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AN INVESTIGATION ON THE EXISTENCE AND UNIQUENESS ANALYSIS OF THE FRACTIONAL VOLTERRA-FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this study, we establish some new investigations into the existence of and uniqueness analysis of the fractional Volterra-Fredholm integro differential equations with beginning conditions that are nonlinear Caputo fractional equations. The desired outcomes are demonstrated using the fractional inequality, a variation of the nonlinear alternative of Leray-Schauder in Banach spaces and the Banach fixed point theorem for nonself mappings. Additionally, by using the contraction mapping principle, the uniqueness results are established. To summarise our key findings, a few examples are shown at the end.

1. Introduction

In areas like aerodynamics, polymer science, fractals and chaos, nonlinear control theory, signal and image processing, bioengineering, and chemical engineering, among others, fractional integro-differential equations have proven to be an excellent tool for mathematical modelling of dynamical systems and real-world issues. Yet, there are numerous different definitions of fractional derivatives and noninteger order integral operators in the literature. Fractional calculus and fractional differential equations have recently been extensively used extensively in physical applications. For instance, the book by Tarasov [**37**] discusses the use of fractional

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calculus in the dynamics of fields and media and includes some recent examples of how fractional calculus is used to model complex physical systems. Applications of the fractional calculus in the dynamics of the Mainardi book [**27**] include. The fractional calculus has used in the dynamics of viscoelasticity. See [**13, 14**] by Das for examples of how fractional calculus is used to describe and regulate physical systems. The most recent advancements in the subject of fractional dynamics are covered in the book by Klafter et al. [**23**]. In addition to the aforementioned works, there are several studies on the use of fractional calculus in thermal systems, heat conduction, viscoelastic materials, biological tissues, signal processing and complex dynamics (see [**28, 29, 38**]). The existence and uniqueness of solutions are topics that have been extensively explored in recent studies on fractional differential and integral equations. We'll talk about a few of them.

In [**6, 8**] Balachandran et al. proved the existence of solutions of integro-differential equations in Banach spaces. Also, we can find existence results for boundary value problems of fractional differential equations in the survey by Agarwal et al. [**2**] and in the papers [**7, 11, 36, 16, 9, 33, 35, 19**]. In [**12**], Cljement et al. proved the existence of Holder continuous solutions for a partial fractional differential equation and in [**22**], Kilbas et al. studied the existence of solutions of several classes of ordinary fractional differential equations. Also, Aghajani et al. [**1**], Aghajani et al. [**3**], Anguraj et al. [**5**], Baleanu and Mustafa [**10**], Diethelm and Ford [**15**], Kilbas and Marzan [**21**], Idczak and Kamocki [**20**], Kosmatov [**24**], Kostijc [**25**], Samko et al. [**34**], Tian and Bai [**39**], Vald'es [**40**], Wei et al. [**41**], Yuste and Acedo [**42**], Pilipovijc and Stojanovijc [**31**], between so many more, have investigated the existence of solutions for various types of fractional differential and integro-differential equations. Furthermore, several analytical and numerical methods have been proposed for approximate solutions of fractional differential equations, e.g. [**15, 16, 17**].

In this study, we investigate the existence and uniqueness of solutions for a problem inspired by previous publications and the absence of research on nonlinear fractional Volterra-Fredholm integro-differential equations with initial conditions.

$$\begin{aligned}
({}^{C}D^{\alpha}_{a^+}u)(x) &= \mathcal{K}(u(x)) \\
&+ f\Big(x, ({}^{C}D^{\beta}_{a^+}u)(x), \int_{a^+}^{x} G_1(x,s,u(s))ds, \int_{a^+}^{b} G_2(x,s,u(s))ds\Big), \qquad (1.1)\\
& u(a) = u_0,\ x = J \in (a,b],
\end{aligned}$$

where ${}^{C}D^{\alpha}_{a^+}$ is the Caputo fractional derivative of order $\alpha, 0 < \alpha < 1, 0 < \beta < 1, \beta < \alpha,\ \mathcal{K} > 0$. Also, $f : [a,b] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $G_1, G_2 : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

The primary objective of this study is to establish new findings regarding the existence and uniqueness of solutions for the aforementioned problem (1.1). This is accomplished through the utilization of the fixed point theorem for nonself mappings, fractional inequality, a modified version of the nonlinear alternative of Leray-Schauder, and the Banach contraction principle.

2. Preliminaries

In this section, we present some preliminaries which will be needed in this paper (for more details see [**22, 26, 40, 41**] .Let $[a, b] \subset \mathbb{R}$ be a finite interval. Also assume that $\alpha, \beta, \gamma \in \mathbb{C}$ and $R(z) = Real(z)$ for $z \in \mathbb{C}$. The Riemann-Liouville fractional integral and derivative of order $\alpha \in \mathbb{C}$ are defined by

$$(I^{\alpha}_{a+}f)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(s)}{(x-s)^{1-\alpha}}ds, \ x > 0, R(\alpha) > 0 \tag{2.1}$$

and

$$(D^{\alpha}_{a+}f)(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x \frac{f(s)}{(x-s)^{\alpha-n+1}}ds \tag{2.2}$$

$$= \frac{d^n}{dx^n}(I^{n-\alpha}_{a+}f)(x), \ , \ x > 0, R(\alpha) > 0 \tag{2.3}$$

respectively, where $n = [R(\alpha)] + 1$ when $\alpha \notin \mathbb{N}_0 = \{0, 1, ...\}$($[\alpha]$ denotes the integer part of α). When $\alpha = n \in \mathbb{N}_0$, then

$$(I^{\alpha}_{a+}f)(x) = \frac{1}{(n-1)!}\int_0^x (x-s)^{1-\alpha}f(s)ds, \tag{2.4}$$

$$n = [R(\alpha)] + 1 \text{ for } \alpha \notin \mathbb{N}_0; n = \alpha \text{ for } \alpha \in \mathbb{N}_0; \tag{2.5}$$

and when $R(\beta) > n$, we can find that the Caputo fractional differentiation operator ${}^{C}D^{\alpha}_{a+}$ of the power function $(t-a)^{\beta-1}$ has the same form. In the next property we recall this facts.

PROPOSITION 2.1. *Let $R(\alpha) > 0$ and n be given by 2.5 . Also let $R(\beta) > 0$, then*

$$(I^{\alpha}_{a+}(t-a)^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(x-a)^{\beta+\alpha-1} \tag{2.6}$$

$$({}^{C}D^{\alpha}_{a+}(t-a)^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(x-a)^{\beta-\alpha-1}, \ R(\beta) > n \tag{2.7}$$

and

$$({}^{C}D^{\alpha}_{a+}(t-a)^{k})(x) = 0, \ k = 0, 1, ..., n-1 \tag{2.8}$$

The following is concerning with boundedness of the fractional integration operator I^{α}_{a+} from the space $C_{\gamma}[a, b]$ into $C[a, b]$.

LEMMA 2.1. *Let $R(\alpha) > 0$ and $0 \leqslant R(\gamma) \leqslant 1$. If $R(\gamma) \leqslant R(\alpha)$, then the fractional integration operator I^{α}_{a+} is bounded from $C_{\gamma}[a, b]$ into $C[a, b]$:*

$$\|I^{\alpha}_{a+}f\|_C \leqslant l\|f\|_{C_{\gamma}} \tag{2.9}$$

$$l = (b-a)^{R(\alpha-\gamma)}\frac{\Gamma(R(\alpha))|\Gamma(1-R(\gamma))|}{|\Gamma(\alpha)|\Gamma(1+R(\alpha-\gamma))}. \tag{2.10}$$

The semigroup property of the fractional integration operator I^{α}_{a+} and the composition relation between the fractional integration operator I^{α}_{a+} and the fractional *difierentiation operator* D^{β}_{a+} *are given by the following lemma (see* [**17**], *Lemma 2.1).*

Lemma 2.2. *Let* $R(\alpha), R(\beta) > 0$ *and* $f(x) \in C[a, b]$. *Then for any* $x \in [a, b]$ *the following assertions are true:*

$$(a)\ (I^{\alpha}_{a+} I^{\beta}_{a+} f)(x) = (I^{\alpha+\beta}_{a^+} f)(x) \tag{2.11}$$

$$(b)\ (D^{\alpha}_{a+} I^{\alpha}_{a+} f)(x) = f(x) \tag{2.12}$$

$$(c)\ \text{If } R(\alpha) > R(\beta) \text{ then} \tag{2.13}$$

$$(D^{\alpha}_{a+} I^{\alpha}_{a+} f)(x) = (I^{\alpha-\beta}_{a^+} f)(x)$$

The following assertion ([**17**], *Lemma 2.2 (a)) yields that the Caputo fractional difierentiation operator* ${}^{C}D^{\alpha}_{a+}$ *is the left inverse of the Riemann-Liouville fractional integration operator* I^{α}_{a+} *when* $R(\alpha) \notin \mathbb{N}_0$ *or* $\alpha \in \mathbb{N}$.

Theorem 2.1. *Let* u *and* f *be continuous and nonnegative functions defined on* $I = [a, b]$ *and let* $n(t)$ *be a continuous, positives and nondecreasing function defined on* I. *Then*

$$u(x) \leqslant n(x) + \int_0^t f(s)u(s)ds, x \in I \tag{2.14}$$

$$\Rightarrow u(x) \leqslant n(x)exp\Big(\int_0^t f(s)ds\Big), x \in I$$

Theorem 2.2. [**32**] *Let* X *be a normed linear space,* K *a convex subset of* X, O *an open subset of* K *and* $\theta \in O($ θ *is the zero element of* X*). Suppose that* $N : \overline{O} \to K$ *is a continuous and compact operator where* $\overline{O}$ *is closure of* O. *Then either*
(i) T *has a fixed point in* $\overline{O}$, *or*
(ii) There exists $u \in \partial O$ *such that* $u = \lambda T u$ *for some* $\lambda \in (0, 1)$, *where* ∂O *is boundary of* O

3. Auxiliary results

In this section, we prove some auxiliary results which will be needed in the next section. The first result is concerning with the composition of the Caputo fractional differentiation operator ${}^{C}D^{\beta}_{a+}$ with the fractional integration operator I^{α}_{a+}

Theorem 3.1. *Let* $f : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *and* $g : [a, b] \times [a, b] \times \mathbb{R}$ *be continuous functions and let* $0 < \alpha < 1, 0 < \beta < 1$ *and* $\beta < \alpha$. *Then a function* $u \in C[a, b]$ *with* $({}^{C}D^{\alpha}_{a+} u)(x) \in C[a, b]$ *is a solution of fractional integro-differential equation* (1.1) *if and only if*

$$u(x) = u_0 + \frac{\mathcal{K}}{\Gamma(\beta)} \int_{a^+}^{x} (x - s)^{\beta-1} u(s)ds \tag{3.1}$$

where $u \in C[a,b]$ *satisfies the integral equation*

$$\begin{aligned} u(x) &= u_0 + \\ &+ \Big(I_{a^+}^{\alpha-\beta}\Big[\mathcal{K}(I_{a^+}^{\beta}u)(s) + f(t,u(t), \int_{a^+}^{t} G_1\big(t,s,u(s)+(I_{a^+}^{\beta}u)(s)\big)ds, \\ &\int_a^b G_2\big(t,s,u(s)+(I_{a^+}^{\beta}u)(s)\big)ds\big)\Big]\Big)(x). \end{aligned} \tag{3.2}$$

Lemma 3.1. *Let* $u : [a,b] \to [0,\infty)$ *be a continuous function and* $A : [a,b] \to [0,\infty)$ *be a nondecreasing and continuous function. Moreover, assume that there exist constants* $\alpha, \beta \in \mathbb{R}$ *with* $0 < min\{\alpha,\beta\} < 1$ *and* $B > 0$ *such that*

$$u(x) \leqslant A(x) + B\Big(\mathcal{K}(I_{a^+}^{\alpha}u)(x) + (I_{a^+}^{\alpha}u)(x) + (I_{a^+}^{\beta}u)(x)\Big),\ x \in [a,b] \tag{3.3}$$

Then we have

$$\begin{aligned} u(x) &\leqslant A(x)\Big(1 + \frac{B\mathcal{K}(b-a)^{\alpha}}{\Gamma(\alpha+1)} + \frac{B(x-a)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big) \\ &\times exp\Big(B\frac{(b-a)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}x\Big), \end{aligned} \tag{3.4}$$

for some $min\{\alpha,\beta\} > 1$.

Proof. Using Lemma (2.2), part (a) and the inequality (3.3) we have

$$\begin{aligned} u(x) &\leqslant A(x) + B\Big(\mathcal{K}(I_{a^+}^{\alpha}A)(x) + (I_{a^+}^{\alpha}A)(x) + (I_{a^+}^{\beta}A)(x)\Big) \\ &+ B\Big(\mathcal{K}(I_{a^+}^{\alpha}u)(x) + (I_{a^+}^{2\alpha}u)(x) + (I_{a^+}^{2\beta}u)(x) + 2(I_{a^+}^{\alpha+\beta}u)(x)\Big). \end{aligned}$$

By an iterative method, the inequality (3.3) implies

$$\begin{aligned} u(x) &\leqslant A(x) + \mathcal{K}(BI_{a^+}^{\alpha+\beta}A)(x) + (BI_{a^+}^{\alpha+\beta}A)(x) \\ &+ \mathcal{K}(BI_{a^+}^{\alpha+\beta}u)(x) + (BI_{a^+}^{(\alpha+\beta}u)(x). \end{aligned} \tag{3.5}$$

Since $A(x)$ is non decreasing, we get

$$(I_{a^+}^{\alpha+\beta}A)(x) \leqslant \frac{\mathcal{K}A(x)(x-a)^{\alpha}}{\Gamma(\alpha+1)} + \frac{A(x)(x-a)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}. \tag{3.6}$$

Now if we choose $\min\{\alpha,\beta\} > 1$, then

$$(I_{a^+}^{(\alpha+\beta}u)(x) \leqslant \frac{\mathcal{K}(b-a)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(b-a)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\int_a^x u(s)ds. \tag{3.7}$$

From (3.5)-(3.7) we obtain

$$\begin{aligned} u(x) &\leqslant A(x)\Big(1 + \frac{B\mathcal{K}(b-a)^{\alpha}}{\Gamma(\alpha+1)} + \frac{B(x-a)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big) \\ &+ \frac{B(b-a)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\int_a^x u(s)ds, \end{aligned}$$

and finally by using Theorem 2.1 we get the inequality (3.4).

Here we recall the definition of signum function. For $x \in \mathbb{R}$

$$sgn(x) = \begin{cases} 1, \ 0 < x, \\ 0, \ x = 0, \\ -1, \ x < 0 \end{cases} \tag{3.8}$$

Now let f and g appeared in Eq. (1.1) satisfy the following hypotheses:

(h_1) $f : [a, b] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and there exist positive constants P_1 and P_2 such that

$$|f(t, x, y, z)| \leqslant P_1(|x| + |y| + |z|) + P_2, \ x, y, z \in \mathbb{R}, \ t \in [a, b]$$

(h_2) $G_1, G_2 : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ are a continuous functions and there exist positive constants $\delta_1^{G_1}, \delta_1^{G_2}, \delta_2^{G_1}$ and $\delta_2^{G_2}$ such that

$$|G_1(t, s, x)| \leqslant \delta_1^{G_1}|x| + \delta_2^{G_1}, \ s, t \in [a, b], x \in \mathbb{R}$$

$$|G_2(t, s, x)| \leqslant \delta_1^{G_2}|x| + \delta_2^{G_2}, \ s, t \in [a, b], x \in \mathbb{R}.$$

Also consider the following equations:

$$\begin{aligned} u(x) &= u_0 + \\ &+ \lambda\Big(I_{a^+}^{\alpha-\beta}\Big[\mathcal{K}(I_{a^+}^{\beta}u)(s) + f\Big(t, u(t), \int_a^t G_1\big(t, s, u(s) + (I_{a^+}^{\beta}u)(s)\big)ds, \\ &\int_a^b G_2\big(t, s, u(s) + (I_{a^+}^{\beta}u)(s)\big)ds\Big)\Big]\Big)(x), \end{aligned} \tag{3.9}$$

where $\lambda \in (0, 1), \ 0 < \alpha < 1, \ 0 < \beta < 1$ and $\beta < \alpha$. Then we have the following theorem. □

THEOREM 3.2. *Assume that* (h_1) *and* (h_2) *hold and* $u(x) \in C[a, b]$ *satisfies* (3.9),*then we have*

$$|u(x)| \leqslant A(x)exp\Big((P_0^{G_1} + P_0^{G_2})\Big[\frac{(b-a)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} + \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\Big]x\Big), \tag{3.10}$$

when $\alpha - \beta \geqslant 1$ *and*

$$\begin{aligned} |u(x)| &\leqslant \lambda A(x)\Big(1 + (P_0^{G_1} + P_0^{G_2})\Big[\frac{\mathcal{K}(b-a)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(b-a)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big]\Big) \\ &\times \ exp\Big((P_0^{G_1} + P_0^{G_2})\frac{(b-a)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}x\Big), \end{aligned} \tag{3.11}$$

when $0 < \alpha - \beta < 1$ *and* $x \in [a, b]$. *Here* $P_0 = P_1 \max\{Q_1^{G_1}, 1\}$, *is a positive integer such that* $(\alpha - \beta) > 1$ *and*

$$\begin{aligned} A(x) &= u_0 + \frac{|u_0|(x-a)^{\alpha}}{\Gamma(1+\alpha)} + P_1Q_1^{G_1}\frac{|u_0|(x-a)^{1+\alpha-\beta}}{\Gamma(\alpha-\beta+2)} + P_1Q_2^{G_2}\frac{|u_0|(x-a)^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)} \\ &+ P_1Q_1^{G_1}\frac{(x-a)^{1+\alpha-\beta}}{\Gamma(\alpha-\beta+2)} + P_1Q_2^{G_2}\frac{(x-a)^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)} + \frac{P_2(b-a)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}. \end{aligned}$$

PROOF. Let u satisfy (3.9). Then using $(h_1),(h_2)$, Property 2.1 and Lemma 2.2, part (a) we find

$$\begin{aligned} u(x) &= \lambda u_0 \\ &+ \lambda\Big(\mathcal{K}(I_{a^+}^{\alpha}|u|)(x) + P_1(I_{a^+}^{\alpha-\beta}|u|)(x) + P_1\Big(I_{a^+}^{\alpha-\beta+1}(Q_1^{G_1}+Q_1^{G_2})|u|)(x) \\ &+ (Q_1^{G_1}+Q_1^{G_2})(I_{a^+}^{\beta}|u|)(x) + (Q_2^{G_1}+Q_2^{G_2})(x)\Big) + P_2(I_{a^+}^{\alpha-\beta}u)(x)\Big) \\ &\leqslant \lambda\Big(u_0 + \frac{\mathcal{K}|u_0|(x-a)^{\alpha}}{\Gamma(1+\alpha)} + P_1(Q_1^{G_1}+Q_1^{G_2})\frac{|u_0|(x-a)^{1+\alpha-\beta}}{\Gamma(\alpha-\beta+2)} \\ &+ P_1(Q_1^{G_1}+Q_1^{G_2})\frac{(x-a)^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)} + \frac{P_2(b-a)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \\ &+ (P_0^{G_1}+P_0^{G_2})\Big[(I_{a^+}^{\alpha-\beta}|u|)(x) + (I_{a^+}^{\alpha+1}|u|)(x)\Big]\Big) \\ &\leqslant \lambda A_0(x) + \lambda(P_0^{G_1}+P_0^{G_2})\Big[(I_{a^+}^{\alpha-\beta}|u|)(x) + (I_{a^+}^{\alpha+1}|u|)(x)\Big],\ x\in[a,b], \end{aligned}$$

where $P_0 = P_1 \max\{Q_1^{G_2}, 1\}$ and for $x \in [a,b]$

$$\begin{aligned} A_0(x) &= y_0 + \frac{\mathcal{K}|u_0|(x-a)^{\alpha}}{\Gamma(1+\alpha)} + P_1(Q_1^{G_1}+Q_1^{G_2})\frac{|u_0|(x-a)^{1+\alpha-\beta}}{\Gamma(\alpha-\beta+2)} \\ &+ P_1(Q_1^{G_1}+Q_1^{G_2})\frac{(x-a)^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)} + \frac{P_2(b-a)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}. \end{aligned}$$

Obviously $A_0(x)$ is non decreasing and since $\frac{P_2(b-a)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} > 0, A_0(x)$ is positive. Similarly we have the following inequality

$$|u(x)| \leqslant \lambda A_1(x) + \lambda(P_0^{G_1}+P_0^{G_2})\Big[(I_{a^+}^{\alpha-\beta}|u|)(x) + (I_{a^+}^{\alpha+1}|u|)(x)\Big],\ x\in[a,b],$$

where

$$\begin{aligned} A_1(x) &= \frac{\mathcal{K}|u_0|(x-a)^{\alpha}}{\Gamma(1+\alpha)} + P_1(Q_1^{G_1}+Q_1^{G_2})\frac{|y_0|(x-a)^{1+\alpha-\beta}}{\Gamma(\alpha-\beta+2)} \\ &+ P_1(Q_1^{G_1}+Q_1^{G_2})\frac{(x-a)^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)} + \frac{P_2(b-a)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \end{aligned}$$

for $x \in [a,b]$. Therefore, we obtain the following inequality

$$\begin{aligned} |u(x)| &\leqslant \lambda A(x) \\ &+ \lambda(P_0^{G_1}+P_0^{G_2})\Big[(I_{a^+}^{\alpha-\beta}|u|)(x) + (I_{a^+}^{\alpha+1}|u|)(x)\Big],\ x\in[a,b] \end{aligned} \tag{3.12}$$

where $A(x)$ is defined by (3.12).
Now if $\alpha-\beta \geqslant 1$, then by (3.12) we have

$$|u(x)| \leqslant \lambda A(x) + \lambda(P_0^{G_1}+P_0^{G_2})\Big[\frac{(b-a)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} + \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\Big]\int_a^x |u(s)|ds,\ x\in[a,b],$$

and finally by Theorem (2.1) we get

$$|u(x)| \leqslant \lambda A(x)\exp\Big((P_0^{G_1}+P_0^{G_2})\Big[\frac{(b-a)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} + \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\Big]x\Big),\ x\in[a,b]$$

For $0 < \alpha - \beta < 1$, implies the inequality (3.11). $\square$

4. Main results

In this section we prove the existence and uniqueness of solutions for Eq. (1.1). First we need the following assumptions:
(h_3) $f : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and there exists positive constant M_1 such that
$|f(t,x_1,y_1,z_1) - f(t,x_2,y_2,z_2)| \leqslant M_1(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|), x_i, y_i, z_i \in \mathbb{R}, i = 1,2,\ t \in [a,b]$
(h_4) $G_1, G_2 : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and there exists positive constant $M_2^{G_1}$, $M_2^{G_2}$ such that

$$|G_1(t,s,x) - G_1(t,s,y)| \leqslant M_2^{G_1}|x-y|,\ s,t \in [a,b], x,y \in R,$$
$$|G_2(t,s,x) - G_2(t,s,y)| \leqslant M_2^{G_2}|x-y|,\ s,t \in [a,b], x,y \in R.$$

Remark 4.1. If (h_3) and (h_4) hold, then

$$|f(t,x,y,z)| \leqslant M_1(|x| + |y| + |z|) + \bar{F},\ x,y,z \in \mathbb{R},\ t \in [a,b] \tag{4.1}$$

$$|G_1(t,s,x)| \leqslant M_2^{G_1}|x| + \bar{G},\ s,t \in [a,b],\ x \in \mathbb{R} \tag{4.2}$$

$$|G_2(t,s,x)| \leqslant M_2^{G_2}|x| + \bar{H},\ s,t \in [a,b],\ x \in \mathbb{R} \tag{4.3}$$

where

$$\bar{F} = \sup\{f(t,0,0,0),\ t \in [a,b]\}, \tag{4.4}$$

$$\bar{G} = \sup\{G_1(t,s,0),\ s,t \in [a,b]\}, \tag{4.5}$$

$$\bar{H} = \sup\{G_2(t,s,0),\ s,t \in [a,b]\}, \tag{4.6}$$

so (h_1) and (h_2) are satisfied.

Theorem 4.1. *Assume that the hypotheses (h_3) and (h_4) hold. Then Eq. (1.1) has at least one solution $u(x) \in C[a,b]$ which $({}^C D_{a^+}^{\alpha} u)(x) \in C[a,b]$.*

Proof. By Theorem 3.1, it is sufficient to show that (3.2) and (3.3) have a solution $u \in C[a,b]$

$$\begin{aligned}(Tu)(x) &= u_0 \\ &+ \Big(I_{a^+}^{\alpha-\beta}\Big[\mathcal{K}(I_{a^+}^{\beta}u)(s) + f\big(t, u(t), \int_a^t G_1\big(t,s,u(s) + (I_{a^+}^{\beta}u)(s)\big)ds, \\ &\quad \int_a^b G_2\big(t,s,u(s) + (I_{a^+}^{\beta}u)(s)\big)ds\big)\Big]\Big)(x)\end{aligned}$$

Also let

$$r_0 = A(b)exp\Big((P_0^{G_1} + P_0^{G_2})\Big[\frac{\mathcal{K}(b-a)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(b-a)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} + \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\Big]b\Big), \tag{4.7}$$

when $\alpha - \beta \geqslant 1$ and

$$\begin{aligned} r_1 &= A(b)\Big(1 + (P_0^{G_1} + P_0^{G_2})\Big[\frac{\mathcal{K}(b-a)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(b-a)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big]\Big) \\ &\times\ exp\Big((P_0^{G_1} + P_0^{G_2})\frac{(b-a)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}b\Big), \end{aligned} \tag{4.8}$$

when $0 < \alpha - \beta < 1$ where $A(x)$ is defined by (3.12) with

$$P_1 = M_1,\ P_2 = \sup\{f(t,0,0,0):\ t \in [a,b]\},$$

$$Q_1^{G_1} = M_2^{G_1},\ Q_2^{G_1} = \sup\{g(t,s,0):\ s,t \in [a,b]\},$$

$$Q_1^{G_2} = M_2^{G_2},\ Q_2^{G_2} = \sup\{g(t,s,0):\ s,t \in [a,b]\},$$

and $P_0^{G_1} = P_1 \max\{1, Q_1^{G_1}\}$, $P_0^{G_2} = P_1 \max\{1, Q_1^{G_2}\}$. Now we set

$$r = \begin{cases} r_1, & if\ 0 < \alpha - \beta < 1 \\ r_0, & if\ 1 \leqslant \alpha - \beta \end{cases} \tag{4.9}$$

$$B_r = \{u \in C[a,b],\ \|u\|_C < r\} \tag{4.10}$$

In the sequel, we show that $T : \overline{B}_r \to C[a,b]$ is a continuous and compact operator. By the continuity of f on $[a,b]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$, G_1, G_2 on $[a,b]\times\mathbb{R}\times\mathbb{R}$ and the operators $I_{a^+}^{\alpha-\beta}, I_{a+}^{\beta} on\ C[a,b]$, one can easily find that $Tu \in C[a,b]$ for $u \in \overline{B}_r$. Let $u, v \in \overline{B}_r$. Then using $(h_3), (h_4)$ and Lemma 2.2, part (a) we obtain

$$\begin{aligned} &\|(Tu)(x) - (Tv)(x)\| \\ &\leqslant \Big(I_{a^+}^{\alpha-\beta}\Big[\mathcal{K}I_{a^+}^{\beta}\|u(t)-v(t)\| + M_1\|u(t)-v(t)\| \\ &+(I_{a^+}^{\beta+1}M_2^{G_1}\|u(s)-v(s)\|)(t) + (I_{a^+}^{\beta+1}M_2^{G_2}\|u(s)-v(s)\|)(t)\Big]\Big)(x) \\ &\leqslant \|u-v\|_C\Big((\mathcal{K}I_{a^+}^{\alpha})(x) + (I_{a^+}^{\alpha-\beta}M_1)(x) + (I_{a^+}^{\beta+1}M_2^{G_1})(x) \\ &+(I_{a^+}^{\beta+1}M_2^{G_2})(x)\Big) \\ &\leqslant \|u-v\|_C\Big(\frac{\mathcal{K}(b-a)^{\alpha}}{\Gamma(\alpha+1)} + \frac{M_1(b-a)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{M_2^{G_1}(b-a)^{\alpha+1}}{\Gamma(\alpha+2)} \\ &+\frac{M_2^{G_2}(b-a)^{\alpha+1}}{\Gamma(\alpha+2)}\Big). \end{aligned}$$

Thus T is continuous on $\overline{B}_r$. Assume that

$$\mathcal{N}_0 = u_0 + \frac{r(b-a)^{\beta}}{\Gamma(\beta+1)} \tag{4.11}$$

$$l^{G_1} = (b-a)\sup\{|G_1(t,s,x)|,\ s,t \in [a,b],\ x \in [-N_0, N_0]\}, \tag{4.12}$$

$$l^{G_2} = (b-a)\sup\{|G_2(t,s,x)|,\ s,t \in [a,b],\ x \in [-N_0, N_0]\}, \tag{4.13}$$

$$M = \sup\{|f(t,x,y,z)|,\ t \in [a,b],\ x \in [-r,r],\ y \in [-l^{G_1}, l^{G_1}],\ z \in [-l^{G_2}, l^{G_2}]\}, \tag{4.14}$$

$$\begin{aligned}(Fu)(t) &= \mathcal{K}(I^{\beta}_{a^+}u)(s) + f\Big(t, u(t), \int_a^t G_1\big(t,s,u(s) + (I^{\beta}_{a^+}u)(s)\big)ds, \\ &\int_a^b G_2\big(t,s,u(s) + (I^{\beta}_{a^+}u)(s)\big)ds\Big).\end{aligned} \tag{4.15}$$

Since f and g are continuous, we find that N_0 and M are finite and $Fu \in C[a,b]$ for $u \in \overline{B}_r$. By (4.11) -(4.15), it is easy to see that $\|Fu\| \leqslant M$ for $u \in \overline{B}_r$. Then the set $\{Tu : u \in \overline{B}_r\}$ is uniformly bounded. Now let $x_1, x_2 \in [a,b]$. Without loss of generality we can assume that $x_1 \leqslant x_2$. For $u \in \overline{B}_r$, we have

$$\begin{aligned}&|(Tu)(x_1) - (Tu)(x_2)| \\ &\leqslant \frac{\mathcal{K}}{\Gamma(\alpha-\beta)}\Big|\int_{x_1}^{x_2} \frac{(Fu)(s)}{(x_2-s)^{1-\alpha+\beta}}ds\Big| + \frac{1}{\Gamma(\alpha-\beta)}\Big|\int_a^{x_1} \frac{(Fu)(s)}{(x_1-s)^{1-\alpha+\beta}}ds\Big| \\ &-\Big|\int_a^{x_1} \frac{(Fu)(s)}{(x_2-s)^{1-\alpha+\beta}}ds\Big| + \frac{1}{\Gamma(\alpha-\beta)}\Big|\int_{x_1}^{x_2} \frac{(Fu)(s)}{(x_2-s)^{1-\alpha+\beta}}ds\Big| \\ &\leqslant \frac{\mathcal{K}M(x_2-x_1)^{\alpha-\beta}}{(\alpha-\beta)\Gamma(\alpha-\beta)} + \frac{M[(x_2-x_1)^{\alpha-\beta} + (x_1-a)^{\alpha-\beta} - (x_2-a)^{\alpha-\beta})]}{(\alpha-\beta)\Gamma(\alpha-\beta)} \\ &+\frac{M[(x_2-x_1)^{\alpha-\beta}]}{(\alpha-\beta)\Gamma(\alpha-\beta)} \\ &\leqslant \frac{(\mathcal{K}+2)M(x_2-x_1)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}.\end{aligned}$$

Therefore, the set $E = \{Tu : u \in \overline{B}_r\}$ is equicontinuous. Then we proved that E is uniformly bounded and equicontinuous. Using Arzela-Ascoli theorem, we find that E is a relatively compact subset of $C[a,b]$ and hence $T : \overline{B}_r \to C[a,b]$ is a continuous and compact operator. Then T satisfies the assumptions of Theorem 2.2 . If we show that the equation $u = \lambda Tu$ doesn't have any solution in ∂B_r for $\lambda \in (a,b)$, then by Theorem 2.2, T has a fixed point in B_r and this shows that Eq. (3.2) has a solution $u \in C[a,b]$. To do this, let $u \in \partial B_r$ and $u = \lambda Tu$. By Remark 4.1, the hypotheses (h_3) and (h_4)imply the hypotheses $(h_1)$and(h_2), therefore the assumptions appeared in Theorem3.2, are satisfied. Also by Theorem 3.2 u satisfies the inequalities (3.11) when $\alpha - \beta \geqslant 1$ and (3.9) when $0 < \alpha - \beta < 1$. According to the definition of r in 4.9 we find that $\|u\|_C < r$ and then $u \notin \partial B_r$ the proof can be done similarly. □

Remark 4.2. In additions to the conditions h_3 and h_4 , if we have

$$\mu = \frac{\mathcal{K}(b-a)^{\alpha}}{\Gamma(\alpha+1)} + \frac{M_1(b-a)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{M_2^{G_1}(b-a)^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{M_2^{G_2}(b-a)^{\alpha+1}}{\Gamma(\alpha+2)} < 1, \tag{4.16}$$

then we can prove the existence and uniqueness of solutions for Eq. (1.1) by using Banach contraction principal.

THEOREM 4.2. *Let the hypotheses* (h_3), (h_4) *and the inequality* (4.16) *hold. Then Eq.* (1.1) *has a unique solution* $u \in C[a,b]$ *such that* $({}^C D^{\alpha}_{a^+} u)(x) \in C[a,b]$.

PROOF. Define the operator T on $C[a,b]$ as:

$$\begin{aligned}(Tu)(x) \quad = \quad & u_0 + \Big(I^{\alpha-\beta}_{a^+} \Big[\mathcal{K}(I^{\beta}_{a^+} u(s) + f\Big(t, u(t), \int_a^t G_1\big(t, s, u(s) + I^{\beta}_{a^+} u(s)\big) ds, \\ & \int_a^b G_2\big(t, s, u(s) + I^{\beta}_{a^+} u(s)\big) ds\Big)\Big]\Big)(x)\end{aligned}$$

The continuity of f on $[a,b] \times \mathbb{R} \times \mathbb{R}$, g on $[a,b] \times \mathbb{R} \times \mathbb{R}$ and the operators $I^{\alpha-\beta}_{a^+}, I^{\beta}_{a^+}$ on $C[a,b]$ imply that $Tu \in C[a,b] for\ any\ u \in C[a,b]$. Using (h_3) and (h_4) for any $u, v \in C[a,b]$ we have

$$\begin{aligned}& \|(Tu)(x) - (Tv)(x)\| \\ & \leqslant \Big(I^{\alpha-\beta}_{a^+} \Big[\mathcal{K} I^{\beta}_{a^+} \|u(t) - v(t)\| + M_1 \|u(t) - v(t)\| \\ & + (I^{\beta+1}_{a^+} M_2^{G_1} \|u(s) - v(s)\|)(t) + (I^{\beta+1}_{a^+} M_2^{G_2} \|u(s) - v(s)\|)(t) \Big] \Big)(x) \\ & \leqslant \|u - v\|_C \Big((\mathcal{K} I^{\alpha}_{a^+})(x) + (I^{\alpha-\beta}_{a^+} M_1)(x) + (I^{\beta+1}_{a^+} M_2^{G_1})(x) \\ & + (I^{\beta+1}_{a^+} M_2^{G_2})(x) \Big) \\ & \leqslant \mu \|u - v\|_C\end{aligned}$$

Thus T is a contraction operator. The Banach contraction principal implies the existence of a unique solution for the integral equation (3.2) in the space $C[a,b]$ has a unique continuous solution on $[a,b]$. Thus using Theorem 3.1 the proof is complete. □

5. Examples

Now we give two examples to illustrate the application of the obtained results.

EXAMPLE 5.1. *Consider the following fractional integro-differential equation*

$$\begin{aligned}(5.1)\quad ({}^C D^{\alpha}_{0^+} u)(x) \quad &= \quad \frac{1}{10} u(x) + \frac{x^4 [({}^C D^{\beta}_{0^+} u)(x)]^3}{e^x (1 + |({}^C D^{\beta}_{0^+} u)(x)|)} \\ & + \int_{0^+}^x \frac{x^2 s \sinh(u(s))}{1+s} ds + \int_{0^+}^1 \frac{xs \sin(u(s))}{1+s} ds, \ x \in [0,1] \\ u(0) \quad &= \quad 0,\end{aligned}$$

where $0 < \beta < \alpha < 1$. *Observe that Eq.*(5.1) *is a special case of Eq.* (1.1). *We deduce that there are satisfied assumptions* (h_3) *and* (h_4) *with* $M_1 = 2, M_2^{G_1} = M_2^{G_2} = 1$ *and* $\mathcal{K} = \frac{1}{10}$. *Then by using Theorem 4.1, the Eq.*(5.1) *has at least one solution* $u(x) \in C[0,1]$ *which* $({}^C D^{\alpha}_{0^+} u)(x) \in C[0,1]$.

EXAMPLE 5.2. *Consider the following fractional integro-differential equation*

$$(5.2)\quad ({}^{C}D_{0^+}^{0.75}u)(x) = \frac{1}{11}u(x) + \frac{x}{4}\sin[({}^{C}D_{0^+}^{0.25}u)(x)] + \frac{1}{2}\int_{0^+}^{x}\frac{x^2 s\sin(u(s))}{s^2+4}ds + \int_{0^+}^{1}\frac{x^2 su(s)}{2s^2+5}ds,\ x\in[0,1],$$
$$u(0) = 0.$$

Notice that the equation (5.2) *can be treated as a special case of Eq.* (1.1)*, we put*

$$\alpha = 0.75,\ \beta = 0.25,\ \mathcal{K} = \frac{1}{11},\ [a,b] = [0,1]$$

Similar to Example 5.1, it is easy to see that the functions f and g satisfy conditions (h_3) *and* (h_4) *with* $M_1 = \frac{1}{2}, M_2^{G_1} = \frac{1}{5}$ *and* $M_2^{G_2} = \frac{1}{7}$*. Also in this example, the value of defined by* (4.16)*, is calculated by*

$$\mu = \frac{\mathcal{K}(b-a)^{\alpha}}{\Gamma(\alpha+1)} + \frac{M_1(b-a)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{M_2^{G_1}(b-a)^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{M_2^{G_2}(b-a)^{\alpha+1}}{\Gamma(\alpha+2)}$$
$$= \frac{1}{11} + \frac{1}{2\Gamma(1.5)} + \frac{1}{5\Gamma(2.75)} + \frac{1}{7\Gamma(2.75)} \cong 0.87 < 1$$

Now applying Theorem 4.2, we infer that Eq. (5.2) *has a unique solution* $u(x) \in C[0,1]$ *such that* $({}^{C}D_{0^+}^{0.75}u)(x) \in C[0,1]$.

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