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BINARY α-LOCALLY CLOSED SETS IN BINARY TOPOLOGICAL SPACES

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Abstract. In this paper, we will define some new class of locally closed sets called binary α-locally closed sets, binary α-$lc^{\star}$, binary α-$lc^{\star\star}$, binary α-dense and binary α-submaximal in binary topological spaces and study some of their characterizations and properties.

1. Introduction and preliminaries

In 1970 Levine [**3**] gives the concept and properties of generalized closed (briefly g-closed) sets and the complement of g-closed set is said to be g-open set. Njasted [**11**] introduced and studied the concept of α-sets. Later these sets are called as α-open sets in 1983. Mashhours et.al [**6**] introduced and studied the concept of α-closed sets, α-closure of set, α-continuous functions, α-open functions and α-closed functions in topological spaces. Maki et.al [**4, 5**] introduced and studied generalized α-closed sets and α-generalized closed sets. In 2011, S.Nithyanantha Jothi and P.Thangavelu [**8**] introduced topology between two sets and also studied some of their properties. Topology between two sets is the binary structure from X to Y which is defined to be the ordered pairs (A, B) where $A \subseteq X$ and $B \subseteq Y$. In this paper, we will define some new class of locally closed sets called binary α-locally closed sets, binary α-$lc^{\star}$, binary α-$lc^{\star\star}$, binary α-dense and binary α-submaximal in binary topological spaces and study some of their characterizations and properties.

Let X and Y be any two nonempty sets. A binary topology [**8**] from X to Y is a binary structure $\mathcal{M} \subseteq \mathbb{P}(X) \times \mathbb{P}(Y)$ that satisfies the axioms namely

(1) (ϕ, ϕ) and $(X, Y) \in \mathcal{M}$,

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(2) $(A_1 \cap A_2, B_1 \cap B_2) \in \mathcal{M}$ whenever $(A_1, B_1) \in \mathcal{M}$ and $(A_2, B_2) \in \mathcal{M}$, and
(3) If $\{(A_\alpha, B_\alpha) : \alpha \in \delta\}$ is a family of members of $\mathcal{M}$, then $(\bigcup_{\alpha\in\delta} A_\alpha, \bigcup_{\alpha\in\delta} B_\alpha) \in \mathcal{M}$.

If $\mathcal{M}$ is a binary topology from X to Y then the triplet $(X, Y, \mathcal{M})$ is called a binary topological space and the members of $\mathcal{M}$ are called the binary open subsets of the binary topological space $(X, Y, \mathcal{M})$. The elements of $X \times Y$ are called the binary points of the binary topological space $(X, Y, \mathcal{M})$. If $Y = X$ then $\mathcal{M}$ is called a binary topology on X in which case we write $(X, \mathcal{M})$ as a binary topological space.

Definition 1.1. [**8**] *Let X and Y be any two nonempty sets and let (A, B) and $(C, D) \in \mathbb{P}(X) \times \mathbb{P}(Y)$. We say that $(A, B) \subseteq (C, D)$ if $A \subseteq C$ and $B \subseteq D$.*

Definition 1.2. [**8**] *Let $(X, Y, \mathcal{M})$ be a binary topological space and $A \subseteq X$, $B \subseteq Y$. Then (A, B) is called binary closed in $(X, Y, \mathcal{M})$ if $(X\backslash A, Y\backslash B) \in \mathcal{M}$.*

Proposition 1.1. [**8**] *Let $(X, Y, \mathcal{M})$ be a binary topological space and $(A, B) \subseteq (X, Y)$.*

Let $(A, B)^{1*} = \cap\{A_\alpha : (A_\alpha, B_\alpha)$ is binary closed and $(A, B) \subseteq (A_\alpha, B_\alpha)\}$ and $(A, B)^{2*} = \cap\{B_\alpha : (A_\alpha, B_\alpha)$ is binary closed and $(A, B) \subseteq (A_\alpha, B_\alpha)\}$. Then $((A, B)^{1*}, (A, B)^{2*})$ is binary closed and $(A, B) \subseteq ((A, B)^{1*}, (A, B)^{2*})$.

Proposition 1.2. [**8**] *Let $(X, Y, \mathcal{M})$ be a binary topological space and $(A, B) \subseteq (X, Y)$. Let $(A, B)^{1*} = \cup\{A_\alpha : (A_\alpha, B_\alpha)$ is binary open and $(A_\alpha, B_\alpha) \subseteq (A, B)\}$ and $(A, B)^{2*} = \cup\{B_\alpha : (A_\alpha, B_\alpha)$ is binary open and $(A_\alpha, B_\alpha) \subseteq (A, B)\}$.*

Definition 1.3. [**8**] *The ordered pair $((A, B)^{1*}, (A, B)^{2*})$ is called the binary closure of (A, B), denoted by b-cl(A, B) in the binary space $(X, Y, \mathcal{M})$ where $(A, B) \subseteq (X, Y)$.*

Definition 1.4. [**8**] *The ordered pair $((A, B)^{1*}, (A, B)^{2*})$ defined in proposition 1.2 is called the binary interior of of (A, B), denoted by b-int(A, B). Here $((A, B)^{1*}, (A, B)^{2*})$ is binary open and $((A, B)^{1*}, (A, B)^{2*}) \subseteq (A, B)$.*

Definition 1.5. *A subset (A, B) of a binary topological space $(X, Y, \mathcal{M})$ is called*

(1) *a binary semi open set [**10**] if $(A, B) \subseteq$ b-cl(b-int(A, B)).*
(2) *a binary pre open set [**2**] if $(A, B) \subseteq$ b-int(b-cl(A, B)),*

Definition 1.6. [**9**] *A subset (A, B) of a binary topological space $(X, Y, \mathcal{M})$ is called a binary g-closed set if b-cl$(A, B) \subseteq (U, V)$ whenever $(A, B) \subseteq (U, V)$ and (U, V) is binary open.*

Definition 1.7. [**1**] *A subset (A, B) of a binary topological space $(X, Y, \mathcal{M})$ is called a binary α-open if $(A, B) \subseteq$ b-int(b-cl(b-int(A, B))).*

Lemma 1.1. (1) [**8**] *Let $((K, L), \mathcal{M}|(K, L))$ be a subspace of $(X, Y, \mathcal{M})$. If (K, L) is binary open in (X, Y), then $\mathcal{M}^\alpha|(K, L) = (\mathcal{M}|(K, L))^\alpha$.*
(2) [**1**] *Let $(A, B) \subseteq (C, D) \subseteq (X, Y)$ and (C, D) be binary α-closed in (X, Y). Then (A, B) is binary α-closed in (C, D) implies (A, B) is binary α-closed in (X, Y).*

(3) [**1**] *If* $(A,B) \subseteq (C,D) \subseteq (X,Y)$ *and* (C,D) *is binary closed in* (X,Y), *then* (A,B) *is binary* α*-open in* (X,Y) *implies* (A,B) *is binary* α*-open in (C,D).*

Definition 1.8. [**7**] *A subset* (A,B) *of a binary topological space* $(X,Y,\mathcal{M})$ *is called*

(1) *binary locally closed if* $(A,B) = (E,F) \cap (G,H)$ *where* (E,F) *is binary open and* (G,H) *is binary closed in* (X,Y).

(2) *binary generalized locally closed (briefly bglc) if* $(A,B) = (E,F) \cap (G,H)$ *where* (E,F) *is binary g-open and* (G,H) *is binary g-closed in* (X,Y).

2. Binary α-locally closed sets

Definition 2.1. *A subset* (A,B) *of* (X,Y) *is called binary* α*-locally closed (briefly b*α*-lc) if* $(A,B) = (E,F) \cap (G,H)$ *where* (E,F) *is binary* α*-open in* (X,Y) *and* (G,H) *is binary* α*-closed in* (X,Y).

The set of all binary α-locally closed sets of $(X,Y,\mathcal{M})$ is denoted by $B\alpha$-$LC(X,Y,\mathcal{M})$ (or $B\alpha$-$LC(X,Y)$ if there is no chance of confusion). This set coincides with the set of all binary locally closed sets in $(X,Y,\mathcal{M}^{\alpha})$. Every binary α-open set (resp. binary α-closed set) is $b\alpha$-lc.

Remark 2.1. Every binary locally closed set is binary α-locally closed but not conversely.

example 2.1. *Let* $X = \{a,b\}$, $Y = \{1,2\}$ *and* $\mathcal{M} = \{(\phi,\phi), (\phi,\{1\}), (\{a\},\{1\}), (\{b\},\{1\}), (X,\{1\}), (X,Y)\}$. *Then* $BLC(X,Y) = \{(\phi,\phi), (\phi,\{1\}), (\phi,\{2\}), (\phi,Y), (\{a\},\phi), (\{a\},\{1\}), (\{a\},\{2\}), (\{b\},\phi), (\{b\},\{1\}), (\{b\},\{2\}), (\{b\},Y), (X,\{1\}), (X,\{2\}), (X,Y)\}$ *and* $B\alpha LC(X,Y) = \mathbb{P}(X)$. *Then the subset* $(\{a\},Y)$ *is binary* α*-locally closed but not binary locally closed in* $(X,Y,\mathcal{M})$.

Proposition 2.1. *Let* $(X,Y,\mathcal{M})$ *be a binary* α*-space. Then*

(1) $B\alpha$-$LC(X,Y) = BLC(X,Y)$

(2) $B\alpha$-$LC(X,Y) \subseteq BGLC(X,Y)$

(3) $B\alpha$-$LC(X,Y) \subseteq BGLSC(X,Y)$.

Proof. (1) Since every binary α-open set is binary open and every binary α-closed set is binary closed in (X,Y), $B\alpha$-$LC(X,Y) \subseteq BLC(X,Y)$ and hence $B\alpha$-$LC(X,Y) = BLC(X,Y)$.

(2) and (3) Since $BLC(X,Y) \subseteq BGLC(X,Y)$ and $BLC(X,Y) \subseteq BGLSC(X,Y)$ for any space (X,Y), the proof follows from (1). □

Corollary 2.1. *If* $BSO(X,Y) = \mathcal{M}$, *then* $B\alpha$-$LC(X,Y) \subseteq BGLSC(X,Y) \subseteq BGLC(X,Y)$.

Proof. $BSO(X,Y) = \mathcal{M}$ implies (X,Y) is an binary α-space. Hence by Proposition 2.1(3) $B\alpha$-$LC(X,Y) \subseteq BGLSC(X,Y)$. Let $(A,B) \subseteq BGLSC(X,Y,\mathcal{M})$. Then $(A,B) = (E,F) \cap (G,H)$ where (E,F) is binary g-open and (G,H) is binary semi-closed. Since $BSO(X,Y) = \mathcal{M}$, (G,H) is binary closed and hence binary g-closed. Thus $(A,B) \in BGLC(X,Y,\mathcal{M})$. □

Proposition 2.2. *If $BGO(X,Y)=\mathcal{M}^{\alpha}$, then*

(1) $B\alpha\text{-}LC(X,Y)=BGLC(X,Y)$
(2) $BGLSC(X,Y)=B\alpha\text{-}LC(X,Y)$.

Proof. proof (1)It follows from definitions and hypothesis.

(2) Let $(A,B)\in BGLSC(X,Y)$. Then $(A,B)=(E,F)\cap(G,H)$ where (E,F) is binary g-open and (G,H) is binary semi-closed. By hypothesis, (E,F) is binary α-open and hence binary semi-open. Therefore $(A,B)\in BSLC(X,Y)=B\alpha\text{-}LC(X,Y)$. □

Definition 2.2. *A subset (A,B) of $(X,Y,\mathcal{M})$ is called*

(1) *binary α-lc* if $(A,B)=(E,F)\cap(G,H)$ where (E,F) is binary α-open in (X,Y) and (G,H) is binary closed in (X,Y).*
(2) *binary α-lc** if $(A,B)=(E,F)\cap(G,H)$ where (E,F) is binary open in (X,Y) and (G,H) is binary α-closed in (X,Y).*

The class of all binary α-lc* (resp. binary α-lc**) sets is denoted by $B\alpha\text{-}LC^{\star}(X,Y)$ (resp. $B\alpha\text{-}LC^{\star\star}(X,Y)$). Every binary α-lc* set is binary α-lc. Also every binary α-lc** set is binary α-lc.

Proposition 2.3. *If $BSO(X,Y)=\mathcal{M}$, then $B\alpha\text{-}LC(X,Y)=B\alpha\text{-}LC^{\star}(X,Y)=B\alpha\text{-}LC^{\star\star}(X,Y)$.*

Proof. Since $\mathcal{M}\subseteq\mathcal{M}^{\alpha}\subseteq BSO(X,Y)$ for any space (X,Y), by hypothesis $\mathcal{M}^{\alpha}=\mathcal{M}$. That is, (X,Y) is a binary α-space and hence $B\alpha\text{-}LC(X,Y)=B\alpha\text{-}LC^{\star}(X,Y)=B\alpha\text{-}LC^{\star\star}(X,Y)$. □

The hypothesis in Proposition 2.3 can still be weakened as follows.

Proposition 2.4. *If $BSO(X,Y)\subseteq BLC(X,Y)$, then $B\alpha\text{-}LC(X,Y)=B\alpha\text{-}LC^{\star}(X,Y)=B\alpha\text{-}LC^{\star\star}(X,Y)$.*

Proof. Let $(A,B)\in B\alpha\text{-}LC(X,Y)$. Then $(A,B)=(E,F)\cap(G,H)$ where (E,F) is binary α-open and (G,H) is binary α-closed. Since $\mathcal{M}^{\alpha}\subseteq BSO(X,Y)$, (G,H) is binary semi-closed. $BSO(X,Y)\subseteq BLC(X,Y)$ implies $BSC(X,Y)\subseteq BLC(X,Y)$,

Therefore (G,H) is binary locally closed. Let $(G,H)=(P,Q)\cap(U,V)$ where (P,Q) is binary open and (U,V) is binary closed. Hence $(A,B)=((E,F)\cap(P,Q))\cap(U,V)$ where $(E,F)\cap(P,Q)$ is binary α-open and (U,V) is binary closed. Hence $(A,B)\in B\alpha\text{-}LC^{\star}(X,Y)$. For any space (X,Y), $B\alpha\text{-}LC^{\star}(X,Y)\subseteq B\alpha\text{-}LC(X,Y)$. Thus $B\alpha\text{-}LC(X,Y)=B\alpha\text{-}LC^{\star}(X,Y)$.

Let $(C,D)\in B\alpha\text{-}LC(X,Y)$, then $(C,D)=(E,F)\cap(G,H)$ where (E,F) is binary α-open and (G,H) is binary α-closed. $BSO(X,Y)\subseteq BLC(X,Y)$ implies $\mathcal{M}^{\alpha}\subseteq BLC(X,Y)$ and so (E,F) is binary locally closed. Let $(E,F)=(P,Q)\cap(U,V)$ where (P,Q) is binary open and (U,V) is binary closed. Hence $(C,D)=(P,Q)\cap((U,V)\cap(G,H))$ where $(U,V)\cap(G,H)$ is binary α-closed. Then $(C,D)\in B\alpha\text{-}LC^{\star\star}(X,Y)$. That is, $B\alpha\text{-}LC(X,Y)\subseteq B\alpha\text{-}LC^{\star\star}(X,Y)$. For any space (X,Y), $B\alpha\text{-}LC^{\star\star}(X,Y)\subseteq B\alpha\text{-}LC(X,Y)$. Thus $B\alpha\text{-}LC(X,Y)=B\alpha\text{-}LC^{\star\star}(X,Y)$. □

PROPOSITION 2.5. *If $(A, B) \subseteq (X, Y)$ is binary pre open and binary α-locally closed, then it is binary semi-open.*

PROOF. As $(A, B) \in B\alpha\text{-}LC(X, Y)$, (A, B) is a binary δ-set. That is, $b\text{-}int(b\text{-}cl(A, B)) \subseteq b\text{-}cl(b\text{-}int(A, B))$. As (A, B) is binary pre open $(A, B) \subseteq b\text{-}int(b\text{-}cl(A, B))$. That is, $(A, B) \subseteq b\text{-}cl(b\text{-}int(A, B))$. Hence (A, B) is binary semi-open. □

THEOREM 2.1. *Let (A, B) and (C, D) be subsets of $(X, Y, \mathcal{M})$.*

(1) *If $(A, B), (C, D) \in B\alpha\text{-}LC(X, Y)$, then $(A, B) \cap (C, D) \in B\alpha\text{-}LC(X, Y)$.*
(2) *If $(A, B), (C, D) \in B\alpha\text{-}LC^{\star}(X, Y)$, then $(A, B) \cap (C, D) \in B\alpha\text{-}LC^{\star}(X, Y)$.*
(3) *If $(A, B), (C, D) \in B\alpha\text{-}LC^{\star\star}(X, Y)$, then $(A, B) \cap (C, D) \in B\alpha\text{-}LC^{\star\star}(X, Y)$.*
(4) *If $(A, B) \in B\alpha\text{-}LC(X, Y)$ and (C, D) is binary α-open (resp. binary α-closed), then $(A, B) \cap (C, D) \in B\alpha\text{-}LC(X, Y)$.*
(5) *If $(A, B) \in B\alpha\text{-}LC^{\star}(X, Y)$ and (C, D) is binary α-open (resp. closed), then $(A, B) \cap (C, D) \in B\alpha\text{-}LC^{\star}(X, Y)$.*
(6) *If $(A, B) \in B\alpha\text{-}LC^{\star\star}(X, Y)$ and (C, D) is binary α-closed (resp. open), then $(A, B) \cap (C, D) \in B\alpha\text{-}LC^{\star\star}(X, Y)$.*
(7) *If $(A, B) \in B\alpha\text{-}LC^{\star}(X, Y)$ and (C, D) is binary α-closed, then $(A, B) \cap (C, D) \in B\alpha\text{-}LC(X, Y)$.*
(8) *If $(A, B) \in B\alpha\text{-}LC^{\star\star}(X, Y)$ and (C, D) is binary α-open, then $(A, B) \cap (C, D) \in B\alpha\text{-}LC(X, Y)$.*
(9) *If $(A, B) \in B\alpha\text{-}LC^{\star\star}(X, Y)$ and $(C, D) \in BGLC^{\star}(X, Y)$, then $(A, B) \cap (C, D) \in BGLSC(X, Y)$.*

PROOF. Since $(X, Y, \mathcal{M}^{\alpha})$ is a binary topology, (1) to (8) hold. (9) Let $(A, B) = (E, F) \cap (G, H)$ where (E, F) is binary open and (G, H) is binary α-closed and let $(C, D) = (P, Q) \cap (U, V)$ where (P, Q) is binary g-open and (U, V) is binary closed. Now, $(A, B) \cap (C, D) = ((E, F) \cap (P, Q)) \cap ((G, H) \cap (U, V))$ where $(E, F) \cap (P, Q)$ is binary g-open and $(G, H) \cap (U, V)$ is binary α-closed and so binary semi-closed. Hence $(A, B) \cap (C, D) \in BGLSC(X, Y)$. □

THEOREM 2.2. (1) *$(A, B) \in B\alpha\text{-}LC(X, Y)$ if and only if there exists a binary α-open set (E, F) such that $(A, B) = (E, F) \cap b\text{-}\alpha cl(A, B)$.*
(2) *$(A, B) \in B\alpha\text{-}LC^{\star}(X, Y)$ if and only if there exists a binary α-open set (E, F) such that $(A, B) = (E, F) \cap b\text{-}cl(A, B)$.*
(3) *$(A, B) \in B\alpha\text{-}LC^{\star\star}(X, Y)$ if and only if there exists a binary open set (E, F) such that $(A, B) = (E, F) \cap b\text{-}\alpha cl(A, B)$.*

PROOF. (1) (Necessity) Let $(A, B) \in B\alpha\text{-}LC(X, Y)$. Then $(A, B) = (E, F) \cap (G, H)$ where (E, F) is binary α-open and (G, H) is binary α-closed. As $(A, B) \subseteq (G, H)$, $b\text{-}\alpha cl(A, B) \subseteq (G, H)$ and so $(E, F) \cap b\text{-}\alpha cl(A, B) \subseteq (A, B)$. $(A, B) \subseteq (E, F)$ and $(A, B) \subseteq b\text{-}\alpha cl(A, B)$ implies $(A, B) \subseteq (E, F) \cap b\text{-}\alpha cl(A, B)$. Therefore $(A, B) = (E, F) \cap b\text{-}\alpha cl(A, B)$.

(Sufficiency) Let $(A, B) = (E, F) \cap b\text{-}\alpha cl(A, B)$ where (E, F) is binary α-open. By definition, $(A, B) \in B\alpha\text{-}LC(X, Y)$. Proofs of (2) and (3) are similar to (1). □

PROPOSITION 2.6. *For a subset (A,B) of $(X,Y,\mathcal{M})$, the following are equivalent.*

(1) *(A,B) is binary α-locally closed*
(2) *$(A,B)=(U,V)\cap b\text{-}\alpha cl(A,B)$ for some binary α-open set (U,V).*
(3) *$b\text{-}\alpha cl(A,B)-(A,B)$ is binary α-closed.*
(4) *$(A,B)\cup((X,Y)-b\text{-}\alpha cl(A,B))$ is binary α-open.*
(5) *$(A,B)\subseteq b\text{-}\alpha int((A,B)\cup((X,Y)-b\text{-}\alpha cl(A,B)))$.*

PROOF. (1) $\Rightarrow$ (2) It follows from Theorem 2.2 (1).

(2) $\Rightarrow$ (3) $(A,B)=(U,V)\cap b\text{-}\alpha cl(A,B)$ implies $b\text{-}\alpha cl(A,B)-(A,B)=b\text{-}\alpha cl(A,B)\cap((X,Y)-(U,V))$ which is binary α-closed since $(X,Y)-(U,V)$ is so.

(3) $\Rightarrow$ (4) $(A,B)\cup((X,Y)-b\text{-}\alpha cl(A,B))=(X,Y)-(b\text{-}\alpha cl(A,B)-(A,B))$. By (3) $(A,B)\cup((X,Y)-b\text{-}\alpha cl(A,B))$ is binary α-open.

(4) $\Rightarrow$ (5) $(A,B)\cup((X,Y)-b\text{-}\alpha cl(A,B))$ is binary α-open implies $(A,B)\cup((X,Y)-b\text{-}\alpha cl(A,B))=b\text{-}\alpha int((A,B)\cup((X,Y)-b\text{-}\alpha cl(A,B)))$. Thus (5) holds.

(5) $\Rightarrow$ (1) $(A,B)=b\text{-}\alpha int((A,B)\cup((X,Y)-b\text{-}\alpha cl(A,B)))\cap b\text{-}\alpha cl(A,B)$, so $(A,B)\in B\alpha\text{-}LC(X,Y)$. □

DEFINITION 2.3. *A subset (A,B) of $(X,Y,\mathcal{M})$ is called binary α-dense if $b\text{-}\alpha cl(A,B)=(X,B)$.*

DEFINITION 2.4. *A space $(X,Y,\mathcal{M})$ is called binary α-submaximal if every binary α-dense subset of (X,Y) is binary α-open in (X,Y).*

EXAMPLE 2.2. *Let $X=\{a,b\}$, $Y=\{1,2\}$ and $\mathcal{M}=\{(\phi,\phi),(\phi,\{1\}),(\{a\},\{1\}),(X,Y)\}$. Here binary α-dense subsets are $(\phi,\{1\})$, (ϕ,Y), $(\{a\},\{1\})$, $(\{a\},Y)$, $(\{b\},\{1\})$, $(\{b\},Y)$, $(X,\{1\})$ and (X,Y). Then $B\alpha O(X,Y)=\{(\phi,\phi),(\phi,\{1\}),(\phi,Y),(\{a\},\{1\}),(\{a\},Y),(\{b\},\{1\}),(\{b\},Y),(X,\{1\}),(X,Y)\}$. Every binary α-dense subset is binary α-open and so (X,Y) is binary α-submaximal.*

THEOREM 2.3. *A space $(X,Y,\mathcal{M})$ is binary α-submaximal if an only if every subset of $(X,Y,\mathcal{M})$ is binary α-locally closed.*

PROOF. (Necessity) Let $(A,B)\in\mathbb{P}(X,Y)$ and $(U,V)=(A,B)\cup((X,Y)-b\text{-}\alpha cl(A,B))$. Then $b\text{-}\alpha cl(U,V)=(X,Y)$. That is, (U,V) is binary α-dense in (X,Y). By hypothesis (U,V) is binary α-open. By Proposition 2.6, (A,B) is binary α-*lc*. Hence $\mathbb{P}(X,Y)\subseteq B\alpha\text{-}LC(X,Y)$ and this implies $\mathbb{P}(X,Y)=B\alpha\text{-}LC(X,Y)$.

(Sufficiency) Let (A,B) be binary α-dense in (X,Y). Then $(A,B)\cup((X,Y)-b\text{-}\alpha cl(A,B))=(A,B)$. By hypothesis (A,B) is binary α-*lc*. Then $(A,B)\cup((X,Y)-b\text{-}\alpha cl(A,B))$ is binary α-open by Proposition 2.6. That is, $(X,Y,\mathcal{M})$ is binary α-submaximal. □

PROPOSITION 2.7. *If $(X,Y,\mathcal{M})$ is binary submaximal, then it is binary α-submaximal.*

PROOF. Let (A,B) be binary α-dense in (X,Y). Then (A,B) is binary dense and so binary open as (X,Y) is binary submaximal. Since every binary open set is binary α-open, (A,B) is binary α-open and hence (X,Y) is binary α-submaximal. $\square$

LEMMA 2.1. *Let (U,V) be binary open in $(X,Y,\mathcal{M})$ and $(E,F)\subseteq(U,V)$. If (E,F) is binary α-open in $((U,V),\mathcal{M}|(U,V))$, then (E,F) is binary α-open in $(X,Y,\mathcal{M})$*

PROOF. Since (E,F) is binary α-open in $((U,V),\mathcal{M}|(U,V))$, $(E,F)\in(\mathcal{M}|(U,V))^{\alpha}$. But $(\mathcal{M}|(U,V))^{\alpha}=\mathcal{M}^{\alpha}|(U,V)$ as (U,V) is binary open in (X,Y) by 1.1. Hence $(E,F)\in\mathcal{M}^{\alpha}|(U,V)$ and so $(E,F)=(C,D)\cap(U,V)$ where (C,D) is binary α-open in (X,Y). Now openness of (U,V) implies $(E,F)\in\mathcal{M}^{\alpha}$. $\square$

PROPOSITION 2.8. *Let (A,B) and (U,V) be subsets of $(X,Y,\mathcal{M})$ such that $(A,B)\subseteq(U,V)$ and (U,V) is binary open and binary α-closed in (X,Y). Then*

(1) *If $(A,B)\in B\alpha\text{-}LC((U,V),\mathcal{M}|(U,V))$, then $(A,B)\in B\alpha\text{-}LC(X,Y,\mathcal{M})$.*
(2) *If $(A,B)\in B\alpha\text{-}LC^{\star}((U,V),\mathcal{M}|(U,V))$, then $(A,B)\in B\alpha\text{-}LC^{\star}(X,Y,\mathcal{M})$.*
(3) *If $(A,B)\in B\alpha\text{-}LC^{\star\star}((U,V),\mathcal{M}|(U,V))$, then $(A,B)\in B\alpha\text{-}LC^{\star\star}(X,Y,\mathcal{M})$.*

PROOF. , (1) Let $(A,B)\in B\alpha\text{-}LC((U,V),\mathcal{M}|(U,V))$. Then $(A,B)=(E,F)\cap(G,H)$ where (E,F) is binary α-open and (G,H) is binary α-closed in $((U,V),\mathcal{M}|(U,V))$. Then by Lemma 1.1(2) and Lemma 2.1, (E,F) is binary α-open and (G,H) is binary α-closed in $(X,Y,\mathcal{M})$. So $(A,B)\in B\alpha\text{-}LC(X,Y,\mathcal{M})$.

(2) If $(A,B)\in B\alpha\text{-}LC^{\star}((U,V),\mathcal{M}|(U,V))$, then $(A,B)=(E,F)\cap b\text{-}cl_{(U,V)}(A,B)$ where (E,F) is binary α-open in $((U,V),\mathcal{M}|(U,V))$ by Theorem 2.2. But $b\text{-}cl_{(U,V)}(A,B)=b\text{-}cl(A,B)\cap(U,V)$, So $(A,B)=((E,F)\cap(U,V))\cap b\text{-}cl(A,B)$. By Lemma 2.1 and the fact that $\mathcal{M}^{\alpha}$ is a binary topology, $(E,F)\cap(U,V)$ is binary α-open in (X,Y). Hence $(A,B)\in B\alpha\text{-}LC^{\star}(X,Y,\mathcal{M})$.

(3) If $(A,B)\in B\alpha\text{-}LC^{\star\star}((U,V),\mathcal{M}|(U,V))$. Then $(A,B)=(E,F)\cap(G,H)$ where (E,F) is binary open and (G,H) is binary α-closed in $((U,V),\mathcal{M}|(U,V))$. Then by Lemma 1.1(2), (E,F) is binary open and (G,H) is binary α-closed in $(X,Y,\mathcal{M})$. So $(A,B)\in B\alpha\text{-}LC^{\star\star}(X,Y,\mathcal{M})$. $\square$

DEFINITION 2.5. *Let $(A,B),(C,D)\subseteq(X,Y)$. Then (A,B) and (C,D) are said to be binary α-separated if $(A,B)\cap b\text{-}\alpha cl(C,D)=(\phi,\phi)$ and $(C,D)\cap b\text{-}\alpha cl(A,B)=(\phi,\phi)$.*

PROPOSITION 2.9. *Let $(A,B),(C,D)\in B\alpha\text{-}LC(X,Y,\mathcal{M})$. If (A,B) and (C,D) are binary α-separated then $(A,B)\cup(C,D)\in B\alpha\text{-}LC(X,Y,\mathcal{M})$.*

PROOF. As $(A,B),(C,D)\in B\alpha\text{-}LC(X,Y,\mathcal{M})$, by Theorem 2.2, there exists binary α-open sets (E,F) and (G,H) of (X,Y) such that $(A,B)=(E,F)\cap b\text{-}\alpha cl(A,B)$ and $(C,D)=(G,H)\cap b\text{-}\alpha cl(C,D)$. Put $(U,V)=(E,F)\cap((X,Y)-b\text{-}\alpha cl(C,D)))$ and $(P,Q)=(J,K)\cap((X,Y)-b\text{-}\alpha cl(A,B))$. Then $(A,B)=(U,V)\cap b\text{-}\alpha cl(A,B)$ and $(C,D)=(P,Q)\cap b\text{-}\alpha cl(C,D)$ hold and $(U,V)\cap b\text{-}\alpha cl(C,D)=(\phi,\phi)$ and $(P,Q)\cap b\text{-}\alpha cl(A,B)=(\phi,\phi)$. Since $\mathcal{M}^{\alpha}$ is a binary topology, (U,V) and (P,Q) are binary α-open sets of (X,Y). Also $(U,V)\cup(P,Q)$ is binary α-open. Now $((U,V)\cup(P,Q))\cap b\text{-}\alpha cl((A,B)\cup(C,D))=(A,B)\cup(C,D)$. Hence $(A,B)\cup(C,D)\in B\alpha\text{-}LC(X,Y,\mathcal{M})$. $\square$

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