BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Bull. Int. Math. Virtual Inst., **13**(2)(2023), 327–336
DOI: 10.7251/BIMVI2302327I

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

ON EQUIVALENCE OF P-HENSTOCK-TYPE INTEGRALS FOR INTERVAL VALUED FUNCTIONS

Victor Odalochi Iluebe, Adesanmi Alao Mogbademu, Joshua Olugbenga Ajilore and Olumide Afolabi Esan

Abstract. In this paper, we use the p-norm to define the interval p-Henstock integral, introduce the interval p-Sequential Henstock integrals and show the equivalence of the interval p-Henstock-type integrals. The p-norm provides an alternative approach to defining the Henstock-type integrals of interval valued functions and the p-integral of interval valued functions.

1. Introduction

Several authors have studied the concept of integration for real valued space functions. The techniques in this integration concept have some difficulties as that of the Lebesgue integral. Some have sought to redefine the rigorous nature of these techniques with a view of avoiding its difficulties by introducing new and reliable integrals. One of the most popular integrals is the Henstock integral. Its definition is obtained by a slight modification of the Riemann's definition. Henstock integral was introduced independently by R. Henstock and J. Kurzweil in 1955 and 1957 respectively. It is well known that Henstock integral is equivalent to the Denjoy and Perron integrals and is easier and more reliable than the Wiener, Feynmann and Lebesgue integrals(see[1-20]) and has been shown by Paxton [15] to be equivalent to the Sequential Henstock integral. In 2018, Ray [16] obtained results dealing with equivalence of Riemann integrals based on p-norm. It is well known that in the usual generalised Riemann integral setting, the Henstock norm or mesh is adopted for the Henstock sums. Interval analysis helps to reduce the uncertainty and error

2020 *Mathematics Subject Classification.* Primary 28B05; Secondary 28B10, 28B15, 46G10.
Key words and phrases. Interval Sequential Henstock integrals, Interval p-Henstock, Continuous functions, Guages, Interval *ap*-Henstock integral.
Communicated by Dusko Bogdanic.

bounds yields in real values computations and measurements as well as find guaranteed solutions to differential equations and optimization problems. In 2000, Wu and Gong[19] introduced the concept of the Henstock (H) integral of interval valued functions and fuzzy number-valued functions and obtained a number of properties. In the same year, Hamid and Elmuiz[4] established the concept of the Henstock Stieltjes (HS) integrals of interval valued functions and fuzzy number-valued functions and obtained some number of properties of these integrals. The equivalence of integrals in real valued space have been studied by several researchers. Paxton[15] gave several variations of the Henstock integral of functions f in $[a, b]$ and show equivalence of same to the Sequential Henstock integral. Recently, the authors [7] showed all sorts of equivalences of Henstock-type integrals and certain Sequential Henstock integrals.

Therefore, it is pertinent to ask the following questions:

Question 1. Is interval p-Sequential Henstock integral equivalent to the interval Sequential Henstock integral?

Question 2. Is interval *ap*-Sequential Henstock integral equivalent to the interval Sequential Henstock integral?

These questions were answered later in this work.

2. Preliminaries

Let $\mathbb{R}$ denote the set of real numbers, $F(X)$ as an interval valued function, F^-, the left endpoint, F^+ as right endpoint, $\{\delta_n(x)\}_{n=1}^{\infty}$, as set of gauge functions, P_n, as set of partitions of subintervals of a compact interval $[a, b]$, X, as non empty interval in $\mathbb{R}$ and $d(X) = X^+ - X^-$, as width of the interval X and $\ll$ as much more smaller. Moreso, we recall the following as given in [15].

• $FPV[a, b] = \bigcup_{n=1}^{\infty} FPV^{(n)}[a, b]$ is the set of all the finite partition vectors of $[a, b]$ whose length is n. i.e.,
$FPV_f^{(n)}[a, b] = \{\bar{\beta} \in R^{n+1} : \beta_0 = a < \beta_1 < ... < \beta_j < \beta_{j+1} < ... < \beta_n = b\}$

• $FPV[a, b] = \cup_{n=1}^{\infty} FPV^{(n)}[a, b]$ is the set of all the finite partition vectors of $[a, b]$. Observe that $FPV[a, b] \subseteq \cup_{n=1}^{\infty} R^{n+1}$.

• $\prod[\bar{\beta}] = \prod_{i=0}^{|\bar{\beta}|-1}[\beta_i - \beta_{i-1}]$. For example, if $\bar{\beta} = (1, 1.2, 1.5, 2.2, 2.6, 3) \in FPV^{(5)}[1, 3]$, then $(1.1, 1.4, 1.5, 2.4, 2.9) \in \prod[\bar{\beta}]$. Its' obvious that $\prod[\bar{\beta}]$ represents the space where the tags are located, given a partition $\bar{\beta}$ of $[a, b]$

• $\|\bar{\beta}_p\| = \sqrt[p]{(\sum_{i=0}^{|\beta^{\epsilon}|-1} |\bar{\beta}_{i+1} - \bar{\beta}_i|)^p}$, where $p > 1$. For example, if $\bar{\beta} = (1, 2, 4, 7, 10)$ then $\|\bar{\beta}\|_3 = \sqrt[3]{(1 + 2^3 + 3^3 + 3^3)} = \sqrt[3]{64}$. By exploiting Minkowski inequality. Hence $\|.\|_p$ is a norm.

• $S_f(\bar{\beta}, \bar{P}_n) = \sum_{i=0}^{|\beta^\epsilon|-1} |\bar{\beta}_{i+1} - \bar{\beta}_i|.f(\bar{P}_i)$, where $\bar{P}_i \in [\beta_i, \beta_{i+1}]$, where $\bar{P}_n$ represents a sequence of tags. i.e $\bar{P} = \prod[\bar{\beta}]$

Definition 2.1. *[14]. A gauge on $[a,b]$ is a positive real-valued function $\delta : [a,b] \to \mathbb{R}^+$. Let $\delta > 0$, then a partition P is δ-fine if every subinterval $[u_{i-1}, u_i]$ satisfies $[u_{i-1}, u_i] \subset [t_i - \delta(t_i), t_i + \delta(t_i)]$.*

Definition 2.2. *[14] A sequence of tagged partition P_n of [a,b] is a finite collection of ordered pairs $P_n = \{(u_{(i-1)_n} u_{i_n}), t_{i_n}\}_{i=1}^{m_n}$ where $[u_{i-1}, u_i] \in [a,b]$, $u_{(i-1)_n} \leqslant t_{i_n} \leqslant u_{i_n}$ and $a = u_0 < u_{i_1} < ... < u_{m_n} = b$.*

Definition 2.3. *[15]. (p-integral). A function $f : [a,b] \to \mathbb{R}$ is p integrable on $[a,b]$ to a number $\alpha \in \mathbb{R}$ if for any $\bar{\varepsilon} > 0$, there exists a $\delta_{\bar{\varepsilon}}^p > 0$ such that for any $\bar{\beta} \in FPV_p(\delta_{\bar{\varepsilon}})$ and for all $\bar{P} \in \prod[\bar{\beta}]$, we have $|S_f(\bar{\beta}, \bar{P}) - \alpha| < \bar{\varepsilon}$. We say that α is a p-integral of f on $[a,b]$ with $\alpha = (p)\int_a^b f$. We use $p[a,b]$ to denote the set of all p-integrable functions defined on $[a,b]$.*

Definition 2.4. *([2]. Let E be a measurable set and let $c \in \mathbb{R}$. The density of E at c is defined by*

$$d_c E = \lim_{h \to 0+} \frac{\mu(E \cap (c-h, c+h))}{2h},$$

provided the limit exists. The point c is called a point of density of E if $d_c E = 1$. The set E^d represents the set of all points $x \in E$ such that x is a point of density of E.

Definition 2.5. *[2]. A function $F : [a,b] \to \mathbb{R}$ is said to be approximately differentiable at $c \in [a,b]$ if there exists a measurable set $E \subseteq [a,b]$ such that $c \in E^d$ and*

$$\lim_{t \to c, t \in E} \frac{F(t) - F(c)}{t - c},$$

exists. The approximate derivative of F at c is denoted by $F'_{ap}(c)$. The concept of sequence of approximate neighborhoods(or ap-nbds) of $t_{i_n} \in [a,b]$ is a measurable set $S_{t_{i_n}} \subseteq [a,b]$ containing t_{i_n} as a sequence of points of density. For every $t_{i_n} \in E \subseteq [a,b]$, choose an ap-nbd $S_{t_{i_n}} \subseteq [a,b]$ of t_{i_n}. Then we say that $S = \{S_{t_{i_n}} : t_{i_n} \in E\}$ is a choice on E. A tagged interval $(t_{i_n}, [c_{i_n}, d_{i_n}])$ is said to be subordinate to the choice $S = S_{t_{i_n}}$ if $c_{i_n}, d_{i_n} \in S_{t_{i_n}}$. Let $P_n = \{(t_{i_n}, [c_{i_n}, d_{i_n}]) : 1 \leqslant i \leqslant m, m \in \mathbb{N}\}$ be a finite collection of non-overlapping tagged intervals. If $(t_{i_n}, [c_{i_n}, d_{i_n}])$ is subordinate to a choice S for each i_n for $i = 1, ..., m$, then we say that P_n is subordinate to S. If P_n is subordinate to S and $[a,b] = \bigcup_{i=1}^{n} [c_{i_n}, d_{i_n}]$, then we say that P_n is a tagged partition of $[a,b]$ that is subordinate to S.

Definition 2.6. *[14] .(Sequential Henstock Integral). A function $f : [a,b] \to \mathbb{R}$ is Sequential Henstock integrable ($SH[a,b]$) on $[a,b]$, to a real number α if for any $\varepsilon > 0$, there exists a sequence of gauge functions $\{\delta_n(x)\}_{n=1}^{\infty}$ such that for any $\delta_n(x) - fine$ tagged partitions $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}_{i=1}^{m_n}$, we have $|\sum_{i=1}^{m_n} f(t_{i_n})(u_{i_n} - u_{(i-1)_n}) - \alpha| < \varepsilon$. We say that α is a Sequential Henstock* *integral of f on $[a,b]$. i.e. $\alpha = SH\int_{[a,b]} f$ and use $SH_f[a,b]$ to denote the set of all Sequential Henstock-integrable functions defined on $[a,b]$*

DEFINITION 2.7. *([12]) (p-Sequential Henstock integral). A function $f:[a,b]\to\mathbb{R}$ is p-Sequential Henstock integrable on $[a,b]$ to a number $\alpha\in\mathbb{R}$ if for any $\bar{\varepsilon}>0$ there exists a sequence of positive functions $\{\delta^p_{n\bar{\varepsilon}}(x)\}_{n=1}^{\infty}$, such that for any $\bar{\beta}_n\in FPV_p(\delta_{n\bar{\varepsilon}}(x))$ and for all $\bar{P}_n\in\prod_{i=1}^{m_n}[\bar{\beta}_n]$, we have $|S_f(\bar{\beta}_n,\bar{P}_n)-\alpha|<\bar{\varepsilon}$.. We say that α is a p-Sequential Henstock integral of f on $[a,b]$. i.e. $p\text{-}SH_f[a,b]=\alpha=\int_{[a,b]} f$. and use $p-SH_f[a,b]$ to denote the set of all p-Sequential Henstock integrable functions defined on $[a,b]$.*

DEFINITION 2.8. *([12]) (ap-Sequential Henstock integral) A function $f:[a,b]\to\mathbb{R}$ is ap-Sequential Henstock integrable on $[a,b]$ to a vector $\alpha\in\mathbb{R}$ if for any $\varepsilon>0$ there exists a sequence of choice $\{S_n(x)\}_{n=1}^{\infty}$ on $[a,b]$ such that for any S_n-fine tagged partitions $P_n=\{(u_{(i-1)_n},u_{i_n}),t_{i_n}\}_{i=1}^{m_n}$ we have*

$$|\sum_{i=1}^{m_n} f(t_{i_n})(u_{i_n}-u_{(i-1)_n})-\alpha|<\varepsilon.$$

We say that α is an ap-Sequential Henstock integral of f on $[a,b]$. We use $apSH_f[a,b]$ to denote the set of all ap-Sequential Henstock integrable functions defined on $[a,b]$.

DEFINITION 2.9. *([11,15]) Let $I_{\mathbb{R}}=\{I=[I^-,I^+]$: I is a closed bounded interval on the real line $\mathbb{R}\}$. For $X,Y\in I_{\mathbb{R}}$, we define*
i. $X\leqslant Y$ if and only if $Y^-\leqslant X^-$ and $X^+\leqslant Y^+$,
ii. $X+Y=Z$ if and only if $Z^-=X^-+Y^-$ and $Z^+=X^++Y^+$,
iii. $X.Y=\{x.y:x\in X,y\in Y\}$, where

$$(X.Y)^-=min\{X^-.Y^-,X^-.Y^+,X^+.Y^-,X^+.Y^+\}$$

Then $d(X,Y)=max(|X^--Y^-|,|X^+-Y^+|)$ is the metric distance between intervals X and Y.

Now, we will define the Sequential Henstock integral of interval valued function.

DEFINITION 2.10. *An interval valued function $F:[a,b]\to I_{\mathbb{R}}$ is Sequential Henstock integrable$(ISH[a,b])$ to $I_0\in I_{\mathbb{R}}$ on $[a,b]$ if for any $\varepsilon>0$ there exists a sequence of positive gauge functions $\{\delta_n(x)\}_{n=1}^{\infty}$ such that for every $\delta_n(x)-fine$ tagged partitions $P_n=\{(u_{(i-1)_n},u_{i_n}),t_{i_n}\}_{i=1}^{m_n}$, we have*

$$d(\sum_{i=1}^{m_n} F(t_{i_n})(u_{i_n}-u_{(i-1)_n}),I_0)<\varepsilon$$

$$=max|d(\sum_{i=1}^{m_n} F^-(t_{i_n})(u_{i_n}-u_{(i-1)_n}),I_0^-)<\varepsilon,d(\sum_{i=1}^{m_n} F^+(t_{i_n})(u_{i_n}-u_{(i-1)_n}),I_0^+)<\varepsilon|$$

We say that $I_0=[I_0^-,I_0^+]$ is the Sequential Henstock integral of F on $[a,b]$. i.e. $(IH)\int_{[a,b]} F=\alpha$ and $F\in ISH[a,b]$.

DEFINITION 2.11. *([5]) (p-Sequential Henstock integral). A function* $f : [a,b] \to \mathbb{R}$ *is p-Sequential Henstock integrable on* $[a,b]$ *to a number* $\alpha \in \mathbb{R}$ *if for any* $\bar{\varepsilon} > 0$ *there exists a sequence of positive functions* $\{\delta^p_{n\bar{\varepsilon}}(x)\}_{n=1}^{\infty}$*, such that for any* $\bar{\beta}_n \in FPV_p(\delta_{n\bar{\varepsilon}}(x))$ *and for all* $\bar{P}_n \in \prod_{i=1}^{m_n}[\bar{\beta}_n]$*, we have* $|S_f(\bar{\beta}_n, \bar{P}_n) - \alpha| < \bar{\varepsilon}$*.. We say that* α *is a p-Sequential Henstock integral of* f *on* $[a,b]$*. i.e.* $\alpha = \int_a^b f$ *i.e p-*$SH_f[a,b] = \alpha = \int_{[a,b]} f$*. and use* $p - SH_f[a,b]$ *to denote the set of all p-Sequential Henstock integrable functions defined on* $[a,b]$*.*

We define newly the following concepts

DEFINITION 2.12. *(p-interval integral). A function* $F : [a,b] \to I_{\mathbb{R}}$ *is p-interval integrable on* $[a,b]$ *to a number* $I_0 \in I_{\mathbb{R}}$ *if for any* $\bar{\varepsilon} > 0$*, then there exists a* $\delta^p_{\bar{\varepsilon}} > 0$ *such that for any* $\bar{\beta} \in FPV_p(\delta_{\bar{\varepsilon}})$ *and for all* $\bar{P} \in \prod[\bar{\beta}]$*, we have*

$$d(S_F(\bar{\beta}, \bar{P}), I_0) < \bar{\varepsilon}.$$

We say that $I_0 = [I_0^-, I_0^+]$ *is a p-interval integral of* F *on* $[a,b]$ *with* $I_0 = (p)\int_a^b F$*. We use* $p[a,b]$ *to denote the set of all p-interval integrable functions defined on* $[a,b]$*.*

DEFINITION 2.13. *([5]) (Interval p-Sequential Henstock integral). A function* $F : [a,b] \to I_{\mathbb{R}}$ *is p-Sequential Henstock integrable on* $[a,b]$ *to a number* $I_0 \in I_{\mathbb{R}}$ *if for any* $\bar{\varepsilon} > 0$ *there exists a sequence of positive functions* $\{\delta^p_{n\bar{\varepsilon}}(x)\}_{n=1}^{\infty}$*, such that for any* $\bar{\beta}_n \in FPV_p(\delta_{n\bar{\varepsilon}}(x))$ *and for all* $\bar{P}_n \in \prod_{i=1}^{m_n}[\bar{\beta}_n]$*, we have*

$$d(S_F(\bar{\beta}_n, \bar{P}_n), I_0) = max|d(S_{F^-}(\bar{\beta}_n, \bar{P}_n), I_0^-) < \bar{\varepsilon}, d(S_{F^+}(\bar{\beta}_n, \bar{P}_n), I_0^+) < \bar{\varepsilon}|.$$

We say that I_0 *is a interval p-Sequential Henstock integral of* F *on* $[a,b]$*. i.e p-*$ISH_F[a,b] = I_0 = \int_{[a,b]} F$*. and use p-*$ISH_F[a,b]$ *to denote the set of all interval p-Sequential Henstock integrable functions defined on* $[a,b]$*.*

DEFINITION 2.14. *([5]) (interval ap-Sequential Henstock integral) A function* $F : [a,b] \to I_{\mathbb{R}}$ *is interval ap-Sequential Henstock integrable on* $[a,b]$ *to a vector* $I_0 \in I_{\mathbb{R}}$ *if for any* $\varepsilon > 0$ *there exists a sequence of positive choice functions* $\{S_n(x)\}_{n=1}^{\infty}$ *on [a,b] such that*

$$d(\sum_{i=1}^{m_n} F(t_{i_n})(u_{i_n} - u_{(i-1)_n}), I_0)$$

$$= max|d(\sum_{i=1}^{m_n \in \mathbb{N}} F^-(t_{i_n})(u_{i_n} - u_{(i-1)_n}), I_0^-) < \varepsilon, d(\sum_{i=1}^{m_n \in \mathbb{N}} F^+(t_{i_n})(u_{i_n} - u_{(i-1)_n}), I_0^+) < \varepsilon|$$

whenever $P_n = \{([u_{(i-1)_n}, u_{i_n}], t_{i_n})\}_{i=1}^{m_n}$ *is a* $S_n(x) - fine$ *ap-interval Sequential Henstock partitions on* $[a,b]$*. We say that* I_0 *is a interval ap-Sequential Henstock integral of* F *on* $[a,b]$ *i.e* $I_0 = (apISH)\int_a^b F$ *and* $F \in apISH[a,b]$ *and use* $apISH_F[a,b]$ *to denote the set of all interval ap-Sequential Henstock integrable functions defined on* $[a,b]$

3. Main results

Now, we give the equivalence of certain interval Henstock-type integrals $IH[a,b]$ using sequential approach. Firstly, we state and prove the following:

THEOREM 3.1. *If $F:[a,b]\to I_{\mathbb{R}}$ is interval Sequential Henstock integrable on $[a,b]$, then it is p-Sequential Henstock integrable there, Infact,*

$$(ISH)\int_a^b F = (p\text{-}IH)\int_a^b F$$

PROOF. Suppose $F\in SH[a,b]$ in the sense of Definition 2.10, we want to show that $F\in p\text{-}SH[a,b]$. Let $\varepsilon>0$ be given, then there exists a $\{\delta_n(x)\}_{n=1}^{\infty}$ and $P_n \ll \delta_n(x)$, such that

$$\max|d(\sum_{i=1}^{m_n} F^-(t_{i_n})(u_{i_n}-u_{(i-1)_n}), I_0^-) < \varepsilon, d(\sum_{i=1}^{m_n} F^+(t_{i_n})(u_{i_n}-u_{(i-1)_n}), I_0^+) < \varepsilon|$$

Choose arbitrarily one $\delta^p_{\mu\varepsilon}(x)$ from $\{\delta^p_{n\varepsilon}(x)\}_{n=1}^{\infty}$ for $0<p<1$, such that for a given $\delta^p_{\mu\varepsilon}(x)>0$ and $\bar{\beta}^{\epsilon}=\sqrt[p]{(S(F,P_n)^p}$ be arbitrary, then

$$\|\beta_p\| = \sqrt[p]{\sum_{i=0}^{|\beta^{\epsilon}|-1} d(\bar{\beta}_{(i+1)_n}, \bar{\beta_{i_n}})^p} \leqslant \|\bar{\beta}^{\epsilon}\|_p = \sqrt[p]{(\sum_{i=0}^{|\beta^{\epsilon}|-1} d(\bar{\beta}^{\epsilon}_{(i+1)_n}, \bar{\beta}^{\epsilon}_{i_n}))^p} < \bar{\delta}^p_{\epsilon}.$$

i.e $SH[a,b]\subseteq FPV_p^{(n)}[a,b]$. Then, the result $F\in SH[a,b]$ follows immediately from

$$ISH[a,b] = p\text{-}IH[a,b].$$

□

THEOREM 3.2. *If $F:[a,b]\to I_{\mathbb{R}}$ is p-interval Sequential Henstock integrable on $[a,b]$, then it is interval Sequential Henstock integrable there. Infact,*

$$(p\text{-}IH)\int_a^b F = (ISH)\int_a^b F.$$

PROOF. Suppose $F\in p\text{-}IH[a,b]$ in the sense of Definition 2.13, we want to show that $F\in ISH[a,b]$.
Let $\varepsilon>0$ be given, there exist sequence of positive functions $\{\delta^p_{n\epsilon}(x)\}_{n=1}^{\infty}$ and $P_n \ll \delta_n(x)$, such that

$$\max|d(S_{F^-}(\bar{\beta}_n,\bar{P}_n), I_0^-) < \bar{\varepsilon}, d(S_{F^+}(\bar{\beta}_n,\bar{P}_n), I_0^+) < \bar{\varepsilon}|. < \frac{1}{n^p}.$$

Let $\varepsilon>0$ be arbitrary, we choose an arbitrary $\bar{\delta}^p_{\mu_\epsilon} \leqslant \delta^p_{\mu_\epsilon}$ from $\{\delta^p_{n\epsilon}(x)\}_{n=1}^{\infty}$ for $n\geqslant\mu$ such that for a given $\delta^p_{\mu_\epsilon}(x)>0$, then we have

$$d(\delta^p_{\mu_\epsilon}(x), \delta^p_{N\epsilon}(x)) < \varepsilon$$

for all $\bar{\beta} \in FPV_p(\delta_{n\bar{\varepsilon}}(x))$ and $\bar{P}_n \in \prod_{i=1}^{m_n}[\bar{\beta}_n]$ $|S_F(\bar{\beta}, \bar{P}_n) - \alpha| < \bar{\varepsilon}$ for some $I_0 \in I_{\mathbb{R}}$. Choose $\bar{\beta}_n^\epsilon \in FPV_p^{(n)}(\bar{\delta}_{\mu_\epsilon}^p)$ where $\bar{\beta}^\epsilon \in P_n(\bar{\beta}^\epsilon)$ is arbitrary, so that for a given $\delta_\mu^p(x) > 0$, we have

$$\|\bar{\beta}_n\|_p = \sqrt[p]{\sum_{j=0}^{|\beta_n|-1} (d(\bar{\beta}_{i+1,\bar{\beta}_i}))^p} \leqslant \|\beta^\epsilon\|_p < \varepsilon, \forall x \in [a,b].$$

Let $\lambda \to 0$, then, our choice of $\delta_\mu(x)$ guarantees that if $\bar{P}_n \ll \delta_{\mu_\epsilon}^p(x)$, then $\bar{P}_n \ll \delta_\mu(x)$. Hence for a given $\bar{P}_n^1 \ll \delta_n(x)$, we can make the Riemann sums for $FPV_p^{(n)}$ and $\bar{P}_n$ arbitrary close(using the same tagged partition) such that

$$d(\bar{\beta}_n^\epsilon, S_F(\bar{\beta}, \bar{P}_n)) < \frac{\varepsilon}{2}.$$

Now, for any $\varepsilon > 0$, there exists a $\delta_{\mu_\epsilon}^p(x) \in \{\delta_n^p(x)\}_{n=1}^\infty$ for $n \geqslant \mu$, we find a $\delta_\mu(x) \in \{\delta_n(x)\}_{n=1}^\infty$ satisfying (1) above and move down the sequence $\{\delta_n(x)\}_{n=1}^\infty$ denoting its' new position as N^*, so that $\frac{1}{N^*} < \frac{\varepsilon}{2}$. If $P_n \ll \delta_n(x)$, then

$$\begin{aligned} d(\bar{\beta}_n^\epsilon, I_0) &\leqslant d(\bar{\beta}_n^\epsilon, S_F(\bar{\beta}, \bar{P}_n)) + d(S_F(\bar{\beta}, \bar{P}_n), I_0) \\ &< \frac{\varepsilon}{2}. + \frac{1}{N^*} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus, p-$IH[a,b] = ISH[a,b]$. □

Corollary 3.1. *A $F : [a,b] \to I_{\mathbb{R}}$ is p-interval Sequential Henstock integrable on $[a,b]$, if and only if, it is interval Sequential Henstock integrable there and*

$$p\text{-}IH[a,b] = ISH[a,b].$$

Proof. The result follows easily from the proof of Theorems 3.1 and 3.2. This completes the proof. □

Theorem 3.3. *If $F : [a,b] \to I_{\mathbb{R}}$ is interval ap-Sequential Henstock integrable on $[a,b]$, then it is interval Sequential Henstock integrable there. Infact,*

$$(apIH)\int_a^b F = (ISH)\int_a^b F.$$

Proof. Let $F \in ap$-$IH[a,b]$ in the sense of Definition 2.14, we want to show that $F \in ISH[a,b]$. Suppose that $\{S_n(x)\}_{n=1}^\infty$ is a decreasing sequence of choice functions such that $S_{n+1} < S_n$ for all $t \in [a,b]$. Let $\varepsilon > 0$ be given, then there exists a $\{S_n(x)\}_{n=1}^\infty$ and $S_n(x) - fine$ partitions $P_n = \{[u_{(i-1)_n}, u_{i_n}, t_{i_n}]\}_{i=1}^{m_n}$, then

$$\max|d(\sum_{i=1}^{m_n} F^-(t_{i_n})(u_{i_n} - u_{(i-1)_n}), I_0^-) < \varepsilon, d(\sum_{i=1}^{m_n} F^+(t_{i_n})(u_{i_n} - u_{(i-1)_n}), I_0^+) < \varepsilon|,$$

where $I_0 = [I_0^-, I_0^+] = \int_a^b F$. For $n = 1, 2, ...,$. Let ε_n be a rational ε such that $0 < \varepsilon < 1$. By Definition 2.10, there exists $\delta_n(x)$ for each ε_n satisfying the last equation. Since Q is a rational number which is countable, then $\{\delta_n(x)\}_{n=1}^\infty$ is a sequence. Given $\varepsilon > 0$, there exists a $\{\delta_{n_v}(x)\}_{n=1}^{\infty}$ and $\delta_n - fine$ partitions $P_n = [(u_{(i-1)_n}, u_{i_n}), t_{i_n}]_{i=1}^{m_n}$, then

$$d(\sum_{i=1}^{m_n} F(t_{i_n})(u_{i_n} - u_{(i-1)_n}), I_0) < \varepsilon.$$

Thus,

$$apIH[a,b] = ISH[a,b].$$

□

THEOREM 3.4. *If $F : [a,b] \to I_{\mathbb{R}}$ is interval Sequential Henstock integrable on $[a,b]$, then it is interval ap-Sequential Henstock integrable there. Infact,*

$$(ISH)\int_a^b F = (apIH)\int_a^b F.$$

PROOF. Suppose $F \in ISH[a,b]$ in the sense of Definition 2.14, we want to show that $F \in ap\text{-}IH[a,b]$.
Suppose there exists a positive $\{\delta_n(x)\}_{n=1}^{\infty}$ such that $P_n = \{[(u_{(i-1)_n}, u_{i_n}), t_{i_n}]\}_{i=1}^{m_n}$ where $u_{(i-1)_n} \leqslant t_{i_n} \leqslant u_{i_n}$. For every δ_n-fine partition of $[a,b]$, then

$$\max|d(\sum_{i=1}^{m_n} F^-(t_{i_n})(u_{i_n} - u_{(i-1)_n}), I_0^-), d(\sum_{i=1}^{m_n} F^+(t_{i_n})(u_{i_n} - u_{(i-1)_n}), I_0^+)| < \frac{1}{N^*}.$$

Let $\mu \in \mathbb{R} > 0$, we choose a $\{\delta_n(x)\}_{n=1}^{\infty}$ such that for a given $\delta_n(x) > 0$

$$d(\delta_{\mu_v}(t), \delta_\mu(x)) < \varepsilon$$

for all $x \in [a,b]$. where $\delta_{\mu_v}(x)$ is a sequence of positive gauge function in $[a,b]$ for $n \geqslant \mathbb{N}$. Then, the choice sequence $S_n(x)$ guarantees that $P_n \ll \delta_\mu(x)$. Hence, for a positive gauge function then $P_n \ll S_{\mu_v}(x)$, We can make the Riemann sums for P_μ and P_n arbitrarily close (using the similar tags on each partition) such that

$$d(S(F,P_\mu), S(F,P_n)) < \frac{\varepsilon}{2}.$$

Since for any $\varepsilon > 0$, there exists a choice $\{S_n(x)\}_{n=1}^{\infty}$ such that for each $S_\mu(x) \in \{S_n(x)\}_{n=1}^{\infty}$ and move down the sequence $\{S_n(x)\}_{n=1}^{\infty}$, we denote its' new position as N^*, so that $\frac{1}{N^*} < \frac{\varepsilon}{2}$. Then if $P_n \ll S_n(x)$, then

$$\begin{aligned} d(S(F,P_n), I_0) &= d(S(F,P_n), S(F,P_\mu) + S(f,P_\mu), I_0) \\ &\leqslant d(S(F,P_n), S(F,P_\mu)) + d(S(F,P_\mu), I_0) \\ &< \frac{\varepsilon}{2}. + \frac{1}{N^*} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus,

$$ISH[a,b] = apIH[a,b].$$

□

Corollary 3.2. *$F : [a, b] \to I_{\mathbb{R}}$ is interval ap-Sequential Henstock integrable on $[a, b]$, if and only if, it is interval Sequential Henstock integrable there and*

$$apIH[a, b] = ISH[a, b].$$

Proof. It follows easily from the proof of Theorems 3.3 and 3.4. This completes the proof. □

Acknowledgements

The authors are thankful to the editors and the anonymous reviewers for their valuable suggestions and comments that helped to improve this manuscript.

References

[1] J. A. Chartfield, Equivalence Of Integrals Proceedings of American Mathematical Society. in Mathematics. Vol. **3**, (1973), 279-285.

[2] Z. Dafang and Y. Guoju. On AP-Henstock Stieljes Integral. Chungcheong Mathematical Society. Vol. **19**, (2006), 1-2.

[3] R. Gordon, The Integral of Lebesgues, Denjoy, Perron and Henstock. Graduate studies in Mathematics. *American Mathematical Society. Providence, RI,* **4**, (1994), 12-30.

[4] M. E. Hamid and A. H. Elmuiz, . On Henstock-Stieltjes Integrals of interval-Valued Functions and Fuzzy-Number-Valued Functions. *Journal of Applied Mathematics and Physics,* **4**. (2016), 779-786.

[5] M. E. Hamid and A. H. Elmuiz. On McShane Integrals of interval-Valued Functions and Fuzzy-Number-Valued Functions on Time Scale. Journal of Progressive Research in Mathematics. Vol. **12**, 1, (2017), 1780-1787.

[6] R. Henstock. The General Theory of Integration. *Oxford University Press, Oxford, UK,* (1991), 4-20.

[7] V. O. Iluebe and A. A. Mogbademu, Equivalence Of Henstock And Certain Sequential Henstock Integral. *Bangmond International Journal of Mathematical and Computational Science.,* **1**.(1 and 2), (2020), 9 - 16.

[8] V. O. Iluebe and A. A. Mogbademu, On Sequential Henstock Integral For $L^p[0, 1]$-Interval Valued Functions. Malaya Journal of Matematik. **10**(02)(2022), 178-186.

[9] V. O. Iluebe and A. A. Mogbademu, On ap-Sequential Henstock Integral for Interval valued Functions. Int. j. Nonlinear Anal. Appl. **13**(2)(2022), 3095-3103..

[10] V. O. Iluebe and A. A. Mogbademu, Dominated and Bounded Convergence Results of Sequential Henstock Stieltjes Integral in Real valued Space. Journal of Nepal Mathematics Society(JNMS), Vol. **3**. Issue 1 (2020). 17-20.

[11] V. O. Iluebe and A. A. Mogbademu, Sequential Henstock Integral For Interval Valued Functions. CJMS. **11** (2022), 358-367.

[12] V. O. Iluebe and A. A. Mogbademu, On ap-Sequential Henstock Stieltjes Integral For Interval Valued Functions. Int. J. Maths. Virtual Institute.**12** (2022), 369-378.

[13] Y. J. Kim, The Equivalence of Perron, Henstock and Variational Stieltjes Integrals. Journal of the Chungcheong Mathematical Society, Vol.**10**, (1997), 29-36.

[14] R. E. Moore, R. B. Kearfott, and J. C. Michael Introduction to Interval Analysis. *Society for Industrial and Applied Mathematics.*(2009), 37-38; 129-135.

[15] L. A. Paxton, Sequential Approach to the Henstock Integral. *Washington State University, arXiv:1609.05454v1* [maths.CA],(2016), 9-13.

[16] M. C. Ray, Equivalence Of Riemann Integral Based on p-Norm. School Of Mathematics and Statistics. Vol.**6**, (2008), 1-13.

[17] R. Sergio and Jr. Canoy. On Equivalence of the ϕ-Integral and the Henstock Integral for TVS-valued Functions. Mathematical Analysis. **8**, (2014), 625-632.

[18] L. Solomon. The Kursweil-Henstock Integral and Its Applications. Marcel Dekker, (2001), 11-25.

[19] C. X. Wu and Gong, Z. T. On Henstock Integrals of interval-Valued Functions and Fuzzy-Number-Valued Functions. *Fuzzy Set and Systems.* **115**, (2016), 377-391.

[20] J. L. Ying, On The Equivalence Of Mcshane and Lebesgue Integrals. Real Analysis Exchange. Vol. 21(2), (1995-96), 767-770.

Received by editors 3.3.2023; Revised version 27.7.2023; Available online 25.9.2023.

Victor Odalochi Iluebe, University of Lagos, Department of Mathematics, Lagos, Nigeria

Email address: victorodalochi1960@gmail.com

Adesanmi Alao Mogbademu, University of Lagos, Department of Mathematics, Lagos, Nigeria

Email address: amogbademu@unilag.edu.ng

Joshua Olugbenga Ajilore, Lagos State University of Science and Technology, Department of Mathematics, Lagos, Nigeria

Email address: joshua104u@yahoo.com

Olumide Afolabi Esan, Yaba College of Technology, Department of Mathematics, Lagos, Nigeria

Email address: esnolu@yahoo.com