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ON PATH-EQUIENERGETIC GRAPHS

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Abstract. The path energy is a recently conceived variant of graph energy, equal to the sum of the absolute values of the eigenvalues of the path matrix [Shikare et. al, 2018]. Two graphs having equal path energy are said to be path-equienergetic. All trees of same order are mutually path-equienergetic. Members of a large class of connected regular graphs of same order and degree are mutually path-equienergetic. In this paper, we construct other types of path-equienergetic graphs and explore their spectral properties.

1. Introduction

The energy of a graph (= sum of absolute values of the eigenvalues of the adjacency matrix) was introduced in 1978 by one of the present authors [**7, 9**]. The concept of equienergetic graphs was conceived by Balakrishnan [**3**] and Brankov et al. [**4**], independently, in 2004. Since then, numerous methods for constructing such graphs were discovered [**9**].

Motivated by the success of the theory of graph energy, a large number of other graph energies were put forward, equal to the sum of absolute values of the eigenvalues of some graph matrix. One of these is the path energy [**14**], based on the concept of path matrix [**11**].

Definition 1.1. [**11**]. *Let G be a simple graph with vertex set $\mathbf{V}(G) = \{v_1, v_2, \dots, v_n\}$. The path matrix $\mathbf{P} = \mathbf{P}(G) = (p_{ij})$ is the square matrix of size n, such that p_{ij} is equal to the number of vertex-disjoint paths from v_i to v_j if $i \neq j$, and $p_{ij} = 0$ if $i = j$.*

Examples illustrating the construction of path matrices are found in [**14, 11, 8**].

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DEFINITION 1.2. [**14**]. *The path energy of a graph G is the sum of the absolute values of eigenvalues of the path matrix $\mathbf{P}(G)$, and is denoted by $PE(G)$.*

For earlier works on path energy see [**14, 13, 6, 8, 1, 2, 10**].

DEFINITION 1.3. *The graphs G_1 and G_2 are said to be path-equienergetic if $PE(G_1) = PE(G_2)$.*

In this paper we are concerned with path-equienergetic graphs. At the first glance, this topic seems to be trivially simple, because of the following two results (Theorems 1.1 and 1.2).

Denote by K_n the complete graph on n vertices, and by $\mathbf{A}(K_n)$ its adjacency matrix. As well known [**5**], the spectrum of $\mathbf{A}(K_n)$ is $\{n-1, -1, \ldots, -1\}$, and the respective energy $2(n-1)$.

THEOREM 1.1. [**14**] *Let T be a tree of order n. Then*

$$\mathbf{P}(T) = \mathbf{A}(K_n)\,.$$

Therefore, the spectrum of path matrix of any tree of order n is $\{n-1,-1,\ldots,-1\}$, and its path energy is equal to $2(n-1)$. Consequently, all n-vertex trees are mutually path-equienergetic.

We say that a regular graph is *nice* if it is connected and if its edge-connectivity is equal to its degree. Note that the majority of regular graphs encountered in graph theory and its applications are nice.

THEOREM 1.2. [**1**] *Let R be a nice regular graph of order n and degree r. Then*

$$\mathbf{P}(R) = r\,\mathbf{A}(K_n)\,.$$

Therefore, the spectrum of the path matrix of any nice regular graph of order n and degree r is $\{r(n-1), -r, \ldots, -r\}$, and its path energy is equal to $2r(n-1)$. Consequently, all n-vertex nice regular graphs of degree r are mutually path-equienergetic.

In what follows we describe path-equienergetic graphs different from those in Theorems 1.1 and 1.2.

2. More path-equienergetic graphs

In [**12**], it was proven that all connected unicyclic graph of equal order and girth are path-cospectral, thus being mutually path-equienergetic. Further, n-vertex bicyclic graphs having the same structure of cycles were shown to be path-equienergetic [**12**].

Theorems 1.1 and 1.2 imply the following corollaries.

COROLLARY 2.1.
(a) If F is an n-vertex forest with c components, then $PE(F) = 2(n-c)$.
(b) If R is an n-vertex regular graph of degree r with c components, all of which being nice, then $PE(R) = 2r(n-c)$.

COROLLARY 2.2. *An n_1-vertex tree and a nice regular graph of order n_2 and degree r are path-equienergetic if and only if $r = (n_1-1)/(n_2-1)$.*

EXAMPLE 2.1. *Denote by P_n and C_n be the path and cycle of order n. Then P_5 and C_3, are path-equienergetic.*

EXAMPLE 2.2. *The complete graph on n vertices and a tree of order p are path-equienergetic if and only if $p = n^2 - 2n + 2$.*

The above examples show that path-equienergetic graphs need not be path-cospectral.

It is known [**14**] that if the path energy is a rational number, then it is an even integer.

EXAMPLE 2.3. *A graph H with path energy $2k$ and any tree of order $k + 1$ are path-equienergetic.*

The following theorem provides a relation between path energy of a nice regular graph and the path energy of its complement.

THEOREM 2.1. *Let R be a nice regular graph on n vertices and $\overline{R}$ its complement. If $\overline{R}$ is also nice, then*

$$PE(R) + PE(\overline{R}) = PE(K_n)\,.$$

PROOF. Let R be r-regular. Then $\overline{R}$ is $(n-r-1)$-regular. Therefore, $PE(R) = 2r(n-1)$ and $PE(\overline{G}) = 2(n-r-1)(n-1)$. Then,

$$PE(R) + PE(\overline{R}) = 2r(n-1) + 2(n-r-1)(n-1) = 2(n-1)^2 = PE(K_n)\,.$$

□

Let $G_1 \cup G_2$ denote the graph consisting of disconnected components G_1 and G_2.

COROLLARY 2.3. *For any regular graph R of order n, satisfying the conditions of Theorem 2.1, $PE(R \cup \overline{R}) = PE(K_n)$.*

Theorem 2.1 can be generalized.

THEOREM 2.2. *Let R_1 and R_2 be nice regular graphs of order n, of degree r_1 and r_2, respectively. Let $r_1 + r_2 = n - 1$. Then $PE(R_1 \cup R_2) = PE(K_n)$.*

The following theorem pertains to a class of non-regular graphs.

THEOREM 2.3. *Let R be a nice regular graph of order n and degree r. Let G be a connected graph of degree n, with one vertex of degree greater than r and all other vertices of degree r. Let the edge-connectivity of G be r (i.e., same as of R). Then G and R are path-equienergetic.*

PROOF. Denote by $p(x, y)$ the number of vertex-disjoint paths between the vertices x and y. Recall that [**11**]

$$p(x, y) \leqslant \min\{\deg(x), \deg(y)\}\,. \tag{2.1}$$

Let w be the vertex of G whose degree is greater than r. Because of r-connectivity of the graph G, in view of Theorem 1.2, for any two vertices u, v different from w, $p(u, v) = r$.

Consider now the element $p(u,w)$ of the path matrix of G, where u is any other vertex of G. Since $\deg(u) = r$ and $\deg(w) > r$, by Eq. (2.1), $p(u,w) \leqslant r$.

Repeating an argument used in the proof of Theorem 8 in [**1**], since G is r-connected, there are at least r internally disjoint paths between u and w, i.e., $p(u,w) \geqslant r$.

Therefore, $p(u,w) = r$. Therefore, all off-diagonal elements of $\mathbf{P}(G)$ are equal to r, i.e., $\mathbf{P}(G) = r\,\mathbf{A}(K_n)$. □

Corollary 2.4. *(a) Any nice* 3*-regular graph on* n *vertices and the wheel* W_n *on* n *vertices are path-equienergetic.*
(b) The graph obtained by coalescing a vertex of the cycles C_n *and* C_m *is path-equienergetic with* C_{n+m-1}.

3. Concluding remarks

All the path-equienergetic graphs having equal number of vertices, described in the above parts of this paper, are path-cospectral. It would be of some interest to seek for pairs of path-equienergetic non-path-cospectral graphs of equal order.

A trivial way to achieve this goal would be to find a pair of path-equienergetic graphs G_1, G_2 of different order, say of n_1 and n_2, $n_1 < n_2$, and to add $n_2 - n_1$ isolated vertices to G_1.

By computer search we checked for connected path-equienergetic non-path-cospectral graphs of equal order n. Up to $n = 5$ no such graphs were found.

Conjecture 3.1. There exist pairs of connected path-equienergetic non-path-cospectral graphs of equal order.

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