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A NOTE ON MULTI-VALUED MAPPINGS

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Abstract. We introduce some notions for multi-valued mappings such as m-act, transitively mG-space, m-orbit, and m-stabilizer. Some algebraic topological properties are given for multivalued functions.

1. Introduction

When we look at the past periods, it has been seen that single-valued mappings are insufficient to reach some desired results. For this reason, a new concept called multi-valued mappings have been introduced. It is possible to think of multi-valued mappings as generalizations of single-valued mappings. Multi-valued mappings are associated to a point in the domain to a set in the image. Today, it is possible to see studies on these mappings in various areas such as statistics, economy, optimal control theory and calculus of variation.

One of the studies on multi-valued mappings is about continuity. The reason for this is that the concept of continuity is one of the most significant topics in the field of algebraic topology. The definition of continuity of multi-valued mappings is introduced for specific cases by [**7**], [**8**], and [**26**]. All of these definitions separately generalize the concept of continuity to multi-valued mappings and are equivalent to each other. Kuratowski [**13**] revealed the notions of lower semi-continuity and upper semi-continuity in his work. In the following years, studies on the continuity of multi-valued functions continued. In his article, Strother [**24**] examined the relationship between the definitions of continuity that emerged as a result of these studies.

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Many subjects have been studied using multi-valued mappings. Kakutani's article [**9**] is one of the most significant. In this article, Brouwer Fixed Point Theorem has been studied. Michael [**14**] created a topology using non-empty closed subsets. With the help of this topology, Michael managed to generalize many properties valid for single-valued mappings. Strother [**22**] has studied on the fixed point theorem and given definitions of fixed point and trace for multi-valued mappings. Also, the notion of homotopy been generalized to multi-valued mappings and is called multi-homotopy. The most important article about multi-homotopy is the article of Strother [**23**]. In this article, the concept of multi-homotopy is defined and some of its properties are given. In addition, Ponomarev [**16**] has studied on the fundamental properties of multi-valued mappings and many applications on the topological space he defined on closed sets. In another work, he [**17**] discovered properties that are preserved under multi-valued mappings. Also, Rhee [**19**] has generalized some theorems from homotopy theory to multi-valued mappings. Ozkan and Karaca [**11**] have introduced a notion of multi-category and a notion homeomorphism using the multi category.

In the first part of this article, some definitions and properties used in proofs are given. We have introduced some notions such as m-act, mG-space, transtive mG-space, m-orbit, and m-stabilizer. Also, we give some new properties about these notions. In short, our goal in this article is to generalize the properties about single-valued mappings to multi-valued mappings.

2. Preliminaries

All of the topological spaces used in this article are Hausdorff topological spaces and Greek alphabet letters are used to represent multi-valued mappings. Assume that P is a non-empty space and Q is any space. If $\alpha(p)$ is a subset of Q for every element $p \in P$, then $\alpha : P \rightrightarrows Q$ is said to be a multi-valued mapping. It is possible to say that single-valued mappings are a special case of multi-valued mappings. Also, considering that $\alpha : P \rightrightarrows Q$ maps $p \in P$ to the set in $\mathcal{P}(Q)$, it can be said that α behaves like a single-valued mapping. $R(\alpha) = \bigcup_{p\in P} \alpha(p)$ is the *range* of a multi-valued mapping $\alpha : P \rightrightarrows Q$ by [**2**]. Also, we get

$$\alpha(P_0) = \bigcup_{p\in P_0}$$

for each $P_0 \subset P$ from [**3**]. If $\alpha(p) \cap \alpha(p^{'}) = \emptyset$ when $p \neq p^{'}$ for any $p, p^{'} \in P$, α is called *injective (one-to-one)* from [**2**]. If $R(\alpha) = Q$, α is called *surjective (onto)*. If $\alpha(p)$ is closed (open) in Q for each $p \in P$, α becomes *closed valued* (*open valued*) by [**5**]. It is the *composition* of $\alpha : P \rightrightarrows Q$ and $\beta : Q \rightrightarrows R$ represented as $\beta \circ \alpha : P \rightrightarrows R$ and, is defined as $(\beta \circ \alpha)(p) = \beta(\alpha(p)) = \bigcup\limits_{q\in\alpha(p)} \beta(q)$.

PROPOSITION 2.1. *Let P and Q be two spaces. If $\alpha : P \rightrightarrows Q$, $p \mapsto \alpha(p)$ is an injective multi-valued mapping, then $f : P \to \mathcal{P}(Q)$, $p \mapsto \alpha(p)$ is one-to-one single-valued mapping.*

PROOF. If α is an one-to-one multi-valued mapping, then for all $p_1, p_2 \in P$ such that $p_1 \neq p_2$, $\alpha(p_1) \cap \alpha(p_2) = \emptyset$. Since $f(p_1) = \alpha(p_1)$ and $f(p_2) = \alpha(p_2)$, $f(p_1) \neq f(p_2)$. Thus f is one-to-one single-valued mapping. $\square$

The next lemma was created by Strother [**23**] on the continuous of multi-valued mappings.

LEMMA 2.1. *Let $\alpha : P \rightrightarrows Q$ be a multi-valued mapping and $\alpha(p)$ be closed in Q for each $p \in P$. α is continuous if and only if the following situations are provided.*

(i) *There exists an open set U of P with $p_0 \in U$ such that $\alpha(p) \cap V \neq \emptyset$ for all $p \in U$, when $p_0 \in P$, V is open in Q and $\alpha(p_0) \cap V \neq \emptyset$.*

(ii) *There exists an open set U with $p_0 \in U$ such that $\alpha(p) \subset V$ for all $p \in U$, when $p_0 \in P$, V is open in Q and $\alpha(p_0) \subset V$.*

The first case in the above Lemma 2.1 is called *the lower semi-continuity*, and the second case is called *the upper semi-continuity.* Let P and Q are compact topological spaces and $\alpha : P \rightrightarrows Q$ is a continuous multi-valued mapping. Then α is closed [**21**]

LEMMA 2.2. [**21**] *If $\alpha : P \rightrightarrows Q$ and $\beta : Q \rightrightarrows R$ are continuous and P, Q, and R are compact, then $\beta \circ \alpha$ is continuous.*

Assume that $\alpha : P \rightrightarrows Q$ is a multi-valued mapping. If $P_0 \subset P$, then $\alpha|_{P_0}$ is defined by $\alpha|_{P_0}(p) = \alpha(p)$ for all $p \in P_0$ and called *the restriction of α to P_0*. Also $\alpha|_{P_0}$ is continuous, when α is continuous [**21**]. The multi-valued mapping $\mathrm{id}_P : P \rightrightarrows P$, $\mathrm{id}_P(p) = \{p\}$ is said to be the *identical multi-valued mapping* of the set P. A mapping $\alpha : P \rightrightarrows Q$ is called *constant mutli-valued mapping* if $\alpha(p) = Q_0$, for all $p \in P$, where Q_0 is a fixed closed subset of Q [**2**].

Let $\alpha : P \rightrightarrows Q$ be a mapping. For any $Q_0 \subset Q$,

$$\begin{aligned} \alpha^-(Q_0) &= \{p \in P \mid \alpha(p) \cap Q_0 \neq \emptyset\} \\ \alpha^+(Q_0) &= \{p \in P \mid \alpha(p) \subset Q_0\} \end{aligned}$$

from [**5**]. We can defined two multi-valued function. These are $\alpha^- : Q \rightrightarrows P$, $q \mapsto \alpha^-(q)$ and $\alpha^+ : Q \rightrightarrows P$, $q \mapsto \alpha^+(q)$. α^- will be considered the inverse of a multi-valued mapping[**3**]. Also, $\alpha^-(q)$ is called the *m-fiber* of q for each $q \in Q$.

PROPOSITION 2.2. [**5**] *Assume that P and Q are topological spaces, $\alpha : P \rightrightarrows Q$ is a multi-valued map, and $Dom(\alpha) = P$. In this case, the situations given below are equivalent to each other.*

(i) *α is upper semi-continuous;*

(ii) *For each open set $V \subset Q$, $\alpha^+(V) \subset P$ is open;*

(iii) *For each closed set $K \subset Q$, $\alpha^-(K) \subset P$ is closed.*

PROPOSITION 2.3. [**5**] *Assume that P and Q are topological spaces, $\alpha : P \rightrightarrows Q$ is a multi-valued map, and $Dom(\alpha) = P$. In this case, the situations given below are equivalent to each other.*

(i) *α is lower semi-continuous;*

(ii) *For each open set $V \subset Q$, $\alpha^-(V) \subset P$ is open;*

(*iii*) *For each closed set $K \subset Q$, $\alpha^+(K) \subset P$ is closed.*

In the next lemmas, it is mentioned under which conditions the composition of multi-valued mapings α, α^- and α^+ will give the identical multi-valued mapping.

Lemma 2.3. [**25**] *If $\alpha : P \rightrightarrows Q$ is one-to-one, $\alpha^- \circ \alpha = id_P$.*

Lemma 2.4. [**12**] *If $\alpha : P \rightrightarrows Q$ is onto, for all $p \in P$ $\alpha(p) \neq \emptyset$ and there is $p \in P$ such that $\alpha(p) \subset \{q\}$ for all $q \in Q$, then $\alpha \circ \alpha^+ = id_B$.*

Now, we give the definition of m-homeomorphism.

Definition 2.1. [**10**] *Let P and Q be two topological spaces. Let $\alpha : P \rightrightarrows Q$ be injective and surjective. α is called an m-*homeomorphism *when α and α^- are continuous.*

Lemma 2.5. [**10**] *Let P, Q, and R be compact topological spaces. If $\alpha : P \rightrightarrows Q$ and $\beta : Q \rightrightarrows R$ are m-homeomorphisms, then $\beta \circ \alpha : P \rightrightarrows R$ is an m-homeomorphism.*

3. Generalizing some properties in algebraic topology to multi-valued mappings

In this part of the article, the definitions and properties in the field of algebraic topology will be generalized to multi-valued functions. Firstly, we will give some definitions.

Definition 3.1. [**23**] *Let P and Q be topological spaces. Let $\alpha : P \rightrightarrows Q$ and $\beta : P \rightrightarrows Q$ are multi-valued mappings. If there exists $\gamma : P \times I \rightrightarrows Q$ such that, $\gamma(p, 0) = \alpha(p)$, $\gamma(p, 1) = \beta(p)$ and it is continuous, we can say that α is m-homotopic to β. If two multi-valued mappings α and β are m-homotopic to each other, it is denoted by $\alpha \simeq_m \beta$.*

If $\mu : I \rightrightarrows P$ is continuous, then it is called an *m-path* for any topological space P. If P_0 is a maximal m-pathwise connected subset of P, then it is called an *m-pathwise component of P*. We say that α is *null m-homotopic*, if α is m-homotopic to a constant multi-valued mapping. Let I^n be a product of unit intervals. We define

$$MQ(n, P, P_0) = \{\alpha : I^n \rightrightarrows P \mid \alpha \text{ is continuous and } \alpha(t) = P_0 \text{ for all } t \in B^{n-1}\}$$

for each topological space P, $P_0 \subset P$ closed and, $n \in \mathbb{N}^+$[**23**].

The subset B^{n-1} of I^n consists of points $(p_1, ..., p_n)$. In this situation, some of the coordinates is zero or one and. Moreover, we can say that B^{n-1} is the boundary of I^n.

Theorem 3.1. [**10**] *The m-homotopy relation is an equivalence relation.*

Proposition 3.1. *Let P and Q be topological spaces. Assume that $\alpha : P \rightrightarrows Q$, $p \mapsto Q_0$ and $\beta : P \rightrightarrows Q$, $p \mapsto Q_1$ are constant multi-valued mappings such that $Q_0, Q_1 \subset Q$ are closed. If $\alpha \simeq_m \beta$, then there is an m-path between Q_0 and Q_1*

PROOF. If $\alpha \simeq_m \beta$, then there is an m-homotopy mapping $\gamma : P \times I \rightrightarrows Q$ such that $\gamma(p,0) = \alpha(p)$ and $\gamma(p,1) = \beta(p)$. Since γ is continuous, $\gamma|_{\{p_0\}\times I}$ is continuous for any $p_0 \in P$. So,

$$\begin{aligned} \gamma|_{\{p_0\}\times I}(p_0,0) &= \alpha(p_0) = Q_0, \\ \gamma|_{\{p_0\}\times I}(p_0,1) &= \alpha(p_0) = Q_1. \end{aligned}$$

Consequently, $\gamma|_{\{p_0\}\times I}$ is an m-path between Q_0 and Q_1. □

DEFINITION 3.2. [**23**] *Let α and $\beta \in MQ(n, P, P_0)$. Define $\gamma = \alpha * \beta$ by*

$$\gamma(t_1, ..., t_n) = \begin{cases} \alpha(2t_1, ..., t_n) & \text{if } 0 \leqslant t_1 \leqslant \frac{1}{2} \\ \beta(2t_1 - 1, ..., t_n) & \text{if } \frac{1}{2} \leqslant t_1 \leqslant 1. \end{cases}$$

$[\alpha]$ is a class of mappings such that mappings in this class are homotopic to mapping α relative to (B^{n-1}, P_0). Also, $[\alpha] * [\beta] = [\alpha * \beta]$. Moreover, γ is well-defined and continuous.

PROPOSITION 3.2. *Let P_0, P_1 and, P_2 be closed subsets of a topological sapce P. If $\alpha : I \rightrightarrows P$ is an m-path from P_0 to P_1 and $\beta : I \rightrightarrows P$ is an m-path from P_1 to P_2, then $\overline{\alpha * \beta} = \overline{\beta} * \overline{\alpha}$.*

PROOF. $\overline{\alpha * \beta}(t) = \alpha * \beta(1-t)$. Then,

$$\begin{aligned} \alpha * \beta(1-t) &= \begin{cases} \alpha(2(1-t)) & \text{if } 0 \leqslant 1-t \leqslant \frac{1}{2} \\ \beta(2(1-t)-1) & \text{if } \frac{1}{2} \leqslant 1-t \leqslant 1, \end{cases} \\ &= \begin{cases} \alpha(2-2t) & \text{if } \frac{1}{2} \leqslant t \leqslant 1 \\ \beta(1-2t) & \text{if } 0 \leqslant t \leqslant \frac{1}{2}. \end{cases} \end{aligned}$$

Similarly, $(\overline{\beta} * \overline{\alpha})(t) = \overline{\beta}(t) * \overline{\alpha}(t) = \beta(1-t) * \alpha(1-t)$. Then,

$$\begin{aligned} \beta(1-t) * \alpha(1-t) &= \begin{cases} \beta(1-2t) & \text{if } 0 \leqslant t \leqslant \frac{1}{2} \\ \alpha(1-2t-1) & \text{if } \frac{1}{2} \leqslant t \leqslant 1, \end{cases} \\ &= \begin{cases} \beta(1-2t) & \text{if } 0 \leqslant t \leqslant \frac{1}{2} \\ \alpha(2-2t) & \text{if } \frac{1}{2} \leqslant t \leqslant 1. \end{cases} \end{aligned}$$

Consequently, $\overline{\alpha * \beta} = \overline{\beta} * \overline{\alpha}$. □

PROPOSITION 3.3. *Let P and Q be compact topological spaces. Assume that $P_0 \subset P$ is closed and $\sigma_P : P \rightrightarrows P$, $p \mapsto P_0$ is a constant multi-valued mapping. If $id_P \simeq_m \sigma_P$ and Q is an m-pathwise connected topological space, then any two mappings from P to Q is m-homotopic.*

PROOF. If $id_P \simeq_m \sigma_X$, then there is an m-homotopy mapping $\gamma : P \times I \rightrightarrows Q$ such that $\gamma(p,0) = id_P(p)$ and $\gamma(p,1) = \sigma_P(p)$. Let α be a continuous multi-valued mapping. Then, $\alpha \circ \sigma : P \times I \rightrightarrows Q$ is continuous and

$$\begin{aligned} \alpha \circ \gamma(p,0) &= \alpha(\gamma(p,0)) = \alpha(id_P(p)) = \alpha(p), \\ \alpha \circ \gamma(p,1) &= \alpha(\gamma(p,1)) = \alpha(\sigma_P(p)) = \alpha(P_0) = Q_0 = \sigma_{Q_0}(p). \end{aligned}$$

Namely, α is null m-homotopic. Given that $Q_1 \subset Q$ is closed. Since Q is an m-pathwise connected topological space, there is an m-path $\mu : I \rightrightarrows Q$ such that $\mu(0) = Q_0$ and $\mu(1) = Q_1$. We can define that $\theta : P \times I \rightrightarrows Q$, $(p,t) \mapsto \mu(t)$. θ is continuous, because of continuity of μ. Since

$$\begin{aligned} \theta(p,0) &= \mu(0) = Q_0 = \sigma_{Q_0}(P) \quad \text{and} \\ \theta(p,1) &= \mu(1) = Q_1 = \sigma_{Q_1}(P), \end{aligned}$$

two constant multi-valued mappings are m-homotopic. We know that any multi-valued mapping is m-homotopic to constant multi-valued mapping. So, two multi-valued mappings from P to Q is m-homotopic. □

Definition 3.3. [**25**] *Assume that $\alpha : P \rightrightarrows Q$ is a continuous multi-valued mapping and P and Q are topological spaces. Assume that U is an open subset in Q. If $\alpha^-(U)$ is an union of D_i in P, D_i's are disjoint sets and, $\alpha|_{D_i} : D_i \rightrightarrows U$ is an m-homeomorphism for every i, then U is called an* evenly m-covered by α. *In this situation, D_i is called* m-sheets.

Definition 3.4. [**25**] *Assume that $\alpha : P \rightrightarrows Q$ is continuous, P is an m-pathwise connected space and Q is a topological space. If $U = U_q$ is an open neighborhood q and it is evenly m-covered by α, then for every $q \in Q$, then (P, α) is an* m-covering space *of Q. In this situation, α is said to be an* m-covering projection*, and $U = U_q$ is said to be an* m-admissible.

Lemma 3.1. *Let (P, α) be an m-covering space of Q, let R be a connected space, and let $\beta : (R, \{r_0\}) \rightrightarrows (Q, Q_0)$ be a continuous multi-valued mapping. Given p_0 in the m-fiber over Q_0, there is at most one continuous $\tilde{\beta} : (R, \{r_0\}) \rightrightarrows (P, P_0)$ with $\alpha \circ \tilde{\beta} = \beta$.*

Proof. Assume that there exists a continuous mapping

$$\beta' : (R, \{r_0\}) \rightrightarrows (P, P_0)$$

such that $\alpha \circ \beta' = \beta$. Let

$$\begin{aligned} A &= \{r \in R \mid \tilde{\beta}(r) = \beta'(r)\} \quad \text{and} \\ B &= \{r \in R \mid \tilde{\beta}(r) \neq \beta'(r)\}. \end{aligned}$$

In this situation, $R = A \cup B$ and $A \cap B = \emptyset$. Let $a \in A$ and U be an m-admissible neighborhood of $\beta(a)$. Let S be an m-sheet of U such that $\tilde{\beta}(a) = \beta'(a)$. $W = \tilde{\beta}^+(S) \cap F'^+(S) \subset R$ is an open neighborhood of a. If $w \in W$, then $\tilde{\beta}(w) \subset S$ and $F'(w) \subset S$. Since $\alpha \circ \tilde{\beta} = \beta$ and $\alpha \circ \beta' = \beta$, $\alpha \circ \tilde{\beta}(w) = \alpha \circ \beta'(w)$. Also, we know that, $\alpha|_S$ is an m-homeomorphism. Thus,

$$(\alpha|_S)^- \circ (\alpha|_S \circ \tilde{\beta}(w)) = (\alpha|_S)^- \circ (\alpha|_S \circ \beta'(w)) \text{ and } \tilde{\alpha}(w) = F'(w).$$

Namely, $w \in A$. So, A is open. Let $b \in B$ and V be m-admissible neighborhood of $\beta(b)$. Let S and S' be m-sheets of V such that $\tilde{\beta}(b) \subset S$ and $\beta'(b) \subset S'$. $W = \tilde{\beta}^+(S) \cap \beta'^+(S') \subset R$ is an open neighborhood of b. If $w \in W$, then $\tilde{\beta}(w) \subset S$ and $\beta'(w) \subset S'$. Since $S \cap S' = \emptyset$, $\tilde{\beta}(w) \neq \beta'(w)$. Namely, $w \in B$. So, B is open.

Since A and B are open subsets, R is connected and $r_0 \in A$, $B = \emptyset$. Consequently, we obtain $\tilde{\beta} = \beta^{'}$. □

Definition 3.5. *Let G be a group and let Q be a topological space. Then G* m-acts *on Q if there is a continuous multi-valued mapping $\alpha : G \times Q \rightrightarrows Q$, denoted by $(g, q) \mapsto \alpha(g, q)$, such that $\alpha((g * g^{'}), q) = \alpha(g, \alpha(g^{'}, q))$ and $\alpha(1, q) = \{q\}$ for all $q \in Q$ and $g, g^{'} \in G$ (here 1 is the indentity element of G).*

We call Q an *mG-space* if G m-acts on Q. One says G *m-acts transitively* on Q if, for each $q, q^{'} \in Q$, there exists $g \in G$ with $\alpha(g, q) = \{q^{'}\}$; call Q a *transitive mG-space* in this case.

Definition 3.6. *Let G m-act on a space Q, and let $q \in Q$. Then the* m-orbit *of q is*

$$m - O(q) = \{\alpha(g, q) \mid g \in G\} \subset \mathcal{P}(Q),$$

and the m-stabilizer *of q is*

$$m - G_q = \{g \in G \mid \alpha(g, q) = \{q\}\}.$$

Lemma 3.2. *If G m-acts transitively on a space Q, then $\bigcup\limits_{A \in m - O(q)} A = Q$ for every $q \in Q$.*

Proof. Assume that G m-acts transitively on a space Q. Then, there is $g \in G$ and a continuous mapping $\alpha : G \times Q \rightrightarrows Q$ such that $\alpha(g, q) = \{q^{'}\}$ for all $q, q^{'} \in Q$. In this case,

$$\begin{aligned} m - O(q) &= \{\alpha(g, q) \mid g \in G\} \\ &= \{\{q^{'}\} \mid q^{'} \in Q\} \cup \{\alpha(g, q) \mid g \in G \text{ and } m(\alpha(g, q)) \neq 1\}. \end{aligned}$$

Consequently, we obtain $\bigcup\limits_{A \in m - o(q)} A = Q$. □

Lemma 3.3. *Assume that a group G m-acts on a topological space Q and $q \in Q$.*

$$\begin{aligned} \phi : m - O(q) &\rightrightarrows G/m - Gq \\ \alpha(g, q) &\mapsto g * m - Gq \end{aligned}$$

is one-to-one.

Proof. For all $\alpha(g_1, q), F(g_2, q) \in m - O(q)$ such that $\alpha(g_1, q) \neq \alpha(g_2, q)$,

$$\begin{aligned} \phi(\alpha(g_1, q)) &= g_1 * m - Gq = \{g_1 * h \mid h \in m - Gq\} \\ \phi(\alpha(g_2, q)) &= g_2 * m - Gq = \{g_2 * h \mid h \in m - Gq\}. \end{aligned}$$

Let $k \in \phi(\alpha(g_1, q)) \cap \phi(\alpha(g_2, q))$. There are $h_1, h_2 \in m - Gq$ such that $k = g_1 * h_1$ and $k = g_2 * h_2$. So, $\alpha(k, q) = \alpha((g_1 * h_1), q)$ and $\alpha(k, q) = \alpha((g_2 * h_2), q)$. In this case,

$$\begin{aligned} \alpha((g_1 * h_1), q) &= \alpha((g_2 * h_2), q) \\ \alpha(g_1, F(h_1, q)) &= \alpha(g_2, \alpha(h_2, q)) \\ \alpha(g_1, \{q\}) &= \alpha(g_2, \{q\}). \end{aligned}$$

It is a contradiction. So, $\phi(\alpha(g_1, q)) \cap \phi(\alpha(g_2, q)) = \emptyset$. Thus, ϕ is one-to-one. □

Proposition 3.4. *Assume that a group G m-acts on topological space Q and $q \in Q$.*

$$\begin{array}{rcl} \phi : m - O(q) & \rightarrow & G/m - Gq \\ \alpha(g,q) & \mapsto & g * m - Gq \end{array}$$

is a surjective single-valued mapping.

Proof. For all $g * m - Gq$, there is $\alpha(g,q) \in m - O(q)$ such that $\phi(\alpha(g,q)) = g * m - Gq$. So, ϕ is a surjective single-valued mapping. □

Lemma 3.4. *Assume that a group G m-acts on a topological space Q and $q \in Q$. In this case, $|m - O(q)| = [G : m - Gq]$.*

Proof.

$$\begin{array}{rcl} \phi : m - O(q) & \rightarrow & G/m - Gq \\ \alpha(g,q) & \mapsto & g * m - Gq \end{array}$$

is an one-to-one and surjective single-valued mapping. Thus,

$$|m - O(q)| = [G : m - Gq].$$

□

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