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ON STRONGLY REGULAR GRAPHS WITH $m_2 = qm_3$ AND $m_3 = qm_2$ for $q = 9, 10$

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Abstract. We say that a regular graph G of order n and degree $r \geqslant 1$ (which is not the complete graph) is strongly regular if there exist non-negative integers τ and θ such that $|S_i \cap S_j| = \tau$ for any two adjacent vertices i and j, and $|S_i \cap S_j| = \theta$ for any two distinct non-adjacent vertices i and j, where S_k denotes the neighborhood of the vertex k. Let $\lambda_1 = r$, λ_2 and λ_3 be the distinct eigenvalues of a connected strongly regular graph. Let $m_1 = 1$, m_2 and m_3 denote the multiplicity of r, λ_2 and λ_3, respectively. We here describe the parameters n, r, τ and θ for strongly regular graphs with $m_2 = qm_3$ and $m_3 = qm_2$ for $q = 9, 10$.

1. Introduction

Let G be a simple graph of order n with vertex set $V(G) = \{1, 2, \ldots, n\}$. The spectrum of G consists of the eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$ of its (0,1) adjacency matrix A and is denoted by $\sigma(G)$. We say that a regular graph G of order n and degree $r \geqslant 1$ (which is not the complete graph K_n) is strongly regular if there exist non-negative integers τ and θ such that $|S_i \cap S_j| = \tau$ for any two adjacent vertices i and j, and $|S_i \cap S_j| = \theta$ for any two distinct non-adjacent vertices i and j, where $S_k \subseteq V(G)$ denotes the neighborhood of the vertex k. We know that a regular connected graph G is strongly regular if and only if it has exactly three distinct eigenvalues [**1**] (see also [**3**]). Let $\lambda_1 = r$, λ_2 and λ_3 denote the distinct eigenvalues of a connected strongly regular graph G. Let $m_1 = 1$, m_2 and m_3 denote the multiplicity of r, λ_2 and λ_3. Further, let $\overline{r} = (n-1) - r$, $\overline{\lambda}_2 = -\lambda_3 - 1$ and $\overline{\lambda}_3 = -\lambda_2 - 1$ denote the distinct eigenvalues of the strongly regular graph $\overline{G}$,

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where $\overline{G}$ denotes the complement of G. Then $\overline{\tau} = n - 2r - 2 + \theta$ and $\overline{\theta} = n - 2r + \tau$ where $\overline{\tau} = \tau(\overline{G})$ and $\overline{\theta} = \theta(\overline{G})$.

REMARK 1.1. (i) if G is a disconnected strongly regular graph of degree r then $G = mK_{r+1}$, where mH denotes the m-fold union of the graph H; (ii) G is a disconnected strongly regular graph if and only if $\theta = 0$.

REMARK 1.2. (i) a strongly regular graph G of order $n = 4k + 1$ and degree $r = 2k$ with $\tau = k - 1$ and $\theta = k$ is called a conference graph; (ii) a strongly regular graph is a conference graph if and only if $m_2 = m_3$ and (iii) if $m_2 \neq m_3$ then G is an integral[1] graph.

We have recently started to investigate strongly regular graphs with $m_2 = qm_3$ and $m_3 = qm_2$, where q is a positive integer [**4**]. In the some work we have described the parameters n, r, τ and θ for strongly regular graphs with $m_2 = qm_3$ and $m_3 = qm_2$ for $q = 2, 3, 4$. In particular, we have described in [**5**] the parameters n, r, τ and θ for strongly regular graphs with $m_2 = qm_3$ and $m_3 = qm_2$ for $q = 5, 6, 7, 8$. We now proceed to establish the parameters of strongly regular graphs with $m_2 = qm_3$ and $m_3 = qm_2$ for $q = 9, 10$, as follows. First,

PROPOSITION 1.1 (Elzinga [**2**]). *Let G be a connected or disconnected strongly regular graph of order n and degree r. Then*

$$r^2 - (\tau - \theta + 1)r - (n - 1)\theta = 0\,. \tag{1.1}$$

PROPOSITION 1.2 (Elzinga [**2**]). *Let G be a connected strongly regular graph of order n and degree r. Then*

$$2r + (\tau - \theta)(m_2 + m_3) + \delta(m_2 - m_3) = 0\,, \tag{1.2}$$

where $\delta = \lambda_2 - \lambda_3$.

REMARK 1.3 (Lepović [**4**]). Using the same procedure applied in [**4**] we can establish the parameters n, r, τ and θ for strongly regular graphs with $m_2 = qm_3$ and $m_3 = qm_2$ for any fixed value $q \in \mathbb{N}$, as follows. First, let $m_3 = p$, $m_2 = qp$ and $n = (q + 1)p + 1$ where $q \in \mathbb{N}$. Using (1.2) we obtain $r = p(|\lambda_3| - q\lambda_2)$. Let $|\lambda_3| - q\lambda_2 = t$ where $t = 1, 2, \ldots, q$. Let $\lambda_2 = k$ where k is a positive integer. Then (i) $\lambda_3 = -(qk + t)$; (ii) $\tau - \theta = -((q - 1)k + t)$; (iii) $\delta = (q + 1)k + t$; (iv) $r = pt$ and (v) $\theta = pt - qk^2 - kt$. Using (ii), (iv) and (v) we can easily see that (1.1) is reduced to

$$(p + 1)t^2 - \big((q + 1)p + 1\big)t + q(q + 1)k^2 + 2qkt = 0\,. \tag{1.3}$$

Second, let $m_2 = p$, $m_3 = qp$ and $n = (q + 1)p + 1$ where $q \in \mathbb{N}$. Using (1.2) we obtain $r = p(q|\lambda_3| - \lambda_2)$. Let $q|\lambda_3| - \lambda_2 = t$ where $t = 1, 2, \ldots, q$. Let $\lambda_3 = -k$ where k is a positive integer. Then (i) $\lambda_2 = qk - t$; (ii) $\tau - \theta = (q - 1)k - t$; (iii) $\delta = (q + 1)k - t$; (iv) $r = pt$ and (v) $\theta = pt - qk^2 + kt$. Using (ii), (iv) and (v) we can easily see that (1.1) is reduced to

$$(p + 1)t^2 - \big((q + 1)p + 1\big)t + q(q + 1)k^2 - 2qkt = 0\,. \tag{1.4}$$

[1]We say that a connected or disconnected graph G is integral if its spectrum $\sigma(G)$ consists only of integral values.

Using (1.3) and (1.4) we can obtain for $t = 1, 2, \ldots, q$ the corresponding classes of strongly regular graphs with $m_2 = qm_3$ and $m_3 = qm_2$, respectively.

2. Main results

REMARK 2.1. Since $m_2(\overline{G}) = m_3(G)$ and $m_3(\overline{G}) = m_2(G)$ we note that if $m_2(G) = qm_3(G)$ then $m_3(\overline{G}) = qm_2(\overline{G})$.

REMARK 2.2. In Theorems 2.1 and 2.2 the complements of strongly regular graphs appear in pairs in (k^0) and $(\overline{k}^0)$ classes, where k denotes the corresponding number of a class.

REMARK 2.3. $\overline{\alpha K_\beta}$ is a strongly regular graph of order $n = \alpha\beta$ and degree $r = (\alpha - 1)\beta$ with $\tau = (\alpha - 2)\beta$ and $\theta = (\alpha - 1)\beta$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -\beta$ with $m_2 = \alpha(\beta - 1)$ and $m_3 = \alpha - 1$.

PROPOSITION 2.1. *Let G be a connected strongly regular graph of order n and degree r with $m_2 = 9m_3$. Then G belongs to the class $(\overline{3}^0)$ or (4^0) or $(\overline{5}^0)$ or (6^0) or (7^0) or (8^0) or (9^0) or $(\overline{10}^0)$ represented in Theorem* 2.1.

PROOF. Let $m_3 = p$, $m_2 = 9p$ and $n = 10p + 1$ where $p \in \mathbb{N}$. Using (1.2) we obtain $r = p(|\lambda_3| - 9\lambda_2)$. Let $|\lambda_3| - 9\lambda_2 = t$ where $t = 1, 2, \ldots, 9$. Let $\lambda_2 = k$ where k is a positive integer. Then according to Remark 1.3 we have (i) $\lambda_3 = -(9k + t)$; (ii) $\tau - \theta = -(8k + t)$; (iii) $\delta = 10k + t$; (iv) $r = pt$ and (v) $\theta = pt - 9k^2 - kt$. In this case we can easily see that (1.3) is reduced to

$$(p + 1)t^2 - (10p + 1)t + 90k^2 + 18kt = 0\,. \tag{2.1}$$

CASE 1. ($t = 1$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(9k + 1)$, $\tau - \theta = -(8k + 1)$, $\delta = 10k + 1$, $r = p$ and $\theta = p - 9k^2 - k$. Using (2.1) we find that $p = 2k(5k + 1)$. So we obtain that G is a strongly regular graph of order $n = (10k + 1)^2$ and degree $r = 2k(5k + 1)$ with $\tau = k^2 - 7k - 1$ and $\theta = k(k + 1)$.

CASE 2. ($t = 2$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(9k + 2)$, $\tau - \theta = -(8k + 2)$, $\delta = 10k + 2$, $r = 2p$ and $\theta = 2p - 9k^2 - 2k$. Using (2.1) we find that $8p - 1 = 9k(5k + 2)$. Replacing k with $4k + 1$ we arrive at $p = 90k^2 + 54k + 8$. So we obtain that G is a strongly regular graph of order $n = 9(10k + 3)^2$ and degree $r = 2(90k^2 + 54k + 8)$ with $\tau = 36k^2 - 4k - 5$ and $\theta = (2k + 1)(18k + 5)$.

CASE 3. ($t = 3$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(9k + 3)$, $\tau - \theta = -(8k + 3)$, $\delta = 10k + 3$, $r = 3p$ and $\theta = 3p - 9k^2 - 3k$. Using (2.1) we find that $7p - 2 = 6k(5k + 3)$. Replacing k with $7k - 1$ we arrive at $p = 210k^2 - 42k + 2$. So we obtain that G is a strongly regular graph of order $n = 21(10k - 1)^2$ and degree $r = 3(210k^2 - 42k + 2)$ with $\tau = 189k^2 - 77k + 5$ and $\theta = 21k(9k - 1)$.

CASE 4. ($t = 4$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(9k+4)$, $\tau - \theta = -(8k+4)$, $\delta = 10k+4$, $r = 4p$ and $\theta = 4p - 9k^2 - 4k$. Using (2.1) we find that $4p - 2 = 3k(5k + 4)$, a contradiction because $4 \nmid 15k^2 + 12k + 2$.

Case 5. ($t = 5$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(9k + 5)$, $\tau - \theta = -(8k + 5)$, $\delta = 10k + 5$, $r = 5p$ and $\theta = 5p - 9k^2 - 5k$. Using (2.1) we find that $5p - 4 = 18k(k + 1)$. Replacing k with $5k + 1$ we arrive at $p = 90k^2 + 54k + 8$. So we obtain that G is a strongly regular graph of order $n = 9(10k + 3)^2$ and degree $r = 5(90k^2 + 54k + 8)$ with $\tau = 225k^2 + 115k + 13$ and $\theta = (5k+2)(45k+13)$. Replacing k with $5k-2$ we arrive at $p = 90k^2 - 54k+8$. So we obtain that G is a strongly regular graph of order $n = 9(10k - 3)^2$ and degree $r = 5(90k^2 - 54k + 8)$ with $\tau = 225k^2 - 155k + 25$ and $\theta = (5k - 1)(45k - 14)$.

Case 6. ($t = 6$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(9k+6)$, $\tau-\theta = -(8k+6)$, $\delta = 10k+6$, $r = 6p$ and $\theta = 6p-9k^2-6k$. Using (2.1) we find that $4p - 5 = 3k(5k + 6)$, a contradiction because $4 \nmid 15k^2 + 18k + 5$.

Case 7. ($t = 7$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(9k + 7)$, $\tau - \theta = -(8k + 7)$, $\delta = 10k + 7$, $r = 7p$ and $\theta = 7p - 9k^2 - 7k$. Using (2.1) we find that $7p - 14 = 6k(5k + 7)$. Replacing k with $7k$ we arrive at $p = 210k^2 + 42k + 2$. So we obtain that G is a strongly regular graph of order $n = 21(10k + 1)^2$ and degree $r = 7(210k^2 + 42k + 2)$ with $\tau = 7(147k^2 + 27k + 1)$ and $\theta = 7(7k + 1)(21k + 2)$.

Case 8. ($t = 8$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(9k + 8)$, $\tau - \theta = -(8k + 8)$, $\delta = 10k + 8$, $r = 8p$ and $\theta = 8p - 9k^2 - 8k$. Using (2.1) we find that $8p - 28 = 9k(5k + 8)$. Replacing k with $4k - 2$ we arrive at $p = 90k^2 - 54k + 8$. So we obtain that G is a strongly regular graph of order $n = 9(10k - 3)^2$ and degree $r = 8(90k^2 - 54k + 8)$ with $\tau = 4(4k - 1)(36k - 13)$ and $\theta = 4(4k - 1)(36k - 11)$.

Case 9. ($t = 9$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(9k+9)$, $\tau-\theta = -(8k+9)$, $\delta = 10k+9$, $r = 9p$ and $\theta = 9p-9k^2-9k$. Using (2.1) we find that $p = 2(k + 1)(5k + 4)$. Replacing k with $k - 1$ we arrive at $p = 2k(5k-1)$. So we obtain that G is a strongly regular graph of order $n = (10k-1)^2$ and degree $r = 18k(5k - 1)$ with $\tau = 81k^2 - 17k - 1$ and $\theta = 9k(9k - 1)$. □

Proposition 2.2. *Let G be a connected strongly regular graph of order n and degree r with $m_3 = 9m_2$. Then G belongs to the class* (3^0) *or* $(\overline{4}^0)$ *or* (5^0) *or* $(\overline{6}^0)$ *or* $(\overline{7}^0)$ *or* $(\overline{8}^0)$ *or* $(\overline{9}^0)$ *or* (10^0) *represented in Theorem* 2.1.

Proof. Let $m_2 = p$, $m_3 = 9p$ and $n = 10p + 1$ where $p \in \mathbb{N}$. Using (1.2) we obtain $r = p(9|\lambda_3| - \lambda_2)$. Let $9|\lambda_3| - \lambda_2 = t$ where $t = 1, 2, \ldots, 9$. Let $\lambda_3 = -k$ where k is a positive integer. Then according to Remark 1.3 we have (i) $\lambda_2 = 9k-t$; (ii) $\tau - \theta = 8k - t$; (iii) $\delta = 10k - t$; (iv) $r = pt$ and (v) $\theta = pt - 9k^2 + kt$. In this case we can easily see that (1.4) is reduced to

$$(p + 1)t^2 - (10p + 1)t + 90k^2 - 18kt = 0\,. \tag{2.2}$$

Case 1. ($t = 1$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 9k - 1$ and $\lambda_3 = -k$, $\tau - \theta = 8k - 1$, $\delta = 10k - 1$, $r = p$ and $\theta = p - 9k^2 + k$. Using (2.2) we find that $p = 2k(5k - 1)$. So we obtain that G is a strongly regular graph of order $n = (10k - 1)^2$ and degree $r = 2k(5k - 1)$ with $\tau = k^2 + 7k - 1$ and $\theta = k(k - 1)$.

Case 2. ($t = 2$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 9k - 2$ and $\lambda_3 = -k$, $\tau - \theta = 8k - 2$, $\delta = 10k - 2$, $r = 2p$ and $\theta = 2p - 9k^2 + 2k$.

Using (2.2) we find that $8p - 1 = 9k(5k - 2)$. Replacing k with $4k - 1$ we arrive at $p = 90k^2 - 54k + 8$. So we obtain that G is a strongly regular graph of order $n = 9(10k - 3)^2$ and degree $r = 2(90k^2 - 54k + 8)$ with $\tau = 36k^2 + 4k - 5$ and $\theta = (2k - 1)(18k - 5)$.

CASE 3. ($t = 3$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 9k - 3$ and $\lambda_3 = -k$, $\tau - \theta = 8k - 3$, $\delta = 10k - 3$, $r = 3p$ and $\theta = 3p - 9k^2 + 3k$. Using (2.2) we find that $7p - 2 = 6k(5k - 3)$. Replacing k with $7k + 1$ we arrive at $p = 210k^2 + 42k + 2$. So we obtain that G is a strongly regular graph of order $n = 21(10k + 1)^2$ and degree $r = 3(210k^2 + 42k + 2)$ with $\tau = 189k^2 + 77k + 5$ and $\theta = 21k(9k + 1)$.

CASE 4. ($t = 4$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 9k - 4$ and $\lambda_3 = -k$, $\tau - \theta = 8k - 4$, $\delta = 10k - 4$, $r = 4p$ and $\theta = 4p - 9k^2 + 4k$. Using (2.2) we find that $4p - 2 = 3k(5k - 4)$, a contradiction because $4 \nmid 15k^2 - 12k + 2$.

CASE 5. ($t = 5$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 9k - 5$ and $\lambda_3 = -k$, $\tau - \theta = 8k - 5$, $\delta = 10k - 5$, $r = 5p$ and $\theta = 5p - 9k^2 + 5k$. Using (2.2) we find that $5p - 4 = 18k(k - 1)$. Replacing k with $5k + 2$ we arrive at $p = 90k^2 + 54k + 8$. So we obtain that G is a strongly regular graph of order $n = 9(10k + 3)^2$ and degree $r = 5(90k^2 + 54k + 8)$ with $\tau = 225k^2 + 155k + 25$ and $\theta = (5k + 1)(45k + 14)$. Replacing k with $5k - 1$ we arrive at $p = 90k^2 - 54k + 8$. So we obtain that G is a strongly regular graph of order $n = 9(10k - 3)^2$ and degree $r = 5(90k^2 - 54k + 8)$ with $\tau = 225k^2 - 115k + 13$ and $\theta = (5k - 2)(45k - 13)$.

CASE 6. ($t = 6$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 9k - 6$ and $\lambda_3 = -k$, $\tau - \theta = 8k - 6$, $\delta = 10k - 6$, $r = 6p$ and $\theta = 6p - 9k^2 + 6k$. Using (2.2) we find that $4p - 5 = 3k(5k - 6)$, a contradiction because $4 \nmid 15k^2 - 18k + 5$.

CASE 7. ($t = 7$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 9k - 7$ and $\lambda_3 = -k$, $\tau - \theta = 8k - 7$, $\delta = 10k - 7$, $r = 7p$ and $\theta = 7p - 9k^2 + 7k$. Using (2.2) we find that $7p - 14 = 6k(5k - 7)$. Replacing k with $7k$ we arrive at $p = 210k^2 - 42k + 2$. So we obtain that G is a strongly regular graph of order $n = 21(10k - 1)^2$ and degree $r = 7(210k^2 - 42k + 2)$ with $\tau = 7(147k^2 - 27k + 1)$ and $\theta = 7(7k - 1)(21k - 2)$.

CASE 8. ($t = 8$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 9k - 8$ and $\lambda_3 = -k$, $\tau - \theta = 8k - 8$, $\delta = 10k - 8$, $r = 8p$ and $\theta = 8p - 9k^2 + 8k$. Using (2.2) we find that $8p - 28 = 9k(5k - 8)$. Replacing k with $4k + 2$ we arrive at $p = 90k^2 + 54k + 8$. So we obtain that G is a strongly regular graph of order $n = 9(10k + 3)^2$ and degree $r = 8(90k^2 + 54k + 8)$ with $\tau = 4(4k + 1)(36k + 13)$ and $\theta = 4(4k + 1)(36k + 11)$.

CASE 9. ($t = 9$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 9k - 9$ and $\lambda_3 = -k$, $\tau - \theta = 8k - 9$, $\delta = 10k - 9$, $r = 9p$ and $\theta = 9p - 9k^2 + 9k$. Using (2.2) we find that $p = 2(k - 1)(5k - 4)$. Replacing k with $k + 1$ we arrive at $p = 2k(5k + 1)$. So we obtain that G is a strongly regular graph of order $n = (10k + 1)^2$ and degree $r = 18k(5k + 1)$ with $\tau = 81k^2 + 17k - 1$ and $\theta = 9k(9k + 1)$. □

REMARK 2.4. We note that $\overline{3K_7}$ is a strongly regular graph with $m_2 = 9m_3$. It is obtained from the class Theorem 2.1 ($\overline{10}^0$) for $k = 0$.

REMARK 2.5. We note that $\overline{9K_9}$ is a strongly regular graph with $m_2 = 9m_3$. It is obtained from the class Theorem 2.1 $(\overline{3}^0)$ for $k = 1$.

THEOREM 2.1. *Let G be a connected strongly regular graph of order n and degree r with $m_2 = 9m_3$ or $m_3 = 9m_2$. Then G is one of the following strongly regular graphs:*

(1^0) *G is the strongly regular graph $\overline{3K_7}$ of order $n = 21$ and degree $r = 14$ with $\tau = 7$ and $\theta = 14$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -7$ with $m_2 = 18$ and $m_3 = 2$;*

(2^0) *G is the strongly regular graph $\overline{9K_9}$ of order $n = 81$ and degree $r = 72$ with $\tau = 63$ and $\theta = 72$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -9$ with $m_2 = 72$ and $m_3 = 8$;*

(3^0) *G is a strongly regular graph of order $n = (10k-1)^2$ and degree $r = 2k(5k-1)$ with $\tau = k^2+7k-1$ and $\theta = k(k-1)$, where $k \geqslant 2$. Its eigenvalues are $\lambda_2 = 9k-1$ and $\lambda_3 = -k$ with $m_2 = 2k(5k-1)$ and $m_3 = 18k(5k-1)$;*

$(\overline{3}^0)$ *G is a strongly regular graph of order $n = (10k-1)^2$ and degree $r = 18k(5k-1)$ with $\tau = 81k^2-17k-1$ and $\theta = 9k(9k-1)$, where $k \geqslant 2$. Its eigenvalues are $\lambda_2 = k-1$ and $\lambda_3 = -9k$ with $m_2 = 18k(5k-1)$ and $m_3 = 2k(5k-1)$;*

(4^0) *G is a strongly regular graph of order $n = (10k+1)^2$ and degree $r = 2k(5k+1)$ with $\tau = k^2-7k-1$ and $\theta = k(k+1)$, where $k \geqslant 8$. Its eigenvalues are $\lambda_2 = k$ and $\lambda_3 = -(9k+1)$ with $m_2 = 18k(5k+1)$ and $m_3 = 2k(5k+1)$;*

$(\overline{4}^0)$ *G is a strongly regular graph of order $n = (10k+1)^2$ and degree $r = 18k(5k+1)$ with $\tau = 81k^2+17k-1$ and $\theta = 9k(9k+1)$, where $k \geqslant 8$. Its eigenvalues are $\lambda_2 = 9k$ and $\lambda_3 = -(k+1)$ with $m_2 = 2k(5k+1)$ and $m_3 = 18k(5k+1)$;*

(5^0) *G is a strongly regular graph of order $n = 9(10k-3)^2$ and degree $r = 2(90k^2-54k+8)$ with $\tau = 36k^2+4k-5$ and $\theta = (2k-1)(18k-5)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 36k-11$ and $\lambda_3 = -(4k-1)$ with $m_2 = 90k^2-54k+8$ and $m_3 = 9(90k^2-54k+8)$;*

$(\overline{5}^0)$ *G is a strongly regular graph of order $n = 9(10k-3)^2$ and degree $r = 8(90k^2-54k+8)$ with $\tau = 4(4k-1)(36k-13)$ and $\theta = 4(4k-1)(36k-11)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 4k-2$ and $\lambda_3 = -(36k-10)$ with $m_2 = 9(90k^2-54k+8)$ and $m_3 = 90k^2-54k+8$;*

(6^0) *G is a strongly regular graph of order $n = 9(10k-3)^2$ and degree $r = 5(90k^2-54k+8)$ with $\tau = 225k^2-155k+25$ and $\theta = (5k-1)(45k-14)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 5k-2$ and $\lambda_3 = -(45k-13)$ with $m_2 = 9(90k^2-54k+8)$ and $m_3 = 90k^2-54k+8$;*

$(\overline{6}^0)$ *G is a strongly regular graph of order $n = 9(10k-3)^2$ and degree $r = 5(90k^2-54k+8)$ with $\tau = 225k^2-115k+13$ and $\theta = (5k-2)(45k-13)$,* *where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 45k - 14$ and $\lambda_3 = -(5k-1)$ with $m_2 = 90k^2 - 54k + 8$ and $m_3 = 9(90k^2 - 54k + 8)$;*

(7^0) *G is a strongly regular graph of order $n = 9(10k+3)^2$ and degree $r = 2(90k^2 + 54k + 8)$ with $\tau = 36k^2 - 4k - 5$ and $\theta = (2k+1)(18k+5)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 4k + 1$ and $\lambda_3 = -(36k+11)$ with $m_2 = 9(90k^2 + 54k + 8)$ and $m_3 = 90k^2 + 54k + 8$;*

($\overline{7}^0$) *G is a strongly regular graph of order $n = 9(10k+3)^2$ and degree $r = 8(90k^2+54k+8)$ with $\tau = 4(4k+1)(36k+13)$ and $\theta = 4(4k+1)(36k+11)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 36k + 10$ and $\lambda_3 = -(4k+2)$ with $m_2 = 90k^2 + 54k + 8$ and $m_3 = 9(90k^2 + 54k + 8)$;*

(8^0) *G is a strongly regular graph of order $n = 9(10k+3)^2$ and degree $r = 5(90k^2+54k+8)$ with $\tau = 225k^2 + 115k + 13$ and $\theta = (5k+2)(45k+13)$, where $k \geqslant 0$. Its eigenvalues are $\lambda_2 = 5k + 1$ and $\lambda_3 = -(45k+14)$ with $m_2 = 9(90k^2 + 54k + 8)$ and $m_3 = 90k^2 + 54k + 8$;*

($\overline{8}^0$) *G is a strongly regular graph of order $n = 9(10k+3)^2$ and degree $r = 5(90k^2+54k+8)$ with $\tau = 225k^2 + 155k + 25$ and $\theta = (5k+1)(45k+14)$, where $k \geqslant 0$. Its eigenvalues are $\lambda_2 = 45k + 13$ and $\lambda_3 = -(5k+2)$ with $m_2 = 90k^2 + 54k + 8$ and $m_3 = 9(90k^2 + 54k + 8)$;*

(9^0) *G is a strongly regular graph of order $n = 21(10k-1)^2$ and degree $r = 3(210k^2 - 42k + 2)$ with $\tau = 189k^2 - 77k + 5$ and $\theta = 21k(9k-1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 7k - 1$ and $\lambda_3 = -(63k-6)$ with $m_2 = 9(210k^2 - 42k + 2)$ and $m_3 = 210k^2 - 42k + 2$;*

($\overline{9}^0$) *G is a strongly regular graph of order $n = 21(10k-1)^2$ and degree $r = 7(210k^2-42k+2)$ with $\tau = 7(147k^2-27k+1)$ and $\theta = 7(7k-1)(21k-2)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 63k - 7$ and $\lambda_3 = -7k$ with $m_2 = 210k^2 - 42k + 2$ and $m_3 = 9(210k^2 - 42k + 2)$;*

(10^0) *G is a strongly regular graph of order $n = 21(10k+1)^2$ and degree $r = 3(210k^2 + 42k + 2)$ with $\tau = 189k^2 + 77k + 5$ and $\theta = 21k(9k+1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 63k + 6$ and $\lambda_3 = -(7k+1)$ with $m_2 = 210k^2 + 42k + 2$ and $m_3 = 9(210k^2 + 42k + 2)$;*

($\overline{10}^0$) *G is a strongly regular graph of order $n = 21(10k+1)^2$ and degree $r = 7(210k^2+42k+2)$ with $\tau = 7(147k^2+27k+1)$ and $\theta = 7(7k+1)(21k+2)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 7k$ and $\lambda_3 = -(63k+7)$ with $m_2 = 9(210k^2 + 42k + 2)$ and $m_3 = 210k^2 + 42k + 2$;*

PROOF. First, according to Remark 2.3 we have $\alpha(\beta - 1) = 9(\alpha - 1)$, from which we find that $\alpha = 3$, $\beta = 7$ or $\alpha = 9$, $\beta = 9$. In view of this we obtain the strongly regular graphs represented in Theorem 2.1 (1^0), (2^0). Next, according to Proposition 2.1 it turns out that G belongs to the class ($\overline{3}^0$) or (4^0) or ($\overline{5}^0$) or (6^0) or (7^0) or (8^0) or (9^0) or ($\overline{10}^0$) if $m_2 = 9m_3$. According to Proposition 2.2 it turns out that G belongs to the class (3^0) or $(\overline{4}^0)$ or (5^0) or $(\overline{6}^0)$ or $(\overline{7}^0)$ or $(\overline{8}^0)$ or $(\overline{9}^0)$ or (10^0) if $m_3 = 9m_2$. □

PROPOSITION 2.3. *Let G be a connected strongly regular graph of order n and degree r with $m_2 = 10m_3$. Then G belongs to the class $(\overline{4}^0)$ or (5^0) or (6^0) or $(\overline{7}^0)$ or (8^0) or $(\overline{9}^0)$ or (10^0) or $(\overline{11}^0)$ or $(\overline{12}^0)$ or (13^0) represented in Theorem* 2.2.

PROOF. Let $m_3 = p$, $m_2 = 10p$ and $n = 11p + 1$ where $p \in \mathbb{N}$. Using (1.2) we obtain $r = p(|\lambda_3| - 10\lambda_2)$. Let $|\lambda_3| - 10\lambda_2 = t$ where $t = 1, 2, \dots, 10$. Let $\lambda_2 = k$ where k is a positive integer. Then according to Remark 1.3 we have (i) $\lambda_3 = -(10k + t)$; (ii) $\tau - \theta = -(9k + t)$; (iii) $\delta = 11k + t$; (iv) $r = pt$ and (v) $\theta = pt - 10k^2 - kt$. In this case we can easily see that (1.3) is reduced to

$$(p+1)t^2 - (11p+1)t + 110k^2 + 20kt = 0\,. \tag{2.3}$$

CASE 1. ($t = 1$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(10k+1)$, $\tau - \theta = -(9k+1)$, $\delta = 11k+1$, $r = p$ and $\theta = p - 10k^2 - k$. Using (2.3) we find that $p = k(11k + 2)$. So we obtain that G is a strongly regular graph of order $n = (11k + 1)^2$ and degree $r = k(11k + 2)$ with $\tau = k^2 - 8k - 1$ and $\theta = k(k + 1)$.

CASE 2. ($t = 2$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(10k+2)$, $\tau - \theta = -(9k+2)$, $\delta = 11k+2$, $r = 2p$ and $\theta = 2p - 10k^2 - 2k$. Using (2.3) we find that $9p - 1 = 5k(11k + 4)$. Replacing k with $3k - 1$ we arrive at $p = 55k^2 - 30k + 4$. So we obtain that G is a strongly regular graph of order $n = 5(11k - 3)^2$ and degree $r = 2(55k^2 - 30k + 4)$ with $\tau = 20k^2 - 33k + 7$ and $\theta = 2k(10k - 3)$.

CASE 3. ($t = 3$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(10k+3)$, $\tau - \theta = -(9k+3)$, $\delta = 11k+3$, $r = 3p$ and $\theta = 3p - 10k^2 - 3k$. Using (2.3) we find that $12p - 3 = 5k(11k + 6)$. Replacing k with $6k - 3$ we arrive at $p = 165k^2 - 150k + 34$. So we obtain that G is a strongly regular graph of order $n = 15(11k - 5)^2$ and degree $r = 3(165k^2 - 150k + 34)$ with $\tau = 3(45k^2 - 54k + 15)$ and $\theta = 3(3k - 1)(15k - 7)$.

CASE 4. ($t = 4$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(10k+4)$, $\tau - \theta = -(9k+4)$, $\delta = 11k+4$, $r = 4p$ and $\theta = 4p - 10k^2 - 4k$. Using (2.3) we find that $14p - 6 = 5k(11k + 8)$. Replacing k with $14k + 6$ we arrive at $p = 770k^2 + 700k + 159$. So we obtain that G is a strongly regular graph of order $n = 70(11k+5)^2$ and degree $r = 4(770k^2+700k+159)$ with $\tau = 2(560k^2+469k+97)$ and $\theta = 28(2k + 1)(20k + 9)$.

CASE 5. ($t = 5$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(10k+5)$, $\tau - \theta = -(9k+5)$, $\delta = 11k+5$, $r = 5p$ and $\theta = 5p - 10k^2 - 5k$. Using (2.3) we find that $3p - 2 = k(11k + 10)$. Replacing k with $3k - 1$ we arrive at $p = 33k^2 - 12k + 1$. So we obtain that G is a strongly regular graph of order $n = 3(11k - 2)^2$ and degree $r = 5(33k^2 - 12k + 1)$ with $\tau = 75k^2 - 42k + 4$ and $\theta = 15k(5k - 1)$.

CASE 6. ($t = 6$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(10k+6)$, $\tau - \theta = -(9k+6)$, $\delta = 11k+6$, $r = 6p$ and $\theta = 6p - 10k^2 - 6k$. Using (2.3) we find that $3p - 3 = k(11k + 12)$. Replacing k with $3k$ we arrive at $p = 33k^2 + 12k + 1$. So we obtain that G is a strongly regular graph of order $n = 3(11k + 2)^2$ and degree $r = 6(33k^2 + 12k + 1)$ with $\tau = 27k(4k + 1)$ and $\theta = 6(3k + 1)(6k + 1)$.

CASE 7. ($t = 7$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(10k + 7)$, $\tau - \theta = -(9k + 7)$, $\delta = 11k + 7$, $r = 7p$ and $\theta = 7p - 10k^2 - 7k$. Using (2.3) we find that $14p-21 = 5k(11k+14)$. Replacing k with $14k-7$ we arrive at $p = 770k^2 - 700k + 159$. So we obtain that G is a strongly regular graph of order $n = 70(11k-5)^2$ and degree $r = 7(770k^2-700k+159)$ with $\tau = 14(245k^2-226k+52)$ and $\theta = 14(7k - 3)(35k - 16)$.

CASE 8. ($t = 8$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(10k + 8)$, $\tau - \theta = -(9k + 8)$, $\delta = 11k + 8$, $r = 8p$ and $\theta = 8p - 10k^2 - 8k$. Using (2.3) we find that $12p-28 = 5k(11k+16)$. Replacing k with $6k+2$ we arrive at $p = 165k^2 + 150k + 34$. So we obtain that G is a strongly regular graph of order $n = 15(11k+5)^2$ and degree $r = 8(165k^2+150k+34)$ with $\tau = 2(480k^2+429k+95)$ and $\theta = 24(2k + 1)(20k + 9)$.

CASE 9. ($t = 9$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(10k + 9)$, $\tau - \theta = -(9k + 9)$, $\delta = 11k + 9$, $r = 9p$ and $\theta = 9p - 10k^2 - 9k$. Using (2.3) we find that $9p - 36 = 5k(11k + 18)$. Replacing k with $3k$ we arrive at $p = 55k^2 + 30k + 4$. So we obtain that G is a strongly regular graph of order $n = 5(11k + 3)^2$ and degree $r = 9(55k^2 + 30k + 4)$ with $\tau = 27(3k + 1)(5k + 1)$ and $\theta = 9(3k + 1)(15k + 4)$.

CASE 10. ($t = 10$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = k$ and $\lambda_3 = -(10k+10)$, $\tau-\theta = -(9k+10)$, $\delta = 11k+10$, $r = 10p$ and $\theta = 10p-10k^2-10k$. Using (2.3) we find that $p = (k + 1)(11k + 9)$. Replacing k with $k - 1$ we arrive at $p = k(11k - 2)$. So we obtain that G is a strongly regular graph of order $n = (11k - 1)^2$ and degree $r = 10k(11k - 2)$ with $\tau = 100k^2 - 19k - 1$ and $\theta = 10k(10k - 1)$. □

PROPOSITION 2.4. *Let G be a connected strongly regular graph of order n and degree r with $m_3 = 10m_2$. Then G belongs to the class* (4^0) *or* $(\overline{5}^0)$ *or* $(\overline{6}^0)$ *or* (7^0) *or* $(\overline{8}^0)$ *or* (9^0) *or* $(\overline{10}^0)$ *or* (11^0) *or* (12^0) *or* $(\overline{13}^0)$ *represented in Theorem* 2.2.

PROOF. Let $m_2 = p$, $m_3 = 10p$ and $n = 11p + 1$ where $p \in \mathbb{N}$. Using (1.2) we obtain $r = p(10|\lambda_3| - \lambda_2)$. Let $10|\lambda_3| - \lambda_2 = t$ where $t = 1, 2, \ldots, 10$. Let $\lambda_3 = -k$ where k is a positive integer. Then according to Remark 1.3 we have (i) $\lambda_2 = 10k-t$; (ii) $\tau - \theta = 9k - t$; (iii) $\delta = 11k - t$; (iv) $r = pt$ and (v) $\theta = pt - 10k^2 + kt$. In this case we can easily see that (1.4) is reduced to

$$(p + 1)t^2 - (11p + 1)t + 110k^2 - 20kt = 0\,. \tag{2.4}$$

CASE 1. ($t = 1$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 10k-1$ and $\lambda_3 = -k$, $\tau - \theta = 9k - 1$, $\delta = 11k - 1$, $r = p$ and $\theta = p - 10k^2 + k$. Using (2.4) we find that $p = k(11k - 2)$. So we obtain that G is a strongly regular graph of order $n = (11k - 1)^2$ and degree $r = k(11k - 2)$ with $\tau = k^2 + 8k - 1$ and $\theta = k(k - 1)$.

CASE 2. ($t = 2$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 10k - 2$ and $\lambda_3 = -k$, $\tau - \theta = 9k - 2$, $\delta = 11k - 2$, $r = 2p$ and $\theta = 2p - 10k^2 + 2k$. Using (2.4) we find that $9p - 1 = 5k(11k - 4)$. Replacing k with $3k + 1$ we arrive at $p = 55k^2 + 30k + 4$. So we obtain that G is a strongly regular graph of order $n = 5(11k + 3)^2$ and degree $r = 2(55k^2 + 30k + 4)$ with $\tau = 20k^2 + 33k + 7$ and $\theta = 2k(10k + 3)$.

Case 3. $(t = 3)$. Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 10k - 3$ and $\lambda_3 = -k$, $\tau - \theta = 9k - 3$, $\delta = 11k - 3$, $r = 3p$ and $\theta = 3p - 10k^2 + 3k$. Using (2.4) we find that $12p - 3 = 5k(11k - 6)$. Replacing k with $6k + 3$ we arrive at $p = 165k^2 + 150k + 34$. So we obtain that G is a strongly regular graph of order $n = 15(11k + 5)^2$ and degree $r = 3(165k^2 + 150k + 34)$ with $\tau = 3(45k^2 + 54k + 15)$ and $\theta = 3(3k + 1)(15k + 7)$.

Case 4. $(t = 4)$. Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 10k - 4$ and $\lambda_3 = -k$, $\tau - \theta = 9k - 4$, $\delta = 11k - 4$, $r = 4p$ and $\theta = 4p - 10k^2 + 4k$. Using (2.4) we find that $14p - 6 = 5k(11k - 8)$. Replacing k with $14k - 6$ we arrive at $p = 770k^2 - 700k + 159$. So we obtain that G is a strongly regular graph of order $n = 70(11k-5)^2$ and degree $r = 4(770k^2-700k+159)$ with $\tau = 2(560k^2-469k+97)$ and $\theta = 28(2k - 1)(20k - 9)$.

Case 5. $(t = 5)$. Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 10k - 5$ and $\lambda_3 = -k$, $\tau - \theta = 9k - 5$, $\delta = 11k - 5$, $r = 5p$ and $\theta = 5p - 10k^2 + 5k$. Using (2.4) we find that $3p - 2 = k(11k - 10)$. Replacing k with $3k + 1$ we arrive at $p = 33k^2 + 12k + 1$. So we obtain that G is a strongly regular graph of order $n = 3(11k + 2)^2$ and degree $r = 5(33k^2 + 12k + 1)$ with $\tau = 75k^2 + 42k + 4$ and $\theta = 15k(5k + 1)$.

Case 6. $(t = 6)$. Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 10k-6$ and $\lambda_3 = -k$, $\tau-\theta = 9k-6$, $\delta = 11k-6$, $r = 6p$ and $\theta = 6p-10k^2+6k$. Using (2.4) we find that $3p-3 = k(11k-12)$. Replacing k with $3k$ we arrive at $p = 33k^2-12k+1$. So we obtain that G is a strongly regular graph of order $n = 3(11k-2)^2$ and degree $r = 6(33k^2 - 12k + 1)$ with $\tau = 27k(4k - 1)$ and $\theta = 6(3k - 1)(6k - 1)$.

Case 7. $(t = 7)$. Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 10k - 7$ and $\lambda_3 = -k$, $\tau - \theta = 9k - 7$, $\delta = 11k - 7$, $r = 7p$ and $\theta = 7p - 10k^2 + 7k$. Using (2.4) we find that $14p - 21 = 5k(11k - 14)$. Replacing k with $14k + 7$ we arrive at $p = 770k^2 + 700k + 159$. So we obtain that G is a strongly regular graph of order $n = 70(11k+5)^2$ and degree $r = 7(770k^2+700k+159)$ with $\tau = 14(245k^2+226k+52)$ and $\theta = 14(7k + 3)(35k + 16)$.

Case 8. $(t = 8)$. Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 10k - 8$ and $\lambda_3 = -k$, $\tau - \theta = 9k - 8$, $\delta = 11k - 8$, $r = 8p$ and $\theta = 8p - 10k^2 + 8k$. Using (2.4) we find that $12p - 28 = 5k(11k - 16)$. Replacing k with $6k - 2$ we arrive at $p = 165k^2 - 150k + 34$. So we obtain that G is a strongly regular graph of order $n = 15(11k-5)^2$ and degree $r = 8(165k^2-150k+34)$ with $\tau = 2(480k^2-429k+95)$ and $\theta = 24(2k - 1)(20k - 9)$.

Case 9. $(t = 9)$. Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 10k - 9$ and $\lambda_3 = -k$, $\tau - \theta = 9k - 9$, $\delta = 11k - 9$, $r = 9p$ and $\theta = 9p - 10k^2 + 9k$. Using (2.4) we find that $9p - 36 = 5k(11k - 18)$. Replacing k with $3k$ we arrive at $p = 55k^2 - 30k + 4$. So we obtain that G is a strongly regular graph of order $n = 5(11k - 3)^2$ and degree $r = 9(55k^2 - 30k + 4)$ with $\tau = 27(3k - 1)(5k - 1)$ and $\theta = 9(3k - 1)(15k - 4)$.

CASE 10. ($t = 10$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 10k - 10$ and $\lambda_3 = -k$, $\tau - \theta = 9k - 10$, $\delta = 11k - 10$, $r = 10p$ and $\theta = 10p - 10k^2 + 10k$. Using (2.4) we find that $p = (k-1)(11k-9)$. Replacing k with $k+1$ we arrive at $p = k(11k+2)$. So we obtain that G is a strongly regular graph of order $n = (11k+1)^2$ and degree $r = 10k(11k+2)$ with $\tau = 100k^2 + 19k - 1$ and $\theta = 10k(10k+1)$. □

REMARK 2.6. We note that the complete bipartite graph $K_{6,6}$ is a strongly regular graph with $m_2 = 10m_3$. It is obtained from the class Theorem 2.2 ($\overline{7}^0$) for $k = 0$.

REMARK 2.7. We note that $\overline{5K_9}$ is a strongly regular graph with $m_2 = 10m_3$. It is obtained from the class Theorem 2.2 ($\overline{9}^0$) for $k = 0$.

REMARK 2.8. We note that $\overline{10K_{10}}$ is a strongly regular graph with $m_2 = 10m_3$. It is obtained from the class Theorem 2.2 ($\overline{4}^0$) for $k = 1$.

THEOREM 2.2. *Let G be a connected strongly regular graph of order n and degree r with $m_2 = 10m_3$ or $m_3 = 10m_2$. Then G is one of the following strongly regular graphs:*

(1^0) *G is the complete bipartite graph $K_{6,6}$ of order $n = 12$ and degree $r = 6$ with $\tau = 0$ and $\theta = 6$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -6$ with $m_2 = 10$ and $m_3 = 1$;*

(2^0) *G is the strongly regular graph $\overline{5K_9}$ of order $n = 45$ and degree $r = 36$ with $\tau = 27$ and $\theta = 36$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -9$ with $m_2 = 40$ and $m_3 = 4$;*

(3^0) *G is the strongly regular graph $\overline{10K_{10}}$ of order $n = 100$ and degree $r = 90$ with $\tau = 80$ and $\theta = 90$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -10$ with $m_2 = 90$ and $m_3 = 9$;*

(4^0) *G is a strongly regular graph of order $n = (11k-1)^2$ and degree $r = k(11k-2)$ with $\tau = k^2 + 8k - 1$ and $\theta = k(k-1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = 10k - 1$ and $\lambda_3 = -k$ with $m_2 = k(11k-2)$ and $m_3 = 10k(11k-2)$;*

($\overline{4}^0$) *G is a strongly regular graph of order $n = (11k-1)^2$ and degree $r = 10k(11k-2)$ with $\tau = 100k^2 - 19k - 1$ and $\theta = 10k(10k-1)$, where $k \in \mathbb{N}$. Its eigenvalues are $\lambda_2 = k - 1$ and $\lambda_3 = -10k$ with $m_2 = 10k(11k-2)$ and $m_3 = k(11k-2)$;*

(5^0) *G is a strongly regular graph of order $n = (11k+1)^2$ and degree $r = k(11k+2)$ with $\tau = k^2 - 8k - 1$ and $\theta = k(k+1)$, where $k \geqslant 9$. Its eigenvalues are $\lambda_2 = k$ and $\lambda_3 = -(10k+1)$ with $m_2 = 10k(11k+2)$ and $m_3 = k(11k+2)$;*

($\overline{5}^0$) *G is a strongly regular graph of order $n = (11k+1)^2$ and degree $r = 10k(11k+2)$ with $\tau = 100k^2 + 19k - 1$ and $\theta = 10k(10k+1)$, where $k \geqslant 9$. Its eigenvalues are $\lambda_2 = 10k$ and $\lambda_3 = -(k+1)$ with $m_2 = k(11k+2)$ and $m_3 = 10k(11k+2)$;*

(6^0) *G is a strongly regular graph of order* $n = 3(11k-2)^2$ *and degree* $r = 5(33k^2-12k+1)$ *with* $\tau = 75k^2-42k+4$ *and* $\theta = 15k(5k-1)$*, where* $k \in \mathbb{N}$*. Its eigenvalues are* $\lambda_2 = 3k-1$ *and* $\lambda_3 = -(30k-5)$ *with* $m_2 = 10(33k^2-12k+1)$ *and* $m_3 = 33k^2-12k+1$ *;*

($\overline{6}^0$) *G is a strongly regular graph of order* $n = 3(11k-2)^2$ *and degree* $r = 6(33k^2-12k+1)$ *with* $\tau = 27k(4k-1)$ *and* $\theta = 6(3k-1)(6k-1)$*, where* $k \in \mathbb{N}$*. Its eigenvalues are* $\lambda_2 = 30k-6$ *and* $\lambda_3 = -3k$ *with* $m_2 = 33k^2-12k+1$ *and* $m_3 = 10(33k^2-12k+1)$ *;*

(7^0) *G is a strongly regular graph of order* $n = 3(11k+2)^2$ *and degree* $r = 5(33k^2+12k+1)$ *with* $\tau = 75k^2+42k+4$ *and* $\theta = 15k(5k+1)$*, where* $k \in \mathbb{N}$*. Its eigenvalues are* $\lambda_2 = 30k+5$ *and* $\lambda_3 = -(3k+1)$ *with* $m_2 = 33k^2+12k+1$ *and* $m_3 = 10(33k^2+12k+1)$ *;*

($\overline{7}^0$) *G is a strongly regular graph of order* $n = 3(11k+2)^2$ *and degree* $r = 6(33k^2+12k+1)$ *with* $\tau = 27k(4k+1)$ *and* $\theta = 6(3k+1)(6k+1)$*, where* $k \in \mathbb{N}$*. Its eigenvalues are* $\lambda_2 = 3k$ *and* $\lambda_3 = -(30k+6)$ *with* $m_2 = 10(33k^2+12k+1)$ *and* $m_3 = 33k^2+12k+1$ *;*

(8^0) *G is a strongly regular graph of order* $n = 5(11k-3)^2$ *and degree* $r = 2(55k^2-30k+4)$ *with* $\tau = 20k^2-33k+7$ *and* $\theta = 2k(10k-3)$*, where* $k \geqslant 2$*. Its eigenvalues are* $\lambda_2 = 3k-1$ *and* $\lambda_3 = -(30k-8)$ *with* $m_2 = 10(55k^2-30k+4)$ *and* $m_3 = 55k^2-30k+4$ *;*

($\overline{8}^0$) *G is a strongly regular graph of order* $n = 5(11k-3)^2$ *and degree* $r = 9(55k^2-30k+4)$ *with* $\tau = 27(3k-1)(5k-1)$ *and* $\theta = 9(3k-1)(15k-4)$*, where* $k \geqslant 2$*. Its eigenvalues are* $\lambda_2 = 30k-9$ *and* $\lambda_3 = -3k$ *with* $m_2 = 55k^2-30k+4$ *and* $m_3 = 10(55k^2-30k+4)$ *;*

(9^0) *G is a strongly regular graph of order* $n = 5(11k+3)^2$ *and degree* $r = 2(55k^2+30k+4)$ *with* $\tau = 20k^2+33k+7$ *and* $\theta = 2k(10k+3)$*, where* $k \in \mathbb{N}$*. Its eigenvalues are* $\lambda_2 = 30k+8$ *and* $\lambda_3 = -(3k+1)$ *with* $m_2 = 55k^2+30k+4$ *and* $m_3 = 10(55k^2+30k+4)$ *;*

($\overline{9}^0$) *G is a strongly regular graph of order* $n = 5(11k+3)^2$ *and degree* $r = 9(55k^2+30k+4)$ *with* $\tau = 27(3k+1)(5k+1)$ *and* $\theta = 9(3k+1)(15k+4)$*, where* $k \in \mathbb{N}$*. Its eigenvalues are* $\lambda_2 = 3k$ *and* $\lambda_3 = -(30k+9)$ *with* $m_2 = 10(55k^2+30k+4)$ *and* $m_3 = 55k^2+30k+4$ *;*

(10^0) *G is a strongly regular graph of order* $n = 15(11k-5)^2$ *and degree* $r = 3(165k^2-150k+34)$ *with* $\tau = 3(45k^2-54k+15)$ *and* $\theta = 3(3k-1)(15k-7)$*, where* $k \in \mathbb{N}$*. Its eigenvalues are* $\lambda_2 = 6k-3$ *and* $\lambda_3 = -(60k-27)$ *with* $m_2 = 10(165k^2-150k+34)$ *and* $m_3 = 165k^2-150k+34$ *;*

($\overline{10}^0$) *G is a strongly regular graph of order* $n = 15(11k-5)^2$ *and degree* $r = 8(165k^2-150k+34)$ *with* $\tau = 2(480k^2-429k+95)$ *and* $\theta = 24(2k-1)(20k-9)$*, where* $k \in \mathbb{N}$*. Its eigenvalues are* $\lambda_2 = 60k-28$ *and* $\lambda_3 = -(6k-2)$ *with* $m_2 = 165k^2-150k+34$ *and* $m_3 = 10(165k^2-150k+34)$ *;*

(11^0) *G is a strongly regular graph of order* $n = 15(11k+5)^2$ *and degree* $r = 3(165k^2+150k+34)$ *with* $\tau = 3(45k^2+54k+15)$ *and* $\theta = 3(3k+1)(15k+7)$, *where* $k \in \mathbb{N}$. *Its eigenvalues are* $\lambda_2 = 60k+27$ *and* $\lambda_3 = -(6k+3)$ *with* $m_2 = 165k^2+150k+34$ *and* $m_3 = 10(165k^2+150k+34)$ *;*

($\overline{11}^0$) *G is a strongly regular graph of order* $n = 15(11k+5)^2$ *and degree* $r = 8(165k^2+150k+34)$ *with* $\tau = 2(480k^2+429k+95)$ *and* $\theta = 24(2k+1)(20k+9)$, *where* $k \in \mathbb{N}$. *Its eigenvalues are* $\lambda_2 = 6k+2$ *and* $\lambda_3 = -(60k+28)$ *with* $m_2 = 10(165k^2+150k+34)$ *and* $m_3 = 165k^2+150k+34$ *;*

(12^0) *G is a strongly regular graph of order* $n = 70(11k-5)^2$ *and degree* $r = 4(770k^2-700k+159)$ *with* $\tau = 2(560k^2-469k+97)$ *and* $\theta = 28(2k-1)(20k-9)$, *where* $k \in \mathbb{N}$. *Its eigenvalues are* $\lambda_2 = 140k-64$ *and* $\lambda_3 = -(14k-6)$ *with* $m_2 = 770k^2-700k+159$ *and* $m_3 = 10(770k^2-700k+159)$ *;*

($\overline{12}^0$) *G is a strongly regular graph of order* $n = 70(11k-5)^2$ *and degree* $r = 7(770k^2-700k+159)$ *with* $\tau = 14(245k^2-226k+52)$ *and* $\theta = 14(7k-3)(35k-16)$, *where* $k \in \mathbb{N}$. *Its eigenvalues are* $\lambda_2 = 14k-7$ *and* $\lambda_3 = -(140k-63)$ *with* $m_2 = 10(770k^2-700k+159)$ *and* $m_3 = 770k^2-700k+159$ *;*

(13^0) *G is a strongly regular graph of order* $n = 70(11k+5)^2$ *and degree* $r = 4(770k^2+700k+159)$ *with* $\tau = 2(560k^2+469k+97)$ *and* $\theta = 28(2k+1)(20k+9)$, *where* $k \geqslant 0$. *Its eigenvalues are* $\lambda_2 = 14k+6$ *and* $\lambda_3 = -(140k+64)$ *with* $m_2 = 10(770k^2+700k+159)$ *and* $m_3 = 770k^2+700k+159$ *;*

($\overline{13}^0$) *G is a strongly regular graph of order* $n = 70(11k+5)^2$ *and degree* $r = 7(770k^2+700k+159)$ *with* $\tau = 14(245k^2+226k+52)$ *and* $\theta = 14(7k+3)(35k+16)$, *where* $k \geqslant 0$. *Its eigenvalues are* $\lambda_2 = 140k+63$ *and* $\lambda_3 = -(14k+7)$ *with* $m_2 = 770k^2+700k+159$ *and* $m_3 = 10(770k^2+700k+159)$ *.*

Proof. First, according to Remark 2.3 we have $\alpha(\beta-1) = 10(\alpha-1)$, from which we find that $\alpha = 2$, $\beta = 6$ or $\alpha = 5$, $\beta = 9$ or $\alpha = 10$, $\beta = 10$. In view of this we obtain the strongly regular graphs represented in Theorem 2.2 (1^0), (2^0), (3^0). Next, according to Proposition 2.3 it turns out that G belongs to the class ($\overline{4}^0$) or (5^0) or (6^0) or ($\overline{7}^0$) or (8^0) or ($\overline{9}^0$) or (10^0) or ($\overline{11}^0$) or ($\overline{12}^0$) or (13^0) if $m_2 = 10m_3$. According to Proposition 2.4 it turns out that G belongs to the class (4^0) or ($\overline{5}^0$) or ($\overline{6}^0$) or (7^0) or ($\overline{8}^0$) or (9^0) or ($\overline{10}^0$) or (11^0) or (12^0) or ($\overline{13}^0$) if $m_3 = 10m_2$. $\square$

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