BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Bull. Int. Math. Virtual Inst., **13**(2)(2023), 249–252
DOI: 10.7251/BIMVI2302249L

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

ON RELATIONS BETWEEN ATOM-BOND SUM-CONNECTIVITY INDEX AND OTHER CONNECTIVITY INDICES

Zhen Lin

Abstract. In 2022, the atom-bond sum-connectivity index is introduced by Ali, Furtula, Redžepović and Gutman inspired by the Randić index, the sum-connectivity index and the atom-bond-connectivity index. Recently, the extreme values of the new index for a graph class is widely studied. In this paper, we pay more attention to the mathematical relations between the atom-bond sum-connectivity index and some other connectivity indices.

1. Introduction

In chemical graph theory, the topological index of a graph, also called molecular structure descriptor, is often used to predict the physico-chemical properties and biological activities of molecules. In particular, a large number of degree-based topological indices have been introduced and extensively studied [**9**] in mathematical chemistry.

Let G be a simple connected undirected graph with the vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, d_v denotes the degree of vertex v in G. The minimum and the maximum degree of G are denoted by δ and Δ, respectively. A pendant vertex is a vertex of degree one and a quasi-pendant vertex is a vertex adjacent to a pendant vertex.

The Randić index [**8**], also called the connectivity index or the branching index, is one of the most famous and important degree-based topological indices, and

2010 *Mathematics Subject Classification.* Primary 05C09; Secondary 05C92; 05C07.
Key words and phrases. Sum-connectivity index, Topological index.
Supported by the National Natural Science Foundation of China (Grant No. 12261074).
Communicated by Dusko Bogdanic.

defined as

$$R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u d_v}}.$$

The harmonic index [**6**], the sum-connectivity index [**10**] and the atom-bond-connectivity index [**5**] are the class of successful variants of the connectivity index, and defined as

$$\begin{aligned} H(G) &= \sum_{uv \in E(G)} \frac{2}{d_u + d_v}, \\ SCI(G) &= \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u + d_v}}, \\ ABC(G) &= \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}}. \end{aligned}$$

In 2022, Ali, Furtula, Redžepović and Gutman [**1**] proposed a novel degree-based topological index called the atom-bond sum-connectivity index (ABS index for short) based on the above connectivity index, which is defined as

$$ABS(G) = \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u + d_v}} = \sum_{uv \in E(G)} \sqrt{1 - \frac{2}{d_u + d_v}}.$$

Recently, the extreme values of the ABS index is widely studied, see [**2, 3, 4, 7**]. A natural problem is the mathematical relationship between these connectivity indices. For a connected graph G with $n \geqslant 4$ vertices, it is not difficult to find that

$$\begin{aligned} H(G) &\leqslant R(G), \quad \text{(as mean value inequality)}, \\ R(G) &\leqslant SCI(G), \quad \text{for } \delta \geqslant 2, \\ SCI(G) &< ABC(G), \\ ABC(G) &\leqslant ABS(G), \quad \text{for } \delta \geqslant 2. \end{aligned}$$

In this paper, the mathematical relations between the ABS index and other connectivity indices are investigated.

2. Main results

THEOREM 2.1. *Let G be a connected graph with the maximum degree Δ and the minimum degree δ. Then*

$$\sqrt{\delta(\delta - 1)} R(G) \leqslant ABS(G) \leqslant \sqrt{\Delta(\Delta - 1)} R(G)$$

with equality if and only if G is regular.

PROOF. Let $f(x, y) = xy - \frac{2xy}{x+y}$. Then we have

$$\begin{aligned} \frac{\partial f(x, y)}{\partial x} &= \frac{y(x^2 + y^2 + 2xy - 2y)}{(x + y)^2} > 0, \\ \frac{\partial f(x, y)}{\partial y} &= \frac{x(x^2 + y^2 + 2xy - 2x)}{(x + y)^2} > 0. \end{aligned}$$

This implies that $f(x, y)$ is increasing for both $x \geqslant 1$ and $y \geqslant 1$. Thus we have

$$\sqrt{\delta(\delta-1)} \leqslant \sqrt{d_u d_v \left(1 - \frac{2}{d_u + d_v}\right)} \leqslant \sqrt{\Delta(\Delta-1)},$$

that is,

$$\frac{\sqrt{\delta(\delta-1)}}{\sqrt{d_u d_v}} \leqslant \sqrt{1 - \frac{2}{d_u + d_v}} \leqslant \frac{\sqrt{\Delta(\Delta-1)}}{\sqrt{d_u d_v}}$$

for $\delta \leqslant d_u \leqslant \Delta$ and $\delta \leqslant d_v \leqslant \Delta$. Further, we have

$$\sqrt{\delta(\delta-1)}R(G) \leqslant ABS(G) \leqslant \sqrt{\Delta(\Delta-1)}R(G)$$

with equality if and only if G is regular. This completes the proof. □

THEOREM 2.2. *Let G be a connected graph. If the degree of quasi-pendant vertices of G is greater than or equal to three, then $ABS(G) > R(G)$.*

PROOF. Let $g(x, y) = x^2 y + xy^2 - 2xy - x - y$ for $x \geqslant 1$ and $y \geqslant 1$. Then we have

$$\begin{aligned} \frac{\partial g(x,y)}{\partial x} &= 2(xy-1) + (y-1)^2 > 0, \\ \frac{\partial g(x,y)}{\partial y} &= 2(xy-1) + (x-1)^2 > 0 \end{aligned}$$

for $x \neq 1$ and $y \neq 1$. Thus $g(x, y)$ is increasing for $x \neq 1$ and $y \neq 1$. Since

$$g(1,3) = 3, \quad g(2,2) = 4,$$

we have

$$g(1,y) \geqslant g(1,3) > 0, \quad g(x,y) \geqslant g(2,2) > 0$$

for $x \geqslant 2$ and $y \geqslant 3$. Further, we have

$$g(d_u, d_v) = d_u^2 d_v + d_u d_v^2 - 2d_u d_v - d_u - d_v > 0,$$

that is,

$$1 - \frac{2}{d_u + d_v} > \frac{1}{d_u d_v}$$

for $d_u \geqslant 2$ and $d_v \geqslant 3$. This means that if the degree of quasi-pendant vertices of G is greater than or equal to three, then

$$ABS(G) > R(G).$$

This completes the proof. □

COROLLARY 2.1. *Let G be a connected graph. If the degree of quasi-pendant vertices of G is greater than or equal to three, then $ABS(G) > H(G)$.*

THEOREM 2.3. *Let G be a connected graph with the maximum degree Δ and the minimum degree δ. Then*

$$\sqrt{\delta(\delta-1)}H(G) \leqslant ABS(G) \leqslant \sqrt{\Delta(\Delta-1)}H(G)$$

with equality if and only if G is regular.

Proof. Let $h(x) = \frac{1}{x^2} - \frac{1}{x}$. Then $h'(x) = \frac{x-2}{x^3}$. This implies that $h(x)$ is decreasing for $x < 2$. For $uv \in E(G)$, we have

$$h\left(\frac{1}{\delta}\right) \leqslant h\left(\frac{2}{d_u + d_v}\right) \leqslant h\left(\frac{1}{\Delta}\right),$$

that is,

$$(\delta^2 - \delta)\left(\frac{2}{d_u + d_v}\right)^2 \leqslant 1 - \frac{2}{d_u + d_v} \leqslant (\Delta^2 - \Delta)\left(\frac{2}{d_u + d_v}\right)^2$$

for $\delta \leqslant d_u \leqslant \Delta$ and $\delta \leqslant d_v \leqslant \Delta$. Further, we have

$$\sqrt{\delta(\delta - 1)}H(G) \leqslant ABS(G) \leqslant \sqrt{\Delta(\Delta - 1)}H(G)$$

with equality if and only if G is regular. This completes the proof. □

By Theorems 2.1 and 2.3, we have

Theorem 2.4. *Let G be a connected graph with the maximum degree Δ and the minimum degree δ. Then*

$$\sqrt{\delta(\delta - 1)}R(G) \leqslant ABS(G) \leqslant \sqrt{\Delta(\Delta - 1)}H(G)$$

with equality if and only if G is regular.

References

1. A. Ali, B. Furtula, I. Redžepović, and I. Gutman, Atom-bond sum-connectivity index, *J. Math. Chem.* **60** (2022), 2081–2093.
2. A. Ali, I. Gutman, and I. Redžepović, Atom-bond sum-connectivity index of unicyclic graphs and some applications, *Electron. J. Math.* **5** (2023), 1–7.
3. T. A. Alraqad, I. Ž. Milovanović, H. Saber, A. Ali, and J. P. Mazorodze, Minimum atom-bond sum-connectivity index of trees with a fixed order and/or number of pendent vertices, *arXiv:2211.05218v1.*
4. T. Alraqad, H. Saber, and A. Ali, On the maximum atom-bond sum-connectivity index of graphs, *arXiv:2302.01905v1.*
5. E. Estrada, L. Torres, and I. Gutman, An atom-bond connectivity index: modelling the enthalpy of formation of alkanes, *Indian J. Chem. Sect. A* **37** (1998), 849–855.
6. S. Fajtlovicz, On conjectures on Graffiti-II, *Congr. Numer.* **60** (1987), 187–197.
7. V. Maitreyi, S. Elumalai, and S. Balachandran, The minimum *ABS* index of trees with given number of pendent vertices, *arXiv:2211.05177v1.*
8. M. Randić, Characterization of molecular branching, *J. Amer. Chem. Soc.* **97** (1975), 6609–6615.
9. R. Todeschini and V. Consonni, *Molecular Descriptors for Chemoinformatics*, Wiley VCH, Weinheim, 2009.
10. B. Zhou and N. Trinajstić, On a novel connectivity index, *J. Math. Chem.* **46** (2009), 1252–1270.

Received by editors 19.2.2023; Revised version 14.7.2023; Available online 25.9.2023.

Zhen Lin, School of Mathematics and Statistics, The State Key Laboratory of Tibetan Intelligent Information Processing and Application, Qinghai Normal University, Xining, 810008, Qinghai, China

Email address: `lnlinzhen@163.com`