?

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