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REGULAR GAMMA NEARNESS SEMIGROUPS

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Abstract. This paper is concerned with basic concepts and some results on regular Γ-nearness semigroup and ideals of a Γ-nearness semigroup. Also, it is given some properties about ideals of a regular Γ-nearness semigroup and an example about the subject. Furthermore, we study relations among ideals of a Γ-nearness semigroup.

1. Introduction

Semigroups were utilized in many areas of theoretical computer science, graph theory, optimization theory and in particular for studying automata, coding theory and formal languages. In 1963, Saito first investigated the notion of regular element as a generalization in the semigroup theory of the concept of inverse element in the group theory [**20**]. This subject of regular element has been effectively used in the ideal theory of semigroups in Miller and Clifford [**6**]. Readers can find all fundamental concepts concerning semigroups in [**3**].

Firstly, Nobusawa introduced Γ-ring in 1964 [**5**]. Also, Barnes studied Γ-ring after Nobusawa in [**1**]. Barnes defined a different concept of Γ-ring and gave related definitions. Many researchers investigated the Γ-ring in the sense of Barnes and Nobusawa and found parallel results in ring theory.

In 1981, the concept of Γ-semigroups in the sense of Nobusawa was introduced and some examples were given by Sen [**22**]. In 1986, Sen and Saha studied Γ-semigroup in the sense of Barnes and gave some examples (See [**23**]). Then Rao introduced Γ-semirings in the sense of Barnes [**18**] as a generalization of Γ-ring and

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semiring. Readers can find all fundamental concepts concerning Γ-semirings and semirings in [**18**], [**19**], [**21**] and [**2**].

Rough set theory studied by Pawlak can be seen as a new mathematical approach to uncertainty [**14**]. The rough set idea is about assumption that every object of the universe of discourse we deal with have some information. Afterwards, Peters defined an indiscernibility relation by using the features of the objects to determine the nearness of the objects [**15**] in 2002 as a generalization of rought set theory. Moreover, he studied approach theory of the nearness of non-empty sets resembling each other [**16, 17**].

In 2012, İnan and Öztürk investigated the concept of nearness semigroups [**4**]. Then, in 2018, Öztürk was defined nearness semirings (i.e., semirings on weak nearness approximation spaces) and to deal with its basic properties [**7**]. Afterward, in 2019, Öztürk et al. studied gamma nearness semigroups [**8**]. After these studies, some mathematicians continued this subject with [**8**], [**9**], [**10**] [**8**], [**9**], and [**10**] studies.

In this paper, we introduced the concept of regular Γ-nearness semigroups, interior Γ-nearness ideals and also studied some properties. Furthermore, we investigated some features of bi-Γ-nearness ideals and quasi Γ-nearness ideals in regular Γ-nearness semigroups.

2. Preliminaries

Definition 2.1. [**22**] *Let S and Γ be any two nonempty sets. S is called a Γ-semigroup (in the sense of Nobusawa) if the following assertions are satisfied: For all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$,*

1) *$a\alpha b \in S$ and $\alpha a \beta \in \Gamma$,*
2) *$(a\alpha b)\beta c = a(\alpha b\beta)c = a\alpha(b\beta c)$.*

Definition 2.2. [**23**] *Let S and Γ be any two nonempty sets. S is called a Γ-semigroup (in the sense of Barnes) if the following assertions are satisfied: For all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$,*

1) *$a\alpha b \in S$,*
2) *$(a\alpha b)\beta c = a\alpha(b\beta c)$.*

Definition 2.3. [**23**] *Let S be a Γ-semigroup. An element a of S is called regular provided $a = a\alpha x\beta a$ for some $x \in S$, $\alpha, \beta \in \Gamma$, i.e., $a \in a\Gamma S\Gamma a$, where $a\Gamma S\Gamma a = \{a\alpha x\beta a \mid x \in S; \alpha, \beta \in \Gamma\}$. A Γ-semigroup S is called a regular Γ-semigroup provided every element is regular.*

A nearness approximation space is a tuple $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B), \nu_{N_r})$ where the nearness approximation space is defined with a set of perceived objects $\mathcal{O}$, set of probe functions $\mathcal{F}$ representing object features, indiscernibility relation $\sim_{B_r}$ defined relative to $B_r \subseteq B \subseteq \mathcal{F}$, collection of partitions (families of neighbour-hoods) $N_r(B)$, and overlap function ν_{N_r}.

Definition 2.4. [**8**] *Let $\mathcal{O}$ be a set of sample objects, $\mathcal{F}$ a set of the probe functions, $\sim_{B_r}$ an indiscernibility relation, and N_r a collection of partitions. Then, $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B))$ is called a weak nearness approximation space.*

THEOREM 2.1. [**8**] *Let $(\mathcal{O},\mathcal{F},\sim_{B_r},N_r(B))$ be a weak nearness approximation space and $X,Y\subset\mathcal{O}$. Then the following statements hold:*
i) $X\subseteq N_r(B)^*X$,
ii) $N_r(B)^*(X\cup Y)=N_r(B)^*X\cup N_r(B)^*Y$,
iii) $X\subseteq Y$ *implies* $N_r(B)^*X\subseteq N_r(B)^*Y$,
iv) $N_r(B)^*(X\cap Y)\subseteq N_r(B)^*X\cap N_r(B)^*Y$.

DEFINITION 2.5. [**8**] *Let $S=\{x,y,z,\ldots\}\subseteq\mathcal{O}$, and $\Gamma=\{\alpha,\beta,\gamma,\ldots\}\subseteq\mathcal{O}'$, where $(\mathcal{O},\mathcal{F},\sim_{B_r},N_r(B))$ and $(\mathcal{O}',\mathcal{F},\sim_{B_r},N_r(B))$ are two different weak near approximation spaces. If the following properties are satisfied, then S is called a Γ-semigroup on weakly approximate approximation spaces $\mathcal{O}-\mathcal{O}'$, or, in short, a Γ-nearness semigroup.*
i) $x\gamma y\in N_r(B)^*S$ *for all* $x,y\in S$ *and* $\gamma\in\Gamma$*;*
ii) $(x\beta y)\gamma z=x\beta(y\gamma z)$ *property holds in* $N_r(B)^*S$ *for all* $x,y\in S$ *and* $\beta,\gamma\in\Gamma$.

Let S be a Γ-semigroup on weakly approximate approximation spaces $\mathcal{O}-\mathcal{O}'$, where $(\mathcal{O},\mathcal{F},\sim_{B_r},N_r(B))$ and $(\mathcal{O}',\mathcal{F},\sim_{B_r},N_r(B))$ are two different weak near approximation spaces. If $\mathcal{O}=\mathcal{O}'$, then S is a Γ-semigroup on weakly approximate approximation spaces $\mathcal{O}$.

DEFINITION 2.6. [**8**] *Let $S\subseteq\mathcal{O}$ and $\Gamma\subseteq\mathcal{O}'$, where $(\mathcal{O},\mathcal{F},\sim_{B_r},N_r(B))$ and $(\mathcal{O}',\mathcal{F},\sim_{B_r},N_r(B))$ are two different weak near approximation spaces, $B_r\subseteq\mathcal{F}$ where $r\leqslant|B|$ and $B\subseteq\mathcal{F}$, $\sim_{B_r}$ be a indiscernibility relation on $\mathcal{O}-\mathcal{O}'$. Then, $\sim_{B_r}$ is called a congruence indiscernibility relation on Γ-nearness semigroup S, if $x\sim_{B_r}y$, where $x,y\in S$ implies $x\gamma a\sim_{B_r}y\gamma a$ and $a\gamma x\sim_{B_r}a\gamma y$ for all $a\in S$ and $\gamma\in\Gamma$.*

PROPOSITION 2.1. [**8**] *Let $S\subseteq\mathcal{O}$ and $\Gamma\subseteq\mathcal{O}'$, where $(\mathcal{O},\mathcal{F},\sim_{B_r},N_r(B))$ and $(\mathcal{O}',\mathcal{F},\sim_{B_r},N_r(B))$ are two different weak near approximation spaces, S be a Γ-nearness semigroup. If $\sim_{B_r}$ is a congruence indiscernibility relation on S, then $[x]_{B_r}\gamma[y]_{B_r}\subseteq[x\gamma y]_{B_r}$ for all $x,y\in S$ and $\gamma\in\Gamma$.*

S be a Γ-nearness semigroup. Let $X\Gamma Y=\{x\gamma y\mid x\in X,\gamma\in\Gamma$, and $y\in Y\}$, where subsets X and Y of S.

LEMMA 2.1. [**8**] *Let $S\subseteq\mathcal{O}$ and $\Gamma\subseteq\mathcal{O}'$, where $(\mathcal{O},\mathcal{F},\sim_{B_r},N_r(B))$ and $(\mathcal{O}',\mathcal{F},\sim_{B_r},N_r(B))$ are two different weak near approximation spaces, S be a Γ-nearness semigroup. If $X,Y\subseteq S$, then $(N_r(B)^*X)\Gamma(N_r(B)^*Y)\subseteq N_r(B)^*(X\Gamma Y)$.*

DEFINITION 2.7. [**8**] *Let $S\subseteq\mathcal{O}$ and $\Gamma\subseteq\mathcal{O}'$, where $(\mathcal{O},\mathcal{F},\sim_{B_r},N_r(B))$ and $(\mathcal{O}',\mathcal{F},\sim_{B_r},N_r(B))$ are two different weak near approximation spaces, $B_r\subseteq\mathcal{F}$ where $r\leqslant|B|$and $B\subseteq\mathcal{F}$, $\sim_{B_r}$be a indiscernibility relation on $\mathcal{O}-\mathcal{O}'$. Then, $\sim_{B_r}$ is called a complete congruence indiscernibility relation on Γ-nearness semigroup S, if $[x]_{B_r}\gamma[y]_{B_r}=[x\gamma y]_{B_r}$ for all $x,y\in S$ and $\gamma\in\Gamma$.*

DEFINITION 2.8. [**11**] *Let $S=\{x,y,z,\ldots\}\subseteq\mathcal{O}$, and $\Gamma=\{\alpha,\beta,\gamma,\ldots\}\subseteq\mathcal{O}'$, where $(\mathcal{O},\mathcal{F},\sim_{B_r},N_r(B))$ and $(\mathcal{O}',\mathcal{F},\sim_{B_r},N_r(B))$ are two different weak near approximation spaces. S is called a Γ-semiring on $\mathcal{O}-\mathcal{O}'$ if the following properties hold:*

$NGSR_1$) $(S,+)$ is an abelian monoid on $\mathcal{O}$ with identity element 0_S,
$NGSR_2$) $(\Gamma,+)$ is an abelian monoid on $\mathcal{O}$ with identity element 0_Γ,
$NGSR_3$) $(S,\cdot)$ is a Γ-monoid on $\mathcal{O}-\mathcal{O}'$ with identity element 1_S,
$NGSR_4$) For all $x,y,z\in S$, and $\gamma,\beta\in\Gamma$

$$\begin{aligned} i)\ & x\gamma\left(y+z\right)=\left(x\gamma y\right)+\left(x\gamma z\right), \\ ii)\ & x(\beta+\gamma)z=\left(x\beta z\right)+\left(y\gamma z\right), \\ iii)\ & (x+y)\gamma z=x\gamma z+y\gamma z. \end{aligned}$$

hold in $N_r\left(B\right)^* S$,
$NGSR_5$) For all $x\in S$ and $\gamma\in\Gamma$,

$$0_S\gamma x=0_S=x\gamma 0_S$$

hold in $N_r\left(B\right)^* S$,
$NGSR_6$) $1_S\neq 0_S$.

Let S be a Γ-semiring on weak near approximation spaces $\mathcal{O}-\mathcal{O}'$. If $\mathcal{O}=\mathcal{O}'$, then S is a Γ-semiring on weakly approximate approximation spaces $\mathcal{O}$, i.e. S is a Γ-nearness semiring.

THEOREM 2.2. [**8**] *Let S be a Γ-nearness semiring, $\sim_{B_r}$ be a complete congruence indiscernibility relation on S, and X, Y be two non-empty subsets of S. Then $(N_r\left(B\right)^* X)\Gamma(N_r\left(B\right)^* Y)=N_r\left(B\right)^*\left(X\Gamma Y\right)$.*

DEFINITION 2.9. ([**8**]) *Let S be a Γ-semigroup on $\mathcal{O}-\mathcal{O}'$, where $(\mathcal{O},\mathcal{F},\sim_{B_r},N_r(B))$ and $(\mathcal{O}',\mathcal{F},\sim_{B_r},N_r(B))$ are two different weak near approximation spaces, and A a non-empty subset of S.*

i) A is called a sub Γ-semigroup of S on $\mathcal{O}-\mathcal{O}'$ if $A\Gamma A\subseteq N_r\left(B\right)^* A$.

ii) A is called a upper-near sub Γ-semigroup of S on $\mathcal{O}-\mathcal{O}'$ if $(N_r\left(B\right)^* A)\Gamma(N_r\left(B\right)^* A)\subseteq N_r\left(B\right)^* A$.

DEFINITION 2.10. ([**8**]) *Let S be a Γ-semigroup on $\mathcal{O}-\mathcal{O}'$, where $(\mathcal{O},\mathcal{F},\sim_{B_r},N_r(B))$ and $(\mathcal{O}',\mathcal{F},\sim_{B_r},N_r(B))$ are two different weak near approximation spaces and A be a sub Γ-semigroup of S on $\mathcal{O}-\mathcal{O}'$.*

i) A is called a right (resp. left) Γ-ideals of S on $\mathcal{O}-\mathcal{O}'$ if $A\Gamma S\subseteq N_r\left(B\right)^* A$ ($S\Gamma A\subseteq N_r\left(B\right)^* A$).

ii) A is called a upper-near right (resp. left) Γ-ideals of S on $\mathcal{O}-\mathcal{O}'$ if $(N_r\left(B\right)^* A)\Gamma S\subseteq N_r\left(B\right)^* A$ (resp. $S\Gamma(N_r\left(B\right)^* A)\subseteq N_r\left(B\right)^* A$).

A Γ-nearness ideal of S is both left as well as right Γ-nearness ideal.

THEOREM 2.3. ([**8**]) *Let S be a Γ-semigroup on $\mathcal{O}-\mathcal{O}'$, where $(\mathcal{O},\mathcal{F},\sim_{B_r},N_r(B))$ and $(\mathcal{O}',\mathcal{F},\sim_{B_r},N_r(B))$ are two different weak near approximation spaces. The following properties hold:*

i) If $\varnothing\neq A\subseteq S$ and $A\Gamma S\subseteq A$ (resp. $S\Gamma A\subseteq A$), then A is a upper-near right (resp. left) Γ-ideal of S on $\mathcal{O}-\mathcal{O}'$.

ii) If A is a right (resp. left) Γ-ideal of S and $N_r\left(B\right)^*\left(N_r\left(B\right)^* A\right)=N_r\left(B\right)^* A$, then A is a upper-near right (resp. left) Γ-ideal of S on $\mathcal{O}-\mathcal{O}'$.

DEFINITION 2.11. ([**8**]) *Let S be a Γ-semigroup on $\mathcal{O}-\mathcal{O}'$, where $(\mathcal{O},\mathcal{F},\sim_{B_r},N_r(B))$ and $(\mathcal{O}',\mathcal{F},\sim_{B_r},N_r(B))$ are two different weak near approximation spaces and A be a sub Γ-semigroup of S.*

i) A is called bi-Γ-nearness ideal if $A\Gamma S\Gamma A\subseteq N_r(B)^* A$.

ii) A is called a upper-near bi-Γ-nearness ideal of S if $(N_r(B)^* A)\Gamma S\Gamma(N_r(B)^* A)\subseteq N_r(B)^* A$.

THEOREM 2.4. ([**8**]) *Let S be a Γ-semigroup on $\mathcal{O}-\mathcal{O}'$, where $(\mathcal{O},\mathcal{F},\sim_{B_r},N_r(B))$ and $(\mathcal{O}',\mathcal{F},\sim_{B_r},N_r(B))$ are two different weak near approximation spaces. The following properties hold:*

i) If $\varnothing\neq A\subseteq S$ and $A\Gamma S\subseteq A$ (resp. $S\Gamma A\subseteq A$), then A is a upper-near right (resp. left) Γ-ideal of S on $\mathcal{O}-\mathcal{O}'$.

ii) If A is a right (resp. left) Γ-ideal of S and $N_r(B)^*(N_r(B)^* A)=N_r(B)^* A$, then A is a upper-near right (resp. left) Γ-ideal of S on $\mathcal{O}-\mathcal{O}'$.

3. Regular Γ-nearness semigroups

DEFINITION 3.1. *Let S be a Γ-nearness semigroup and I be a sub Γ-nearness semigroup of S.*

i) I is called interior Γ-nearness ideal of S if $S\Gamma I\Gamma S\subseteq N_r(B)^* I$.

ii) I is called upper-near interior Γ-nearness ideal of S if $S\Gamma(N_r(B)^* I)\Gamma S\subseteq N_r(B)^* I$.

EXAMPLE 3.1. *Let $\mathcal{O}=\{o,a,b,c,d,e,f,i\}$ be a set of perceptual objects, where*

$$o=\begin{bmatrix}0&0&0\end{bmatrix},a=\begin{bmatrix}1&0&0\end{bmatrix},b=\begin{bmatrix}0&1&0\end{bmatrix},c=\begin{bmatrix}0&0&1\end{bmatrix},$$
$$d=\begin{bmatrix}1&1&0\end{bmatrix},e=\begin{bmatrix}0&1&1\end{bmatrix},f=\begin{bmatrix}1&0&1\end{bmatrix},i=\begin{bmatrix}1&1&1\end{bmatrix}$$

for universal set $U=\{\ [a_{ij}]_{1x3}\mid a_{ij}\in\mathbb{Z}_2\}$, $\mathcal{O}'=\{\theta,\alpha,\beta,\gamma,\lambda,\mu,\delta,\sigma\}$ be a set of perceptual objects, where

$$\theta=\begin{bmatrix}0\\0\\0\end{bmatrix},\alpha=\begin{bmatrix}1\\0\\0\end{bmatrix},\beta=\begin{bmatrix}0\\1\\0\end{bmatrix},\gamma=\begin{bmatrix}0\\0\\1\end{bmatrix},$$
$$\lambda=\begin{bmatrix}1\\1\\0\end{bmatrix},\mu=\begin{bmatrix}0\\1\\1\end{bmatrix},\delta=\begin{bmatrix}1\\0\\1\end{bmatrix},\sigma=\begin{bmatrix}1\\1\\1\end{bmatrix}$$

for universal set $U'=\{\ [a_{ij}]_{3x1}\mid a_{ij}\in\mathbb{Z}_2\}$. $B=\{\psi_1,\psi_2,\psi_3\}\subseteq\mathcal{F}$ be a set of probe functions, $S=\{a,c,d,i\}\subset\mathcal{O}$, $I=\{a,d\}\subseteq S$, and $\Gamma=\{\alpha,\gamma,\sigma\}\subset\mathcal{O}'$. For $r=1$, values of the probe functions

$$\psi_1:\mathcal{O}\to V_1=\{\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4\},$$
$$\psi_2:\mathcal{O}\to V_2=\{\vartheta_1,\vartheta_3,\vartheta_4,\vartheta_5\},$$
$$\psi_3:\mathcal{O}\to V_3=\{\vartheta_2,\vartheta_3,\vartheta_4,\vartheta_5,\vartheta_6\}$$

are given in the following table.

	o	a	b	c	d	e	f	i
ψ_1	ϑ_1	ϑ_2	ϑ_3	ϑ_4	ϑ_2	ϑ_4	ϑ_3	ϑ_4
ψ_2	ϑ_1	ϑ_3	ϑ_1	ϑ_3	ϑ_4	ϑ_5	ϑ_4	ϑ_5
ψ_3	ϑ_2	ϑ_5	ϑ_6	ϑ_3	ϑ_4	ϑ_2	ϑ_2	ϑ_5

In this case,

$$\begin{aligned}
[o]_{\psi_1} &= \{x \in \mathcal{O} \mid \psi_1(x) = \psi_1(o) = \vartheta_1\} = \{o\}, \\
[a]_{\psi_1} &= \{x \in \mathcal{O} \mid \psi_1(x) = \psi_1(a) = \vartheta_2\} = \{a, d\} \\
&= [d]_{\psi_1}, \\
[b]_{\psi_1} &= \{x \in \mathcal{O} \mid \psi_1(x) = \psi_1(b) = \vartheta_3\} = \{b, f\} \\
&= [f]_{\varphi_1}, \\
[c]_{\psi_1} &= \{x \in \mathcal{O} \mid \psi_1(x) = \psi_1(c) = \vartheta_4\} = \{c, e, i\} \\
&= [e]_{\psi_1} = [i]_{\psi_1}.
\end{aligned}$$

Then, we get that $\xi_{\psi_1} = \left\{[o]_{\psi_1}, [a]_{\psi_1}, [b]_{\psi_1}, [c]_{\psi_1}\right\}$.

$$\begin{aligned}
[o]_{\psi_2} &= \{x \in \mathcal{O} \mid \psi_2(x) = \psi_2(o) = \vartheta_1\} = \{o, b\} \\
&= [b]_{\psi_2}, \\
[a]_{\psi_2} &= \{x \in \mathcal{O} \mid \psi_2(x) = \psi_2(a) = \vartheta_3\} = \{a, c\}, \\
&= [c]_{\varphi_2}, \\
[d]_{\psi_2} &= \{x \in \mathcal{O} \mid \psi_2(x) = \psi_2(d) = \vartheta_4\} = \{d, f\} \\
&= [f]_{\psi_2}, \\
[e]_{\psi_2} &= \{x \in \mathcal{O} \mid \psi_2(x) = \psi_2(e) = \vartheta_5\} = \{e, i\} \\
&= [i]_{\psi_2}.
\end{aligned}$$

We have $\xi_{\psi_2} = \left\{[o]_{\psi_2}, [a]_{\psi_2}, [d]_{\psi_2},\ [e]_{\psi_2}\right\}$.

$$\begin{aligned}
[o]_{\psi_3} &= \{x \in \mathcal{O} \mid \psi_3(x) = \psi_3(o) = \vartheta_2\} = \{o, d, f\} \\
&= [d]_{\psi_3} = [f]_{\psi_3}, \\
[a]_{\psi_3} &= \{x \in \mathcal{O} \mid \psi_3(x) = \psi_3(a) = \vartheta_5\} = \{a, i\} \\
&= [i]_{\varphi_3}, \\
[b]_{\psi_3} &= \{x \in \mathcal{O} \mid \psi_3(x) = \psi_3(b) = \vartheta_6\} = \{b\}, \\
[c]_{\psi_3} &= \{x \in \mathcal{O} \mid \psi_3(x) = \psi_3(c) = \vartheta_3\} = \{c\}, \\
[d]_{\psi_3} &= \{x \in \mathcal{O} \mid \psi_3(x) = \psi_3(d) = \vartheta_4\} = \{d\}.
\end{aligned}$$

From hence, we obtain that $\xi_{\psi_3} = \left\{ [o]_{\psi_3}, [a]_{\psi_3}, [b]_{\psi_3}, [c]_{\psi_3}, [d]_{\psi_3} \right\}$*. Therefore, for* $r = 1$*, a set of partitions of* $\mathcal{O}$ *is* $N_r(B) = \{\xi_{\psi_1}, \xi_{\psi_2}, \xi_{\psi_3}\}$*. Then, we can write*

$$\begin{aligned} N_1(B)^* S &= \bigcup_{[x]_{\varphi_i} \cap S \neq \varnothing} [x]_{\varphi_i} \\ &= [a]_{\psi_1} \cup [c]_{\psi_1} \cup [a]_{\psi_2} \cup [d]_{\psi_2} \cup [e]_{\psi_2} \\ &\quad \cup [o]_{\psi_3} \cup [a]_{\psi_3} \cup [c]_{\psi_3} \cup [d]_{\psi_3} \\ &= \{o, a, c, d, e, f, i\}. \end{aligned}$$

Considering the following tables of operations:

α	a	c	d	i
a	a	c	d	i
c	o	o	o	o
d	a	c	d	i
i	a	c	d	i

γ	a	c	d	i
a	o	o	o	o
c	a	c	d	i
d	o	o	o	o
i	a	c	d	i

σ	a	c	d	i
a	a	c	d	i
c	a	c	d	i
d	o	o	o	o
i	a	c	d	i

S *is a* Γ*-nearnes semigroup. Afterwards, it is taken* $I = \{a, d, i\} \subseteq S$*.*

$$\begin{aligned} N_1(B)^* I &= \bigcup_{[x]_{\varphi_i} \cap I \neq \varnothing} [x]_{\varphi_i} \\ &= [a]_{\psi_1} \cup [c]_{\psi_1} \cup [a]_{\psi_2} \cup [d]_{\psi_2} \cup [e]_{\psi_2} \\ &\quad \cup [o]_{\psi_3} \cup [a]_{\psi_3} \cup [d]_{\psi_3} \\ &= \{o, a, c, d, e, f, i\}. \end{aligned}$$

Since $I\Gamma I \subseteq N_1(B)^* I$*,* I *is a sub* Γ*-semigroup of* S*. In addition, the condition* $S\Gamma I\Gamma S \subseteq N_r(B)^* I$ *is satisfied by* I *and* I *is a interior* Γ*-nearness ideal of* S*.*

THEOREM 3.1. *Let* S *be a* Γ*-nearness semigroup. Every* Γ*-nearness ideal* I *of* S *is an interior* Γ*-nearness ideal* I *of* S *if* $N_r(B)^* (N_r(B)^* I) = N_r(B)^* I$*.*

PROOF. Let S be a Γ-nearness semigroup and I be Γ-nearness ideal of S. Then, $S\Gamma I \subseteq N_r(B)^* I \Rightarrow S\Gamma I\Gamma S \subseteq (N_r(B)^* I)\Gamma S$. By using Theorem 2.1.($i$) and Lemma 2.1,

$$\begin{aligned} S\Gamma I\Gamma S &\subseteq (N_r(B)^* I)\Gamma S \\ &\subseteq (N_r(B)^* I)\Gamma(N_r(B)^* S) \\ &\subseteq N_r(B)^* (I\Gamma S) \\ &\subseteq N_r(B)^* (N_r(B)^* I) \\ &= N_r(B)^* I. \end{aligned}$$

Therefore, $S\Gamma I\Gamma S \subseteq N_r(B)^* I$ and I is interior Γ-nearness ideal of S. □

We will give the following theorem, which is the same proof as the proof of Theorem 2.3.

THEOREM 3.2. *Let* S *be a* Γ*-nearness semigroup. Then the following properties hold:*

i) If $\varnothing \neq I \subseteq S$ *and* $S\Gamma I\Gamma S \subseteq I$*, then* I *is a upper-near interior* Γ*-nearness ideal of* S*.*

ii) If I is an interior Γ-nearness ideal of S and $N_r(B)^*(N_r(B)^* I) = N_r(B)^* I$, then I is a upper-near interior Γ-nearness ideal of S.

THEOREM 3.3. *Let S be a Γ-nearness semigroup, and $\{I_i | i \in \Delta\}$ be set of interior Γ-nearness ideals of S such that $N_r(B)^*(N_r(B)^* I_i) = N_r(B)^* I_i$ for all $i \in \Delta$ with index set Δ. If $N_r(B)^*(\bigcap_{i\in\Delta} I_i) = \bigcap_{i\in\Delta} N_r(B)^* I_i$, then $\bigcap_{i\in\Delta} I_i = \emptyset$ or $\bigcap_{i\in\Delta} I_i$ is an interior nearness ideal of S.*

PROOF. Let $\bigcap_{i\in\Delta} I_i = I$. Now, we show that I is either empty or an interior Γ-nearness ideal of S. Assume that I is non-empty. Since I_i is an interior Γ-nearness ideals of S for all $i \in \Delta$, I is sub Γ-nearness semigroup of S and $S\Gamma I_i \Gamma S \subseteq N_r(B)^* I_i$ for all $i \in \Delta$. Then,

$$S\Gamma I\Gamma S = S\Gamma(\bigcap_{i\in\Delta} I_i)\Gamma S \subseteq \bigcap_{i\in\Delta} \left(N_r(B)^* I_i\right)$$

for all $i \in \Delta$. Since $\bigcap_{i\in\Delta} N_r(B)^* I_i = N_r(B)^*(\bigcap_{i\in\Delta} I_i)$, we obtain $S\Gamma I\Gamma S \subseteq N_r(B)^* I$. Hence, I is an interior Γ-nearness ideal of S. □

DEFINITION 3.2. *Let S be a Γ-nearness semigroup. The element $x \in S$ is called regular element if there exists $y \in S$ and $\gamma, \beta \in \Gamma$ such that the property $x\beta y\gamma x = x$ satisfies in $N_r(B)^* S$. Moreover, S is called regular Γ-nearness semigroup if all its elements of S are regular.*

EXAMPLE 3.2. *Let's take Γ-nearness semigroup S in Example 3.1. In this case, for all $x \in S$ there exists $y \in S$ and $\gamma, \beta \in \Gamma$ such that the property $x\beta y\gamma x = x$ provides in $N_r(B)^* S$. In Example 3.1,*

for $a \in S$ there exists $c \in S$ and $\alpha, \sigma \in \Gamma$ such that $a\alpha c\sigma a = a$,
for $c \in S$ there exists $a \in S$ and $\gamma, \sigma \in \Gamma$ such that $c\gamma a\sigma c = c$,
for $d \in S$ there exists $a \in S$ and $\alpha, \gamma \in \Gamma$ such that $d\alpha a\gamma d = d$,
for $i \in S$ there exists $c \in S$ and $\alpha, \gamma \in \Gamma$ such that $i\alpha c\gamma i = i$.

Therefore, S is a regular Γ-nearness semigroup.

THEOREM 3.4. *Let S be a regular Γ-nearness semigroup, $\sim_{B_r}$ be a complete congruence indiscernibility relation on S and I be an interior Γ-nearness ideal of S. If $N_r(B)^*(N_r(B)^* I) = N_r(B)^* I$, then $N_r(B)^* I = N_r(B)^*(S\Gamma I\Gamma S)$.*

PROOF. Let S be a regular Γ-nearness semigroup and I be an interior Γ-nearness ideal of S. Since I is an interior Γ-nearness ideal of S, $S\Gamma I\Gamma S \subseteq N_r(B)^* I$. Thus, we have $N_r(B)^*(S\Gamma I\Gamma S) \subseteq N_r(B)^*(N_r(B)^* I)$ by Theorem 2.1.(i). Then, we get that $N_r(B)^*(S\Gamma I\Gamma S) \subseteq N_r(B)^* I$ by the hypothesis. On the other hand, let $x \in N_r(B)^* I$. In this case, $[x]_{B_r} \cap I \neq \emptyset$. Then, there exists an element $y \in [x]_{B_r}$ and $y \in I$. Since S is regular Γ-nearness semigroup, for $y \in S$, there exists $z \in S$ and $\gamma, \beta \in \Gamma$ such that the property $y\gamma z\beta y = y$ holds. Thus,

$y \in y\Gamma S\Gamma y\Gamma S\Gamma y \subseteq S\Gamma(S\Gamma I\Gamma S)\Gamma S \subseteq S\Gamma(N_r(B)^* I)\Gamma S$, for I is an interior Γ-nearness ideal of S. Therefore, $y \in S\Gamma(N_r(B)^* I)\Gamma S$. In this case, we get that $y \in [x]_{B_r}$ and $y \in S\Gamma(N_r(B)^* I)\Gamma S$ and so, $y \in [x]_{B_r} \cap S\Gamma(N_r(B)^* I)\Gamma S$. Thus, we have $[x]_{B_r} \cap S\Gamma(N_r(B)^* I)\Gamma S \neq \emptyset$ and $x \in N_r(B)^*(S\Gamma(N_r(B)^* I)\Gamma S)$. Since $\sim_{B_r}$ is a complete congruence indiscernibility relation and from Theorem 2.2,

$$\begin{aligned} N_r(B)^*(S\Gamma(N_r(B)^* I)\Gamma S) &= (N_r(B)^* S)\Gamma(N_r(B)^*(N_r(B)^* I))\Gamma(N_r(B)^* S) \\ &= (N_r(B)^* S)\Gamma(N_r(B)^* I)\Gamma(N_r(B)^* S) \\ &= N_r(B)^*(S\Gamma I\Gamma S). \end{aligned}$$

Thereby, $x \in N_r(B)^*(S\Gamma I\Gamma S)$ and we get $N_r(B)^* I \subseteq N_r(B)^*(S\Gamma I\Gamma S)$. Consequently, $N_r(B)^* I = N_r(B)^*(S\Gamma I\Gamma S)$. □

Theorem 3.5. *Let S be a regular Γ-nearness semigroup, $\sim_{B_r}$ a complete congruence indiscernibility relation on S, and I a bi-Γ-nearness ideal of S. If $N_r(B)^*(N_r(B)^* I) = N_r(B)^* I$, then $N_r(B)^* I = N_r(B)^*(I\Gamma S\Gamma I)$.*

Proof. Let S be a regular Γ-nearness semigroup and I be a bi-Γ-nearness ideal of S. In this case, we have $I\Gamma S\Gamma I \subseteq N_r(B)^* I$. Thus, we get

$$N_r(B)^*(I\Gamma S\Gamma I) \subseteq N_r(B)^*(N_r(B)^* I) = N_r(B)^* I$$

by Theorem 2.2. Therefore, we get that $N_r(B)^*(I\Gamma S\Gamma I) \subseteq N_r(B)^* I$. Otherwise, let $x \in N_r(B)^* I$. From here, $[x]_{B_r} \cap I \neq \emptyset$ and there exists an element $y \in [x]_{B_r}$ and $y \in I$. Since S is regular Γ-nearness semigroup, for $y \in S$, there exists $z \in S$ and $\gamma, \beta \in \Gamma$ such that $y\gamma z\beta y = y$ satisfies on S. In this case, $y \in y\Gamma S\Gamma y\Gamma S\Gamma y \subseteq I\Gamma S\Gamma(I\Gamma S\Gamma I) \subseteq I\Gamma S\Gamma(N_r(B)^* I)$. Thus, $y \in I\Gamma S\Gamma(N_r(B)^* I)$. From here, $y \in [x]_{B_r}$ and $y \in I\Gamma S\Gamma(N_r(B)^* I)$. Then, $y \in [x]_{B_r} \cap I\Gamma S\Gamma(N_r(B)^* I)$ and we have $[x]_{B_r} \cap I\Gamma S\Gamma(N_r(B)^* I) \neq \emptyset$. Therefore, $x \in N_r(B)^*(I\Gamma S\Gamma(N_r(B)^* I))$. Since $\sim_{B_r}$ is a complete congruence indiscernibility relation and from Theorem 2.2,

$$\begin{aligned} N_r(B)^*(I\Gamma S\Gamma(N_r(B)^* I)) &= (N_r(B)^* I)\Gamma(N_r(B)^* S)\Gamma(N_r(B)^*(N_r(B)^* I)) \\ &= (N_r(B)^* I)\Gamma(N_r(B)^* S)\Gamma(N_r(B)^* I) \\ &= N_r(B)^*(I\Gamma S\Gamma I) \end{aligned}$$

In this case, $x \in N_r(B)^*(I\Gamma S\Gamma I)$ and so, $N_r(B)^* I \subseteq N_r(B)^*(I\Gamma S\Gamma I)$. Finally, $N_r(B)^* I = N_r(B)^*(I\Gamma S\Gamma I)$. □

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