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NORMAL MATRICES WITH REFERENCE TO INDEFINITE INNER PRODUCT

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Abstract. The concept of normal matrix in indefinite inner product space is introduced as a special type of range symmetric matrix in indefinite inner product space and as an analogue of complex normal matrix. Equivalent conditions for a matrix to be normal in indefinite inner product space are obtained. Some properties of normal matrices in indefinite inner product space have been derived.

1. Introduction

An indefinite inner product is a conjugate symmetric sesquilinear form $[x, y]$ together with the regularity condition that $[x, y] = 0, \forall y \in \mathbb{C}^n$ only when $x = 0$. Any indefinite inner product is associated with a unique invertible complex matrix J (called a weight) such that $[x, y] = \langle x, Jy\rangle$, where $\langle \cdot, \cdot\rangle$ denotes the Euclidean inner product on $\mathbb{C}^n$. We also make an additional assumption on J, that is, $J^2 = I$.

Investigations of linear maps on indefinite inner product utilize the usual multiplication of matrices which is induced by the Euclidean inner product of vectors ([**1**],[**17**]). This causes a problem as there are two different values for dot product of vectors. To overcome this difficulty, Ramanathan et al. introduced a new matrix product called indefinite matrix multiplication and investigated some of its properties in [**17**]. More precisely, the indefinite matrix product of two matrices A and B of sizes $m \times n$ and $n \times l$ complex matrices, respectively, is defined to be the matrix $A \circ B = AJ_nB$. The adjoint of A, denoted by $A^{[*]}$ is defined to be the

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matrix $J_nA^*J_m$, where J_m and J_n are weights. Further indefinite matrix product with properties were established.

The concept of a normal matrix with entries from the complex field were introduced by Toeplitz in 1918. The concept of EP matrices over the field of complex number was introduced by Schwerdtfeger as a generalization of normal matrices. In 1965, Katz and Pearl developed the structure theory of EP_r matrices over an arbitrary field F and the concept of zero-type EP_r matrix was introduced. In 1987 Grone et al. [**4**] developed the concept of normal matrices. Gohberg and Reichstein [**3**] in 1990 investigated on classification of normal matrices in an indefinite scalar product. Updated normal matrices were extended by Elsner and Ikramov [**2**] in 1998. Classes of normal matrices in indefinite inner product were extended by Mehl and Rodman [**13**]. Baksalary and Trenkler in 2008 studied the characterization of EP, normal and Hermitian matrices. Normal matrices in which all principal submatrices are normal are said to be principally normal were established by Sherman and Smith [**19**] in 2013. Indefinite matrix product concept was discussed further by many researcher in [**5, 6, 7, 8, 9, 10, 11, 16, 17**].

In this paper we have constructed the concept of normal matrix in indefinite inner product space is introduced as a special type of range symmetric matrix in indefinite inner product space and as an analogue of complex normal matrix. Equivalent conditions for a matrix to be normal in indefinite inner product space are obtained. Some properties of normal matrices in indefinite inner product space has been derived. The notations $rk(A)$, $Ra(A)$ and $Nu(A)$ stand, respectively, for the rank, range space and null space of A respectively.

2. Preliminaries

We begin this section with the notion of an indefinite matrix multiplication. We refer various properties and advantages of this product in [**5, 12, 17**].

Definition 2.1. *Let A and B be $m\times n$ and $n\times l$ complex matrices, respectively. Let J_n be an arbitrary but fixed $n\times n$ complex matrix such that $J_n = J_n^* = J_n^{-1}$. The indefinite matrix product of A and B (relative to J_n) is defined by $A\circ B = AJ_nB$.*

Definition 2.2. *Let A be an $m\times n$ complex matrix. The adjoint of A (relative to J_n, J_m) is defined by $A^{[*]} = J_nA^*J_m$.*

Definition 2.3. *Let A be an $m \times n$ complex matrix. Then the range space of A is defined by $Ra(A) = \{y = A\circ x \in \mathbb{C}^m : x \in \mathbb{C}^n\}$ and the null space of A is defined by $Nu(A) = \{x \in \mathbb{C}^n : A\circ x = 0\}$.*

Definition 2.4. *$A \in C^{n\times n}$ is range symmetric in $\mathcal{I}$ if and only if $Ra(A) = Ra(A^{[*]})$ (or) equivalently $Nu(A) = Nu(A^{[*]})$.*

Remark 2.1. In particular for $J = I_n$, this reduces to the definition of range symmetric matrix in unitary space (or) equivalently to an EP matrix.

Definition 2.5. *Let $A \in \mathbb{C}^{n\times n}$. A is said to be J-invertible if there exists $X \in \mathbb{C}^{n\times n}$ such that $A\circ X = X\circ A = J$.*

PROPERTY 1

(i) $(A^{[*]})^{[*]} = A$.
(ii) $(A^{[\dagger]})^{[\dagger]} = A$.
(iii) $(AB)^{[*]} = B^{[*]}A^{[*]}$.
(iv) $Ra(A^{[*]}) = Ra(A^{[\dagger]})$.
(v) $Ra(A \circ A^{[*]}) = Ra(A),\ Ra(A^{[*]} \circ A) = Ra(A^{[*]})$.
(vi) $Nu(A \circ A^{[*]}) = Nu(A^{[*]}),\ Nu(A^{[*]} \circ A) = Nu(A)$.

3. Normal matrices in $\mathcal{I}$

Let $\mathcal{I}$ denotes the indefinite inner product space, with weight J, under an indefinite matrix multiplication. In [3], the concept of a range symmetric matrix in $\mathcal{I}$ is introduced and developed. A matrix $A \in \mathbb{C}^n$ is said to be normal in unitary space if $AA^* = A^*A$. Here we introduce the concept of normal matrices in $\mathcal{I}$ as a special type of range symmetric matrices in $\mathcal{I}$ and as a generalization of symmetric matrices in $\mathcal{I}$. Some basic algebraic properties related with range symmetric matrices are derived. The characterization of normal matrices in I are determined

DEFINITION 3.1. *A matrix $A \in \mathbb{C}^{n\times n}$ is said to be normal in $\mathcal{I}$* $\Leftrightarrow$ $A^{[*]} \circ A = A \circ A^{[*]}$.

THEOREM 3.1. *For $A \in \mathbb{C}^{n\times n}$, the following conditions are equivalent*
(i) A *is normal in* $\mathcal{I}$
(ii) JA *is normal*
(iii) AJ *is normal*
(iv) $A^{[*]}$ *is normal in* $\mathcal{I}$
(v) $||A \circ x|| = ||A^{[*]} \circ x||$ *for all* $x \in \mathbb{C}^{n\times n}$
(vi) $A - \lambda J$ *is normal in* $\mathcal{I}$.

Proof. $(i) \Leftrightarrow (ii)$: A is normal in $\mathcal{I} \Leftrightarrow A \circ A^{[*]} = A^{[*]} \circ A$
$\Leftrightarrow AJ(JA^*J) = (JA^*J)JA$
$\Leftrightarrow (AJ^2A^*)J = (JA^*)J^2A$
$\Leftrightarrow AA^*J = JA^*A$ (Since $J^2 = I$)

Premultiply by J on bothside
$\Leftrightarrow JAA^*J^* = J^2A^*A$ (By Definition 2.1)
$\Leftrightarrow JA(JA)^* = A^*J^2A$ (Since $J^2 = I$)
$\Leftrightarrow JA(JA)^* = A^*J^*JA$
$\Leftrightarrow (JA)(JA)^* = (JA)^*(JA)$
$\Leftrightarrow JA$ is normal.

$(i) \Leftrightarrow (iii)$ can be proved in the same manner as that of (i)$\Leftrightarrow$(ii).

$(i) \Leftrightarrow (iv)$: A is normal in $\mathcal{I} \Leftrightarrow A^{[*]} \circ A = A \circ A^{[*]}$
$\Leftrightarrow A^{[*]} \circ (A^{[*]})^{[*]} = (A^{[*]})^{[*]} \circ A^{[*]}$ (By Property 1)
$\Leftrightarrow A^{[*]}$ is normal in $\mathcal{I}$.

$(ii) \Leftrightarrow (v)$: JA is normal $\Leftrightarrow (JA)(JA)^* = (JA)^*(JA)$
$\Leftrightarrow (JA(JA)^*x, x) = ((JA)^*JAx, x)$
$\Leftrightarrow ((JA)^*x, (JA)^*x) = (JAx, (JA)x)$

$$\Leftrightarrow (A^{[*]}\circ x, A^{[*]}\circ x) = (A\circ x, A\circ x)\ (By\ (i) \Leftrightarrow (ii))$$
$$\Leftrightarrow ||A^{[*]}\circ x||^2 = ||A\circ x||^2$$
$$\Leftrightarrow ||A^{[*]}\circ x|| = ||A\circ x||.$$

$(i) \Leftrightarrow (vi):$ A is normal in $\mathcal{I} \Leftrightarrow JA$ is normal
$\Leftrightarrow JA - \lambda I$ is normal
$\Leftrightarrow J(A-\lambda J)$ is normal
$\Leftrightarrow A - \lambda J$ is normal in $\mathcal{I}$.

Definition 3.2. *For $A \in \mathbb{C}^{n\times n},\ x \in \mathbb{C}^n$ and λ a scalar if $Ax = \lambda Jx$, then x is said to be a J-eigen vector of A corresponding to the eigen value λ.*

Remark 3.1. For $A \in \mathbb{C}^{n\times n},\ x \in \mathbb{C}^n$, x is a J-eigen vector of A in $\mathcal{I}$ $\Leftrightarrow$ x is an eigen vector of JA.

Theorem 3.2. *Let A be normal in $\mathcal{I}$, then x is a J-eigen vector of A in $\mathcal{I}$ with eigen value λ $\Leftrightarrow$ x is a J-eigen vector of $A^{[*]}$ with eigen value $\bar{\lambda}$.*

Proof. Since A is normal in $\mathcal{I}$, by Theorem 3.1 JA is normal. It is well known that, x is an eigen vector of JA with eigen value $\lambda \Leftrightarrow$ x is a eigen vector of $(JA)^* = A^*J$ with eigen value $\bar{\lambda}$. By Remark 3.1 it follows that x is a J-eigen vector of A in $\mathcal{I}$ with eigen value λ $\Leftrightarrow$ x is a J-eigen vector of $A^{[*]}$ with eigen value $\bar{\lambda}$. □

Theorem 3.3. *Let $A_1, A_2 \in \mathbb{C}^{n\times n}$ be normal in $\mathcal{I}$ with $A_1 \circ A_2^{[*]} = A_2^{[*]} \circ A_1$, then $A_1 + A_2$ is normal in $\mathcal{I}$.*

Proof. Since $A_1 \circ A_2^{[*]} = A_2^{[*]} \circ A_1$, by $J^2 = I$ and $J^{[*]} = J$ we get $A_2 \circ A_1^{[*]} = A_1 \circ A_2^{[*]}$.

$$\begin{aligned}
\text{Now, } (A_1+A_2)\circ(A_1+A_2)^{[*]} &= (A_1+A_2)\circ(A_1^{[*]}+A_2^{[*]})\\
&= A_1\circ A_1^{[*]} + A_2\circ A_1^{[*]} + A_1\circ A_2^{[*]} + A_2\circ A_2^{[*]}\\
&= A_1^{[*]}\circ A_1 + A_1^{[*]}\circ A_2 + A_2^{[*]}\circ A_1 + A_2^{[*]}\circ A_2\\
&= A_1^{[*]}\circ(A_1+A_2) + A_2^{[*]}\circ(A_1+A_2)\\
&= (A_1^{[*]}+A_2^{[*]})\circ(A_1+A_2)\\
&= (A_1+A_2)^{[*]}\circ(A_1+A_2).
\end{aligned}$$

Thus $A_1 + A_2$ is normal in $\mathcal{I}$. □

Theorem 3.4. *The product of normal matrices is normal in $\mathcal{I}$ if each commute with the adjoint of the other.*

Proof. Let $\{A_i | i \in \mathcal{S}\}$ for $\mathcal{S} = \{1, 2, \cdots, n\}$ be a class of normal matrices in $\mathcal{I}$. By hypothesis $A_iA_j^{[*]} = A_j^{[*]}A_i$ for $i \neq j$. We prove the theorem by induction on n.

Case(i): For n=2. Since $A_1,\ A_2$ are normal matrices in $\mathcal{I}$, $A_1 \circ A_1^{[*]} = A_1^{[*]} \circ A_1$

and $A_2 \circ A_2^{[*]} = A_2^{[*]} \circ A_2$. By hypothesis $A_1 A_2^{[*]} = A_2^{[*]} A_1$ and $A_2 A_1^{[*]} = A_1^{[*]} A_2$.

$$\begin{aligned}
\text{Now } (A_1A_2)^{[*]} \circ (A_1A_2) &= A_2^{[*]} A_1^{[*]} \circ (A_1A_2) \\
&= A_2^{[*]} A_1 \circ A_1^{[*]} A_2 \\
&= A_1 A_2 \circ A_2^{[*]} A_1^{[*]} \\
&= (A_1A_2) \circ (A_1A_2)^{[*]}.
\end{aligned}$$

Thus A_1A_2 is normal in $\mathcal{I}$.

Case(ii): For n=3. Let $B = A_1A_2$, by case(i), B is normal.

To prove : $A_1A_2A_3$ is normal. It is enough to verify $BA_3^{[*]} = A_3^{[*]}B$ and $A_3B^{[*]} = B^{[*]}A_3$ is normal.

$$\begin{aligned}
BA_3^{[*]} &= A_1A_2A_3^{[*]} \\
&= A_1A_3^{[*]}A_2 \\
&= A_3^{[*]}A_1A_2 \\
BA_3^{[*]} &= A_3^{[*]}B. \\
(BA_3^{[*]})^{[*]} &= (A_3^{[*]}B)^{[*]} \\
(A_3^{[*]})^{[*]}B^{[*]} &= B^{[*]}(A_3^{[*]})^{[*]} \\
A_3B^{[*]} &= B^{[*]}A_3.
\end{aligned}$$

Hence $A_1A_2A_3 = BA_3$ is normal in $\mathcal{I}$.

Now assume that product of (*k-1*) normal matrices in $\mathcal{I}$ is normal and we prove it is true for n=k. Let $A = A_1A_2 \cdots A_{k-1}$ by induction A is normal. Now $AA_k^{[*]} = (A_1A_2 \cdots A_{k-1})A_k^{[*]} = A_k^{[*]}(A_1A_2 \cdots A_{k-1}) = A_k^{[*]}A$.

Therefore AA_k is normal ($A_1A_2 \cdots A_{k-1}A_k$ is normal). □

THEOREM 3.5. *If $A \in \mathbb{C}^{n\times n}$ is normal in $\mathcal{I}$ then A is range symmetric in $\mathcal{I}$.*

PROOF. Since A is normal $A^{[*]} \circ A = A \circ A^{[*]}$. Clearly $Nu(A) \subseteq Nu(A^{[*]} \circ A)$. Since $rk(A^{[*]} \circ A) = rk(A \circ A^{[*]}) = rk(A)$. Hence $Nu(A) = Nu(A^{[*]} \circ A)$. Also $Nu(A^{[*]}) \subseteq Nu(A \circ A^{[*]})$ and $rk(A^{[*]}) = rk(A) = rk(A^{[*]} \circ A) = rk(A \circ A^{[*]})$, it follows that $Nu(A^{[*]}) = Nu(A \circ A^{[*]})$. Hence $Nu(A^{[*]}) = Nu(A \circ A^{[*]}) = Nu(A^{[*]} \circ A) = Nu(A)$. Thus A is range symmetric in $\mathcal{I}$. □

Example: The converse of the above Theorem fails.

For $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ then A is range symmetric but not normal in $\mathcal{I}$.

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