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SOFT $\mathcal{F}-$ METRIC SPACES AND THEIR FIXED POINTS

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Abstract. In this paper, we proposed the notion of soft $\mathcal{F}-$ metric spaces which generalizes the soft metric space concept. Also, we give the topology generated by the soft $\mathcal{F}-$ metric. Furthermore, we introduce a new style of the Banach contraction principle in these spaces. Additionally, some examples are presented to make our work more comprehensive.

1. Introduction

Soft set theory was introduced by [**22**]. He applied this theory to some areas for example medical and social science, economi etc. In the following years, many authors have work about soft set and its applications. [**18, 19**] applied the soft set theory to decision-making problems. Some researchers have worked in this theory and have gived the soft set topologies [**6, 7, 11, 15, 16, 21, 23**]. Samanta and Das defined the notion of soft element, soft real and complex numbers on this sets [**12**]. Samanta et al. studied the concepts of soft norm, soft metric etc.. Also Samanta et al. introduced contraction theorem [**20**]. Furthermore, many authors worked on the soft set theory and its applications [**3, 4, 5, 10, 13, 14, 17, 24**].

On the other hand, many researchers have gived a lot of interesting extensions of the metric space for example the notions of soft metric, $\mathcal{F}-$ metric etc.. Furthermore the authors have also given new studies about some fixed point theorems using different contractions in this various generalized metric spaces [**1, 2, 8, 9**].

In this study, we define the concept of soft $\mathcal{F}-$metric space and give the topology generated by the soft $\mathcal{F}-$metric. Also, we present a new style of the Banach contraction principle in this metric space and some examples are gived. Assume

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that F is an initial universe set, A is a parameters set, $P(F)$ is the family of all subsets of F and $E \subseteq A$. Firstly, we give some definitions.

Definition 1.1 ([**22**]). *The pair (H, E) called as a soft set on F where H is a mapping given by $H : E \to P(F)$.*

Definition 1.2 ([**19**]). *Let (H, E) be a soft set on F. If $H(\beta) = \emptyset$, for all $\beta \in E$, (H, E) is defined as empty soft set and denoted with $\widetilde{\Phi}$. If $H(\beta) = F$ for all $\beta \in E$, (H, E) is defined as universal (absolute) soft set denoted with $\widetilde{H}$.*

Remark 1.1. In this study, the soft sets $\widetilde{\Phi}$ and (H, E) on F, for every $\beta \in E$ such that $H(\beta) \neq \emptyset$ will be discussed. Also, $S(\widetilde{F})$ and $SS(\widetilde{F})$ denote the class of soft sets and soft subsets of F, respectively.

Definition 1.3 ([**18**]). *Let (H, E) and (G, D) are soft sets on F.*

- i *If $H(\beta) \subseteq G(\beta)$, for all $\beta \in E$ and $E \subseteq D$, then (H, E) is a soft subset of (G, D) and denoted by $(H, E) \widetilde{\subseteq} (G, D)$.*
- ii *The soft sets (G, D) and (H, E) on F are called soft equal if $(H, E) \widetilde{\subseteq} (G, D)$ and $(G, D) \widetilde{\subseteq} (H, E)$.*
- iii *If, where $C = E \cup D$ and for all $\beta \in C$,*
$$K(\beta) = \begin{cases} H(\beta) & , \quad \beta \in E \setminus D \\ G(\beta) & , \quad \beta \in D \setminus E \\ H(\beta) \cup G(\beta) & , \quad \beta \in E \cap D \end{cases}$$
then, the union of two soft sets of $(H, E), (G, D)$ is the soft set (K, C) and denoted by $(H, E) \cup (G, D) = (K, C)$.
- iv *Where $C = E \cap D$ and for all $\beta \in C$, $K(\beta) = H(\beta) \cap G(\beta)$ then, the intersection of two soft sets of $(H, E), (G, D)$ is the soft set (K, C) and denoted by $(H, E) \cap (G, D) = (K, C)$.*

Definition 1.4 ([**20**]). *The complement of a soft set (H, E) is denoted by $(H, E)^c = (H^c, E)$, where $H^c : E \to P(F)$ is a mapping given by $H^c(\beta) = F \setminus H(\beta)$.*

Definition 1.5 ([**17**]). *Let $F \neq \emptyset$ be a set. Then, for each $\beta \in F$ and $E \subset F$, we define a function as $h_\beta^E : \beta \to P(F)$, $\beta \to E$. Each function h_β^E is called a fixed point function of the pair (β, E). The set $(\beta, h_\beta^E(\beta)) = (\beta, E)$ is defined as a soft single point set on F. Also, for all $\beta \in F$, the set $(\beta, \emptyset)$ is called an empty soft single point set. The set of all soft single points is denoted by S_F and is defined as $S_F = \{(\beta, h_\beta^E(\beta)) : \beta \in F, E \subset F\}$.*

Remark 1.2. Each function h_β^E is unique and different for each $\beta \in F, E \subset F$ and consists of one element (β, E) (see [**17**]).

Definition 1.6 ([**17**]). *Let R be the set of real numbers. For all $\alpha \in R$ and $E \subseteq R$,*
$$h_\alpha^E : \underset{\alpha}{\{\alpha\}} \underset{\to}{\to} \underset{E}{P(R)}.$$

Then, the set $\widetilde{R} = \{(\alpha, E) : \alpha \in R, E \subseteq R\}$ *is called the set of soft real points. Similarly the set* $\widetilde{R^+} = \{(\alpha, E) : \alpha \in R^+, E \subseteq R\}$ *and* $\widetilde{R^-} = \{(\alpha, E) : \alpha \in R^-, E \subseteq R\}$ *is called the set of positive soft real points and negative soft real points, respectively.*

EXAMPLE 1.1. $\{(1,(0,1))\}$, $\{\frac{1}{3},(4,\infty)\}$, $\{(\sqrt{3},\{0\})\}$ *are the examples of soft real points (see* [**17**]*).*

DEFINITION 1.7 ([**17**]). *If* $\alpha = \alpha^{'}$ *and* $E = E^{'}$ *for* $\alpha, \alpha^{'} \in R$ *and* $E, E^{'} \subset R$ *the soft real points* (α, E) *and* $(\alpha^{'}, E^{'})$ *are equal and denoted by* $(\alpha, E) \cong (\alpha^{'}, E^{'})$*.*

DEFINITION 1.8 ([**17**]). *For* $\alpha, \alpha^{'} \in R$ *and* $E, E^{'} \subset R$*,*

- i *If* $\{(\alpha, E)\} \widetilde{+} \{(\alpha^{'}, E^{'})\} \cong \{(\alpha + \alpha^{'}, E \cup E^{'})\}$ *then the operation* "$\widetilde{+}$" *is defined the common sum of two soft real points* (α, E) *and* $(\alpha^{'}, E^{'})$*.*
- ii *If* $\{(\alpha, E)\} \widetilde{-} \{(\alpha^{'}, E^{'})\} \cong \{(\alpha - \alpha^{'}, E \cup E^{'})\}$ *then the operation* "$\widetilde{-}$" *is defined the common difference of two soft real points* (α, E) *and* $(\alpha^{'}, E^{'})$*.*
- iii *If* $\{(\alpha, E)\} \widetilde{.} \{(\alpha^{'}, E^{'})\} \cong \{(\alpha . \alpha^{'}, E \cup E^{'})\}$ *then the operation* "$\widetilde{.}$" *is defined the common product of two soft real points* (α, E) *and* $(\alpha^{'}, E^{'})$*.*

DEFINITION 1.9 ([**17**]). *For* (α, E), $(\alpha^{'}, E^{'}) \in \widetilde{R}$ *an exponential soft real point is defined by* $\{(\alpha, E)\}^{\{(\alpha^{'}, E^{'})\}} \cong \{(\alpha^{\alpha^{'}}, E \cup E^{'})\}$*.*

DEFINITION 1.10 ([**17**]). *For* (α, E), $(\alpha^{'}, E^{'}) \in \widetilde{R}$ *with* $\alpha < \alpha^{'}$ *and* $E \subset E^{'}$*. Then, the soft real point* $\{(\alpha, E)\}$ *is a soft real point less than a soft real point* $\{(\alpha^{'}, E^{'})\}$ *is denoted by* $\{(\alpha, E)\} \widetilde{<} \{(\alpha^{'}, E^{'})\}$*.*

DEFINITION 1.11 ([**17**]). *Let* $\widetilde{R}$ *be the set of all soft real points and a mapping* $p : \widetilde{R} \times \widetilde{R} \to \widetilde{R}^{+} \cup \{\widetilde{0}\}$ *satisfies the following conditions for each* $\{(\alpha, E)\}$, $\{(\alpha^{'}, E^{'})\}$, $\{(\alpha^{''}, E^{''})\}$,

- i $p(\{(\alpha, E)\}, \{(\alpha^{'}, E^{'})\}) \cong \{(0, \emptyset)\} \Leftrightarrow \{(\alpha, E)\} \cong \{(\alpha^{'}, E^{'})\}$
- ii $p(\{(\alpha, E)\}, \{(\alpha^{'}, E^{'})\}) \cong p(\{(\alpha^{'}, E^{'})\}, \{(\alpha, E)\})$
- iii $p(\{(\alpha, E)\}, \{(\alpha^{'}, E^{'})\}) \widetilde{\leqslant} p(\{(\alpha, E)\}, \{(\alpha^{'}, E^{'})\}) \widetilde{+} p(\{(\alpha^{'}, E^{'})\}, \{(\alpha^{''}, E^{''})\})$.

Then, $(\widetilde{R}, p)$ *is said to be soft real point metric space.*

DEFINITION 1.12 ([**17**]). *Suppose that* $\{(a_n, E)\}$ *is a sequence of soft real points in* $(\widetilde{R}, p)$*. The sequence* $\{(a_n, E)\}$ *is convergent to* $\{(a, E)\}$*, such that there is a* $n_0 \in N$ *and* $p(\{(a_n, E)\}, \{(a, E)\}) \widetilde{<} \{(e, E)\}$ *for any* $\{(0, \emptyset)\} \widetilde{<} \{(e, E)\} \in \widetilde{R^+}$*. Then, we denote by* $\{(a_n, E)\} \widetilde{\to} \{(a, E)\}$ *as* $n \to \infty$*.*

2. New generalized soft metric spaces

First of all, let's give the definition and proposition that will be necessary in the later sections of our work.

Definition 2.1. *For a constant number* $\alpha \in R^{+} \setminus \{1\}$, $\{(\alpha, E)\} \in \widetilde{R^{+}}$ *and for* $\gamma \in R$, $\{(\gamma, E)\} \in \widetilde{R}$. *A function* $f : \widetilde{R} \to \widetilde{R}^{+}$ *is defined by* $f((\gamma, E)) = \{(\alpha, E)\}^{\{(\gamma,E)\}} \cong \{(\alpha^{\gamma}, E)\}$.

Proposition 2.1. *The function f defined in the above definition is bijection.*

Proof. (i) Let us take two soft real points $\{(\gamma_1, E)\}$, $\{(\gamma_2, E)\} \in \widetilde{R}$ and $\{(\gamma_1, E)\} \neq \{(\gamma_2, E)\}$. Then, $f((\gamma_1, E)) = \{(\alpha, E)\}^{\{(\gamma_1,E)\}} \cong \{(\alpha^{\gamma_1}, E)\} \widetilde{\neq} \{(\alpha^{\gamma_2}, E)\} \cong \{(\alpha, E)\}^{\{(\gamma_2,E)\}} = f((\gamma_2, E))$.

(ii) Let us take $\{(\psi, E)\} \in \widetilde{R}^{+}$. In this case $\psi \in R^{+}$ and there exists a real number $\gamma = \log_{\alpha}\psi \in R$. Then, $f((\gamma, E)) = \{(\alpha, E)\}^{\{(\gamma,E)\}} \cong \{(\alpha^{\gamma}, E)\}$ and since $\gamma = \log_{\alpha}\psi \in R$, $\alpha^{\gamma} = \alpha^{\log_{\alpha}\psi} = \psi$ i.e. $f((\gamma, E)) \cong \{(\psi, E)\}$. Therefore, this function has an inverse defined as $f^{-1}((\gamma, E)) \cong \{(\psi, E)\}$. □

Remark 2.1. In this work, for simplicity, the notation $\widetilde{\alpha}$ will be used for a soft real point (α, E).

Definition 2.2. *Assume that E is a parameters set and a mapping* $p : S_F \times S_F \to \widetilde{R}^{+} \cup \{\widetilde{0}\}$ *satisfies the following conditions for each* $\widetilde{a}, \widetilde{b} \in S_F$,

$\mathcal{G}_1$**:** $p(\widetilde{a}, \widetilde{b}) \cong \widetilde{0} \Leftrightarrow \widetilde{a} \cong \widetilde{b}$

$\mathcal{G}_2$**:** $p(\widetilde{a}, \widetilde{b}) \cong p(\widetilde{b}, \widetilde{a})$

$\mathcal{G}_3$**:** *There exists* $\widetilde{L} \in \widetilde{R}^{+}$, $\widetilde{L} \widetilde{\geqslant} \widetilde{1}$, *for every* $n \in N$, $n \geqslant 2$ *and for every* $(\widetilde{s_i})_{i=1}^{n} \widetilde{\subset} S_F$ *with* $(\widetilde{s_1}, \widetilde{s_n}) = (\widetilde{a}, \widetilde{b})$, *we have*

$$p(\widetilde{a}, \widetilde{b}) \widetilde{\leqslant} \widetilde{L} \tilde{\cdot} \sum_{i=1}^{n-1} p(\widetilde{s_i}, \widetilde{s_{i+1}}).$$

Then, p is said to be generalized soft metric on F, (F, p, E) *is a generalized soft metric space.*

Definition 2.3. *Let* $\mathcal{F}$ *be the set of functions* $\alpha : \widetilde{R}^{+} \to \widetilde{R}$ *satisfying the following conditions for* $\widetilde{k}, \widetilde{l} \in \widetilde{R}^{+}$

α_1**:** *α is non decreasing that is* $\widetilde{0} \widetilde{\leqslant} \widetilde{k} \widetilde{\leqslant} \widetilde{l} \Rightarrow \alpha(\widetilde{k}) \widetilde{\leqslant} \alpha(\widetilde{l})$

α_2**:** *For every sequence* $\{\widetilde{k_n}\} \widetilde{\subset} \widetilde{R}^{+}$, *we have* $\widetilde{k_n} \widetilde{\to} \widetilde{0}$, $n \to \infty \Leftrightarrow \alpha(\widetilde{k_n}) \widetilde{\to} -\widetilde{\infty}$, $n \to \infty$.

Definition 2.4. *Assume that E is a parameters set and F is a universal set. Let consider a mapping* $p : S_F \times S_F \to \widetilde{R}^{+} \cup \{\widetilde{0}\}$ *that satisfies the following conditions for* $\alpha \in \mathcal{F}$ *and* $\widetilde{t} \in \widetilde{R}^{+}$. *For* $\widetilde{a}, \widetilde{b} \in S_F$,

$\mathcal{F}_1$**:** $p(\widetilde{a},\widetilde{b}) \cong \widetilde{0} \Leftrightarrow \widetilde{a} \cong \widetilde{b}$,

$\mathcal{F}_2$**:** $p(\widetilde{a},\widetilde{b}) \cong p(\widetilde{b},\widetilde{a})$,

$\mathcal{F}_3$**:** *For every* $n \in N, n \geqslant 2$ *and for every* $(\widetilde{s_i})_{i=1}^n \widetilde{\subset} S_F$ *with* $(\widetilde{s_1},\widetilde{s_n}) \cong (\widetilde{a},\widetilde{b})$, *we have*

$$p(\widetilde{a},\widetilde{b}) \widetilde{>} \widetilde{0} \Rightarrow \alpha(p(\widetilde{a},\widetilde{b})) \widetilde{\leqslant} \alpha(\widetilde{\sum_{i=1}^{n-1}} p(\widetilde{s_i},\widetilde{s_{i+1}})) \widetilde{+} \widetilde{t}.$$

Then, p is said to be soft $\mathcal{F}$ *metric on F and the* (F,p,E) *is a soft* $\mathcal{F}$ *metric space.*

Remark 2.2. Throughout the article, p is a soft $\mathcal{F}$ metric that meets the condition $\{p(\widetilde{a},\widetilde{b})(\beta) \cong p(\widetilde{a}(\beta),\widetilde{b}(\beta)) : \widetilde{a}(\beta),\widetilde{b}(\beta) \in S_F\}$ for $\beta \in E$ and $\widetilde{a}(\beta),\widetilde{b}(\beta) \widetilde{\in} S_F$ is singleton.

Remark 2.3. Observe that any soft metric on F is a soft $\mathcal{F}$ metric on F. Indeed, if p is a soft metric on F. Obviously it satisfies $\mathcal{G}_1$ and $\mathcal{G}_2$. On the other hand by the triangle inequality, for every $\widetilde{a}(\beta),\widetilde{b}(\beta) \in S_F$ for every $n \in N$, $n \geqslant 2$ and for every $(\widetilde{s_i})_{i=1}^n \widetilde{\subset} S_F$ with $(\widetilde{s_1},\widetilde{s_n}) \cong (\widetilde{a},\widetilde{b})$,

$$p(\widetilde{a},\widetilde{b}) \cong p(\widetilde{s_1},\widetilde{s_n}) \widetilde{\leqslant} p(\widetilde{s_i},\widetilde{s_{i+1}}).$$

Therefore,

$p(\widetilde{a},\widetilde{b}) \widetilde{>} \widetilde{0} \Rightarrow \alpha(p(\widetilde{a},\widetilde{b})) \widetilde{\leqslant} \alpha(\widetilde{\sum_{i=1}^{n-1}} p(\widetilde{s_i},\widetilde{s_{i+1}})) \widetilde{+} \widetilde{0}$. That is if we take $\widetilde{t} \cong \widetilde{0}$, p satisfies $\mathcal{G}_3$.

Example 2.1. *Let* $p : \widetilde{R} \times \widetilde{R} \to \widetilde{R}^+$ *is the mapping defined by*

$$p(\widetilde{a},\widetilde{b}) \cong \begin{cases} (\widetilde{k} \widetilde{-} \widetilde{l})^{\widetilde{2}}, \widetilde{k},\widetilde{l} \in \widetilde{R},\ \widetilde{0} \widetilde{\leqslant} \widetilde{k},\ \widetilde{l} \widetilde{\leqslant} \widetilde{4} \\ \mid \widetilde{k} \widetilde{-} \widetilde{l} \mid, \widetilde{k},\widetilde{l} \in \widetilde{R},\ \widetilde{k},\widetilde{l} \notin [\widetilde{0},\widetilde{4}] \end{cases}.$$

In this case p satisfies the first two conditions required to be a soft metric. But $p(\widetilde{1},\widetilde{4}) \cong \widetilde{9} \widetilde{>} p(\widetilde{1},\widetilde{3}) \widetilde{+} p(\widetilde{3},\widetilde{4}) \cong \widetilde{4} \widetilde{+} \widetilde{1} \cong \widetilde{5}$.

This show that the function p is not a soft metric on $\widetilde{R}^+$. *Now, take* $\widetilde{k},\widetilde{l} \in \widetilde{R}^+$ *such that* $p(\widetilde{k},\widetilde{l}) \widetilde{>} \widetilde{0}$. *For every* $n \in N$, $n \geqslant 2$ *and for every* $(\widetilde{s_i})_{i=1}^n \widetilde{\subset} S_F$ *with* $(\widetilde{s_1},\widetilde{s_n}) \cong (\widetilde{k},\widetilde{l})$.

Let $I = \{i = 1,2,...,n-1 : (\widetilde{s_i},\widetilde{s_{i+1}})$, *for* $1 \leqslant i \leqslant n$, $\widetilde{0} \widetilde{\leqslant} \widetilde{s_i} \widetilde{\leqslant} \widetilde{4}\}$ *and*
$J = \{i = 1,2,...,n-1\} \backslash I$. *Therefore*

$$\begin{aligned} \widetilde{\sum_{i=1}^{n-1}} p(\widetilde{s_i},\widetilde{s_{i+1}}) &\cong \widetilde{\sum_{i\in I}} p(\widetilde{s_i},\widetilde{s_{i+1}}) \widetilde{+} \widetilde{\sum_{j\in J}} p(\widetilde{s_j},\widetilde{s_{j+1}}) \\ &\cong \widetilde{\sum_{i\in I}} (\widetilde{s_{i+1}} \widetilde{-} \widetilde{s_i})^{\widetilde{2}} \widetilde{+} \widetilde{\sum_{j\in J}} \mid \widetilde{s_j} \widetilde{-} \widetilde{s_{j+1}} \mid . \end{aligned}$$

There are two cases.

Case 1: *If* $\widetilde{k},\widetilde{l}\in\widetilde{R}^{+}$, $\widetilde{k},\widetilde{l}\notin[\widetilde{0},\widetilde{4}]$.

Then, we have

$$\begin{aligned}
p(\widetilde{k},\widetilde{l}) &\cong |\widetilde{k}\widetilde{-}\widetilde{l}| \\
&\widetilde{\leqslant} \sum_{i=1}^{\widetilde{n-1}} |\widetilde{s_{i+1}}\widetilde{-}\widetilde{s_i}| \\
&\cong \widetilde{\sum_{i\in I}} |\widetilde{s_{i+1}}\widetilde{-}\widetilde{s_i}| \widetilde{+} \widetilde{\sum_{j\in J}} |\widetilde{s_{j+1}}\widetilde{-}\widetilde{s_j}|
\end{aligned}$$

Furthermore, $|\widetilde{s_{i+1}}\widetilde{-}\widetilde{s_i}|\widetilde{\leqslant}(\widetilde{s_{j+1}}\widetilde{-}\widetilde{s_j})^{\widetilde{2}}$, $i\in I$.

Consequently,

$$\begin{aligned}
p(\widetilde{k},\widetilde{l}) &\widetilde{\leqslant} \widetilde{\sum_{i\in I}}(\widetilde{s_{i+1}}\widetilde{-}\widetilde{s_i})^{\widetilde{2}} \widetilde{+} \widetilde{\sum_{j\in J}} |\widetilde{s_{j+1}}\widetilde{-}\widetilde{s_j}| \\
&\cong \sum_{i=1}^{\widetilde{n-1}} p(\widetilde{s_i},\widetilde{s_{i+1}})
\end{aligned}$$

Case 2: *If* $\widetilde{k},\widetilde{l}\in\widetilde{R}^{+}$, $\widetilde{0}\widetilde{\leqslant}\widetilde{k},\ \widetilde{l}\widetilde{\leqslant}\widetilde{4}$.

Then, we have

$$\begin{aligned}
p(\widetilde{k},\widetilde{l}) &\cong |\widetilde{k}\widetilde{-}\widetilde{l}|^{\widetilde{2}} \\
&\widetilde{\leqslant} \widetilde{4}\widetilde{\cdot}|\widetilde{k}\widetilde{-}\widetilde{l}| \\
&\widetilde{\leqslant} \widetilde{4}\widetilde{\cdot}\widetilde{\sum_{i\in I}} |\widetilde{s_{i+1}}\widetilde{-}\widetilde{s_i}| \widetilde{+} \widetilde{\sum_{j\in J}} |\widetilde{s_{j+1}}\widetilde{-}\widetilde{s_j}| \\
&\cong \widetilde{4}\widetilde{\cdot}\sum_{i=1}^{\widetilde{n-1}} p(\widetilde{s_i},\widetilde{s_{i+1}})
\end{aligned}$$

Therefore, for every $\widetilde{k},\widetilde{l}\in\widetilde{R}^{+}$, *for every* $n\in N$, $n\geqslant 2$ *and for every* $(\widetilde{s_i})_{i=1}^{n}\widetilde{\subset}\widetilde{R}^{+}$ *with* $(\widetilde{s_1},\widetilde{s_n})\cong(\widetilde{k},\widetilde{l})$, *we have* $p(\widetilde{k},\widetilde{l})\widetilde{>}\widetilde{0}\Rightarrow p(\widetilde{k},\widetilde{l})\widetilde{\leqslant}\widetilde{4}\widetilde{\cdot}\sum_{i=1}^{\widetilde{n-1}}p(\widetilde{s_i},\widetilde{s_{i+1}})$. *For* $\alpha(\widetilde{k})\cong\widetilde{4}^{\widetilde{k}}$, $\alpha\in\mathcal{F}$ *and then* $\alpha^{-1}(p(\widetilde{k},\widetilde{l}))\widetilde{\leqslant}\alpha^{-1}(\sum_{i=1}^{\widetilde{n-1}}p(\widetilde{s_i},\widetilde{s_{i+1}}))\widetilde{+}\alpha^{-1}(\widetilde{4})$.

Consequently, p *is a soft* $\mathcal{F}$ *metric on* F *and call* (F,p,E) *a soft* $\mathcal{F}$ *metric space.*

Remark 2.4. Note that, any generalized soft metric on F is a soft $\mathcal{F}$ metric on F.

Remark 2.5. We can see that every soft metric is a generalized soft metric on F for $\widetilde{L} \cong \widetilde{1}$. Furthermore, every soft $b-$metric is a generalized soft metric on F.

3. Topology of the soft $\mathcal{F}-$ metric space.

Definition 3.1. *Let (F,p,E) be a soft $\mathcal{F}$ metric space, $\widetilde{a} \in S_F$ and $\widetilde{e} \in \widetilde{R^+}$. The set of soft elements; $B(\widetilde{a},\widetilde{e}) = \{\widetilde{c} \in S_F : p(\widetilde{a},\widetilde{c}) \widetilde{<} \widetilde{e}\} \widetilde{\subset} S_F$ is called as an open ball and $(B_{\widetilde{e}},E) = SS(B(\widetilde{a},\widetilde{e}))$ is called as a soft $\mathcal{F}$-open ball. Then, $(B_{\widetilde{e}},E)(\beta) = B(\widetilde{a}(\beta),\widetilde{e}(\beta))$ and is an open ball in the soft $\mathcal{F}$ metric space. The set of soft elements $\bar{B}(\widetilde{a},\widetilde{e}) = \{\widetilde{c} \in S_F : p(\widetilde{a},\widetilde{c}) \widetilde{\leqslant} \widetilde{e}\} \widetilde{\subset} S_F$ is called as a closed ball and $(\bar{B}_{\widetilde{e}},A) = SS(\bar{B}(\widetilde{a},\widetilde{e}))$ is called as a soft $\mathcal{F}$-closed ball.*

Definition 3.2. *Let (F,p,E) be a soft $\mathcal{F}$ metric space, $\varphi \widetilde{\subset} S_F$. If there is a $\varphi \widetilde{<} \widetilde{e} \in \widetilde{R^+}$ that satisfies $\widetilde{a} \in \bar{B}(\widetilde{a},\widetilde{e}) \widetilde{\subset} \varphi$, then $\widetilde{a} \in \varphi$ called as a soft $\mathcal{F}$ -interior point.*

Now, we take a soft subset $(H,E) \in S_F$ in (F,p,E). If there is $\widetilde{0} \widetilde{<} \widetilde{e} \in \widetilde{R^+}$ that satisfies $\widetilde{a} \in B(\widetilde{a},\widetilde{e}) \subset SS(H,E)$, then the element $\widetilde{a}$ is a soft $\mathcal{F}$ -interior point of (H,E).

The class of all soft $\mathcal{F}-$interior points in (H,E) is show that $int_{\mathcal{F}}(H,E)$. In this case the soft $\mathcal{F}-$interior is defined as $SS(int_{\mathcal{F}}(H,E)) = (H,E)^{\circ}$.

Definition 3.3. *The set φ, each element of which is a soft $\mathcal{F}-$interior point, is an open class in (F,p,E). So we can define the soft $\mathcal{F}-$open set in (F,p,E), if there is a class φ like above and $(H,E) = SS(\varphi)$.*

Remark 3.1. Obviously, in the soft $\mathcal{F}$ metric (F,p,E), $(B_{\widetilde{e}},A)$ is a soft $\mathcal{F}$ open set.

Theorem 3.1. *Assume that (F,p,E) is a soft $\mathcal{F}$ metric space. Therefore the following axioms are provided,*

(i) *S_F, $\widetilde{\Phi}$ are two soft $\mathcal{F}-$open sets,*
(ii) *Arbitrary elementary union of soft $\mathcal{F}-$open sets is a soft $\mathcal{F}-$open set,*
(iii) *Finite elementary intersection of soft $\mathcal{F}-$open sets is a soft $\mathcal{F}-$open set.*

Proof. (i) For every $\widetilde{a} \in S_F$ there is a $\widetilde{0} \widetilde{<} \widetilde{e} \in \widetilde{R^+}$ such that $\widetilde{a} \in B(\widetilde{a},\widetilde{e}) \widetilde{\subset} S_F$. Thus S_F is a soft $\mathcal{F}-$open set. Obviously, $\widetilde{\Phi}$ is a soft $\mathcal{F}-$open set.

(ii) Let for every $i \in I$, $\widetilde{A_i}$ is a soft $\mathcal{F}-$open set that is for every $\widetilde{a_i} \in \widetilde{A_i}$, $\widetilde{a_i} \in B(\widetilde{a_i},\widetilde{e_i}) \widetilde{\subset} \widetilde{A_i}$. Assume that we take any $\widetilde{a} \in \bigcup_{i\in I} \widetilde{A_i}$. By definition of union, there is an $i_0 \in I$ and $\widetilde{a} \in \widetilde{A_{i_0}}$. Since $\widetilde{A_{i_0}}$ is a soft $\mathcal{F}-$open set, there is a $\widetilde{0} \widetilde{<} \widetilde{e} \in S_F$ such that $B(\widetilde{a}, \widetilde{e}) \widetilde{\subset} \widetilde{A_{i_0}} \widetilde{\subset} \underset{i\in I}{\cup} \widetilde{A_i}$. Consequently $\underset{i\in I}{\cup} \widetilde{A_i}$ is a soft $\mathcal{F}-$open set.

(iii) Let for every $i = 1, 2, ..., n$, $\widetilde{A_i}$ be a soft $\mathcal{F}-$open set. Assume that $\overset{n}{\underset{i=1}{\cap}} \widetilde{A_i} \neq \widetilde{\Phi}$ and we take any $\widetilde{a} \in \overset{n}{\underset{i=1}{\cap}} \widetilde{A_i}$. By definition of intersection, $\widetilde{a} \in \widetilde{A_i}$ for $i = 1, 2, ..., n$. If we choose the smallest of this $\widetilde{0} \widetilde{<} \widetilde{e} \in \widetilde{R^+}$ for $i = 1, 2, ..., n$ and denote by $\widetilde{e}$, then $B(\widetilde{a}, \widetilde{e}) \widetilde{\subset} \overset{n}{\underset{i=1}{\cap}} \widetilde{A_i}$. Consequently $\overset{n}{\underset{i=1}{\cap}} \widetilde{A_i}$ is a soft $\mathcal{F}-$open set.

□

Theorem 3.2. *A soft $\mathcal{F}-$metric space is an elementary soft topological space.*

Proof. Suppose that $\tau_{\mathcal{F}} \widetilde{\subset} S_F$ is the class of soft $\mathcal{F}-$open sets in (F, p, E). From above theorem $\tau_{\mathcal{F}}-$is a topology on F. □

Definition 3.4. *The topology $\tau_{\mathcal{F}}$ is called by the elementary soft $\mathcal{F}-$metric topology on F and $(F, \tau_{\mathcal{F}}, E)$ the elementary soft $\mathcal{F}-$metric topological space.*

Definition 3.5. *A soft set $(H, A) \in S_F$ is soft $\mathcal{F}-$closed in the soft $\mathcal{F}-$ metric space (F, p, E) if $(H, A)^c \in \tau_{\mathcal{F}}$.*

Theorem 3.3. *Assume that (F, p, E) is a soft $\mathcal{F}-$ metric space. Therefore the following axioms are provided,*

(i) *S_F, $\widetilde{\Phi}$ are two soft $\mathcal{F}-$closed sets,*
(ii) *Arbitrary elementary intersection of soft $\mathcal{F}-$closed sets is a soft $\mathcal{F}-$closed set,*
(iii) *Finite elementary union of soft $\mathcal{F}-$closed sets is a soft $\mathcal{F}-$closed set.*

Definition 3.6. *Suppose that $\{\widetilde{a_n}\}$ is a sequence of soft points in (F, p, E) and $\widetilde{a} \in S_F$.*

(i) *$\{\widetilde{a_n}\}$ is $\mathcal{F}-$convergent to $\widetilde{a}$, if there is a $n_0 \in N$ and $p(\widetilde{a_n}, \widetilde{a}) \widetilde{<} \widetilde{e}$ for any $\widetilde{0} \widetilde{<} \widetilde{e} \in \widetilde{R^+}$. Then, we denote by $\widetilde{a_n} \widetilde{\to}_{\mathcal{F}} \widetilde{a}$ as $n \to \infty$.*
(ii) *$\{\widetilde{a_n}\}$ is $\mathcal{F}-$Cauchy in (F, p, E), if there is a $n_0 \in N$ and $p(\widetilde{a_n}, \widetilde{a_m}) \widetilde{<} \widetilde{e}$ for all $n, m > n_0$ and any $\widetilde{0} \widetilde{<} \widetilde{e} \in \widetilde{R^+}$.*
(iii) *(F, p, E) is $\mathcal{F}-$complete, if any $\mathcal{F}-$Cauchy sequence $\{\widetilde{a_n}\}$ converges a soft point in F.*

Proposition 3.1. *Assume that (F, p, E) is a soft $\mathcal{F}-$ metric space and $\widetilde{a}, \widetilde{b} \in S_F$. Then,*

(i) *A sequence $\{\widetilde{a_n}\}$ of soft points of F converges to $\widetilde{a}$ if and only if $p(\widetilde{a_n}, \widetilde{a}) \widetilde{\to}_{\mathcal{F}} \widetilde{0}$, $n \to \infty$.*

(ii) *Two sequence* $\{\widetilde{a_n}\}$ *and* $\{\widetilde{b_n}\}$ *of soft points of* F *with* $\widetilde{a_n} \tilde{\to}_{\mathcal{F}} \widetilde{a}$ *and* $\widetilde{b_n} \tilde{\to}_{\mathcal{F}} \widetilde{b}$, $n \to \infty$. *Then,* $p(\widetilde{a_n}, \widetilde{b_n}) \tilde{\to}_{\mathcal{F}} (\widetilde{a}, \widetilde{b})$, $n \to \infty$.

4. Fixed point theory in soft $\mathcal{F}-$metric spaces

DEFINITION 4.1. *Let* (F, p, E) *be a soft* $\mathcal{F}-$*metric space and* $T : S_F \to S_F$ *is a mapping. We call an* $\widetilde{a_0}$ *is a soft fixed point of* T, *if satisfies the condition* $T(\widetilde{a_0}) \tilde{=} \widetilde{a_0}$.

DEFINITION 4.2. *If there is a soft real point* $\widetilde{k}$, $\widetilde{0} \tilde{<} \widetilde{k} \tilde{<} \widetilde{1}$ *and for each* $\widetilde{a}, \widetilde{b} \in S_F$, $p(T\widetilde{a}, T\widetilde{b}) \tilde{\leqslant} \widetilde{k} \tilde{\cdot} p(\widetilde{a}, \widetilde{b})$ *then* T *is called as a soft* $\mathcal{F}-$*contractive mapping on* F.

THEOREM 4.1. *Let* (F, p, E) *be a* $\mathcal{F}-$*complete soft* $\mathcal{F}-$*metric space and* $T : S_F \to S_F$ *is a soft* $\mathcal{F}-$*contractive mapping on* F. *Then,* T *has a unique soft fixed point.*

PROOF. Let $\alpha \in \mathcal{F}$, $\widetilde{t} \in \widetilde{R^+} \cup \{\widetilde{0}\}$ and

$p(\widetilde{a}, \widetilde{b}) \tilde{\geqslant} \widetilde{0} \Rightarrow \alpha(p(\widetilde{a}, \widetilde{b})) \tilde{\leqslant} \alpha(\tilde{\sum}_{i=1}^{n-1} p(\widetilde{s_i}, \widetilde{s_{i+1}})) \tilde{+} \widetilde{t}$. Let take a soft fixed $\widetilde{\varepsilon} \in \widetilde{R^+}$. From the condition α_2, there exists $\widetilde{\delta} \in \widetilde{R^+}$ and

$$\widetilde{0} \tilde{<} \widetilde{l} \tilde{<} \widetilde{\delta} \Rightarrow \alpha(\widetilde{l}) \tilde{<} \alpha(\widetilde{\varepsilon}) \tilde{-} \widetilde{t}.$$

Let $\widetilde{a_0} \in S_F$ be an arbitrary element and the sequence $\{\widetilde{a_n}\} \tilde{\subset} S_F$ is an iterative sequence. Assume that $p(\widetilde{a_0}, \widetilde{a_1}) \tilde{>} \widetilde{0}$. T is a soft $\mathcal{F}-$contractive mapping on F, we have for $n \in N$, $p(\widetilde{a_n}, \widetilde{a_{n+1}}) \tilde{\leqslant} \widetilde{k^{\widetilde{n}}} \tilde{\cdot} p(\widetilde{a_0}, \widetilde{a_1})$.
Therefore for $m > n$,

$$\begin{aligned}
p(\widetilde{a_n}, \widetilde{a_m}) &\tilde{\leqslant} p(T^n\widetilde{a}_0, T^m\widetilde{a_0}) \\
&\tilde{=} p(T^n\widetilde{a}_0, T^nT^{m-n}\widetilde{a_0}) \\
&\tilde{\leqslant} \widetilde{k^{\widetilde{n}}} \tilde{\cdot} p(\widetilde{a}_0, T^{m-n}\widetilde{a_0}) \\
&\tilde{=} \widetilde{k^{\widetilde{n}}} \tilde{\cdot} p(\widetilde{a}_0, \widetilde{a_{m-n}}) \\
&\tilde{\leqslant} \widetilde{k^{\widetilde{n}}} \tilde{\cdot} [p(\widetilde{a}_0, \widetilde{a}_1) \tilde{+} \ldots \tilde{+} p(\widetilde{a}_{m-n-1}, \widetilde{a_{m-n}})] \\
&\tilde{\leqslant} \widetilde{k^{\widetilde{n}}} \tilde{\cdot} p(\widetilde{a}_0, \widetilde{a}_1) \tilde{\cdot} [\widetilde{1} \tilde{+} \widetilde{k} \tilde{+} \widetilde{k^{\widetilde{2}}} \tilde{+} \ldots \tilde{+} \widetilde{k}_{m-n-1}] \\
&\tilde{\leqslant} \frac{\widetilde{k^{\widetilde{n}}} \tilde{\cdot} p(\widetilde{a}_0, \widetilde{a}_1)}{\widetilde{1} \tilde{-} \widetilde{k}}
\end{aligned}$$

Since $\frac{\widetilde{k^{\widetilde{n}}}\,\widetilde{\cdot}\,p(\widetilde{a_0},\widetilde{a_1})}{\widetilde{1}\widetilde{-}\widetilde{k}} \widetilde{\to}_{\mathcal{F}} \widetilde{0}$, there exists some $n_0 \in N$ as $n \geqslant n_0$, $\widetilde{0} \widetilde{<} \frac{\widetilde{k^{\widetilde{n}}}\,\widetilde{\cdot}\,p(\widetilde{a_0},\widetilde{a_1})}{\widetilde{1}\widetilde{-}\widetilde{k}} \widetilde{<} \widetilde{\delta}$.

So, $\alpha(\frac{\widetilde{k^{\widetilde{n}}}\,\widetilde{\cdot}\,p(\widetilde{a_0},\widetilde{a_1})}{\widetilde{1}\widetilde{-}\widetilde{k}}) \widetilde{<} \alpha(\widetilde{\varepsilon}) \widetilde{-} \widetilde{t}$ and from (α_1),

$\alpha(\sum_{i=n}^{\widetilde{m-1}} p(\widetilde{s_i}, \widetilde{s_{i+1}})) \widetilde{<} \alpha(\frac{\widetilde{k^{\widetilde{n}}}\,p(\widetilde{a_0},\widetilde{a_1})}{\widetilde{1}\widetilde{-}\widetilde{k}}) \widetilde{<} \alpha(\widetilde{\varepsilon}) \widetilde{-} \widetilde{t}$, for $m > n \geqslant n_0$. Then, we get

$$p(\widetilde{a_n}, \widetilde{a_m}) \widetilde{>} \widetilde{0},\ m > n \geqslant n_0 \Rightarrow \alpha(p(\widetilde{a_n}, \widetilde{a_m})) \widetilde{\leqslant} \alpha(\sum_{i=n}^{\widetilde{m-1}} p(\widetilde{a_i}, \widetilde{a_{i+1}})) \widetilde{+} \widetilde{t} \widetilde{<} \alpha(\widetilde{\varepsilon}).$$

From (α_1), $p(\widetilde{a_n}, \widetilde{a_m}) \widetilde{<} \widetilde{\varepsilon}$, for $m > n \geqslant n_0$.

That is, $\{\widetilde{a_n}\}$ is a $\mathcal{F}-$Cauchy sequence. Since (F, p, E) is $\mathcal{F}-$complete, there exists $\widetilde{a} \in S_F$ such that $\{\widetilde{a_n}\}$ is $\mathcal{F}-$convergent to $\widetilde{a}$ i.e. $\widetilde{a}_n \widetilde{\to}_{\mathcal{F}} \widetilde{a}$.

Now, we show that $\widetilde{a} \in S_F$ is a fixed point of T. From the condition $(\mathcal{F}_3)$, we get for $n \in N$, $\alpha(p(T\widetilde{a}, \widetilde{a})) \widetilde{\leqslant} \alpha(p(T\widetilde{a}, T\widetilde{a_n}) \widetilde{+} p(T\widetilde{a_n}, \widetilde{a})) \widetilde{+} \widetilde{t}$.

Hence $\alpha(p(T\widetilde{a}, \widetilde{a})) \widetilde{\leqslant} \alpha(\widetilde{k}\,\widetilde{\cdot}\,p(\widetilde{a}, \widetilde{a_n}) \widetilde{+} p(\widetilde{a_{n+1}}, \widetilde{a})) \widetilde{+} \widetilde{t}$.

Using $\alpha \in \mathcal{F}$, $\alpha(\widetilde{k}\,\widetilde{\cdot}\,p(\widetilde{a}, \widetilde{a_n}) \widetilde{+} p(\widetilde{a_{n+1}}, \widetilde{a})) \widetilde{+} \widetilde{t} \widetilde{\to}_{\mathcal{F}} -\widetilde{\infty}$. This contradicts the fact that $p(T\widetilde{a}, \widetilde{a}) \widetilde{>} \widetilde{0}$. Consequently $p(T\widetilde{a}, \widetilde{a}) \widetilde{=} \widetilde{0}$ that is $T\widetilde{a} \widetilde{=} \widetilde{a}$. Now let $\widetilde{a}, \widetilde{b} \in S_F$ are two soft fixed points of T and $\widetilde{a} \widetilde{\neq} \widetilde{b}$. Then, $p(\widetilde{a}, \widetilde{b}) \widetilde{>} \widetilde{0}$, $T\widetilde{a} \widetilde{=} \widetilde{a}$, $T\widetilde{b} \widetilde{=} \widetilde{b}$. Thus we have $p(\widetilde{a}, \widetilde{b}) \widetilde{=} p(T\widetilde{a}, T\widetilde{b}) \widetilde{\leqslant} \widetilde{k}\,\widetilde{\cdot}\,p(\widetilde{a}, \widetilde{b}) \widetilde{<} p(\widetilde{a}, \widetilde{b})$ and this is a contradiction. □

REMARK 4.1. Note that the sequence $\{\widetilde{a_n}\} \widetilde{\subset} S_F$ for any $\widetilde{a_0} \in S_F$ defined by $\widetilde{a_{n+1}} \widetilde{=} T\widetilde{a_n}$, $n \in N$ is $\mathcal{F}-$convergent to $\widetilde{a}$.

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