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A STUDY ON PYTHAGOREAN FUZZY BI $\Gamma-$HYPERIDEALS IN $\Gamma-$HYPERSEMIGROUPS

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Abstract. In this paper, we study the notion of Pythagorean fuzzy $\Gamma-$hyperideals in $\Gamma-$hypersemigroups. We also study Pythagorean fuzzy $\Gamma-$ subsemihypergroup and Pythagorean fuzzy bi $\Gamma-$hyperideals in $\Gamma-$hypersemigroups and discuss some properties. Relations between these Γ-hyperideals are also studied. Inverse image of Pythagorean fuzzy $\Gamma-$ subsemihypergroup and Pythagorean fuzzy bi $\Gamma-$hyperideals are discussed.

1. Introduction

Fuzzy set theory was proposed by Zadeh[**14**]. Atanassov[**3**] introduced a new generalization, intuitionistic fuzzy set in 1986. He defined some new operations on intuitionistic fuzzy sets[**4**]. The theory of Pythagorean fuzzy set was propounded by Yager[**11**]. It is a generalization of intuitionistic fuzzy set. In recent days Pythagorean fuzzy set got more attention among the researchers. It plays an important role to tackle the uncertainities. Kumar et al.[**9**] approached transportation decision making problems using Pythagorean fuzzy set. Yager[**12**] and Zhang[**15**] were applied the concept of Pythagorean fuzzy set in decision making problem.

Marty[**10**] extended the algebraic structures to algebraic hyperstructures. In classic structure, the product of two elements is an element again but in hyperstructure it will be a set. Anvariyeh et al.[**2**] studied Γ–hyperideals in Γ–hypersemigroups. Hussain et al.[**8**] applied the concept of rough set in Pythagorean fuzzy ideals in semigroups. Chinram et al.[**6**] extended the idea of [**8**] to ternary semigroups. Akram[**1**] established the properties of fuzzy lie algebras. Davvaz[**7**] studied fuzzy

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hyperideals and intuitionistic fuzzy hyperideals in hypersemigroups. In this paper we introduce Pythagorean fuzzy left(right) $\Gamma-$hyperideals, Pythagorean fuzzy Γ–subsemihypergroup and Pythagorean fuzzy bi Γ–hyperideal in Γ–hypersemigroups. We find the relationship between these $\Gamma-$hyperideals. We also study the inverse image of Pythagorean fuzzy set in $\Gamma-$hypersemigroups.

2. Pythagorean left(right) $\Gamma-$hyperideals in $\Gamma-$hypersemigroups

In this section we study the Pythagorean left(right) $\Gamma-$hyperideals in $\Gamma-$hypersemigroups and discuss some properties.

DEFINITION 2.1. [**2**] *Let Y and Γ be two non-empty sets. Y is called a Γ - hypersemigroup if*
(i) $c\gamma d \in Y$ and
(ii) $c\gamma(d\eta e) = (c\gamma d)\eta e$
for every $c, d \in Y$ and $\gamma, \eta \in \Gamma$ are hyperoperations on Y.

Let Y_1 and Y_2 be two non-empty subsets of Y and $\gamma \in \Gamma$ we define
$Y_1\Gamma Y_2 = \bigcup\{c\gamma d : c \in Y_1 \text{ and } d \in Y_2\}$
Also

$$Y_1\Gamma Y_2 = \bigcup_{(c,d)\in Y_1\times Y_2} c\gamma d \tag{2.1}$$

DEFINITION 2.2. [**2**] *Let Y and Γ be any two sets. A non-empty set B of a $\Gamma-$hypersemigroup Y is said to be a*
(i) left(right) $\Gamma-$hyperideal if $Y\Gamma B \subseteq B (B\Gamma Y \subseteq B)$.
(ii) $\Gamma-$subsemihypergroup if $B\Gamma B \subseteq B$.
(iii) bi $\Gamma-$hyperideal if $B\Gamma Y\Gamma B \subseteq B$.

DEFINITION 2.3. *Let Y be a $\Gamma-$hypersemigroup. A Pythagorean fuzzy set $P = (\varphi, \varrho)$ is defined by*
$P = \{(x/\varphi(x), \varrho(x)) : x \in Y\}$ *such that* $0 \leqslant \varphi(x)^2 + \varrho(x)^2 \leqslant 1$.
where $\varphi(x)$ is a membership function from the universe set Y to the closed interval $[0, 1]$ and $\varrho(x)$ is a non-membership function from the universe set Y to the closed interval $[0, 1]$.

DEFINITION 2.4. *If $P = (\varphi, \varrho)$ and $Q = (\varphi', \varrho')$ are any two Pythagorean fuzzy sets of Y then the following operations are defined as:*
(i) $P \cap \mathcal{Q} = \{(x/\varphi(x) \wedge \varphi'(x), \varrho(x) \vee \varrho'(x)) : x \in Y\}$
(ii) $P \cup \mathcal{Q} = \{(x/\varphi(x) \vee \varphi'(x), \varrho(x) \wedge \varrho'(x)) : x \in Y\}$
(iii) $\Box P = (\varphi, \hat{\varphi})$, *where* $\hat{\varphi} = 1 - \varphi$.

DEFINITION 2.5. *A Pythagorean fuzzy set $P = (\varphi, \varrho)$ of Y is said to be a Pythagorean fuzzy left $\Gamma-$hyperideal if for $z \in Y$ we have*
(i) $\varphi(d) \leqslant \inf\limits_{z\in c\gamma d} \varphi(z)$ *and*
(ii) $\varrho(d) \geqslant \sup\limits_{z\in c\gamma d} \varrho(z)$ *for all $c, d \in Y$.*
A Pythagorean fuzzy set $P = (\varphi, \varrho)$ of Y is said to be a Pythagorean fuzzy right

$\Gamma-$hyperideal if for $z \in Y$ we have

(i) $\varphi(c) \leqslant \inf\limits_{z \in c\gamma d} \varphi(z)$ *and*

(ii) $\varrho(c) \geqslant \sup\limits_{z \in c\gamma d} \varrho(z)$ *for all* $c, d \in Y$.

A Pythagorean fuzzy set $P = (\varphi, \varrho)$ of Y is said to be a Pythagorean fuzzy $\Gamma-$hyperideal if P is a both Pythagorean fuzzy left $\Gamma-$hyperideal and Pythagorean fuzzy right $\Gamma-$hyperideal of $\Gamma-$hypersemigroup.

Example 2.1. *Let $Y = \{a, b, c, d\}$ and $\Gamma = \{\gamma\}$ be any two sets then Y is a $\Gamma-$hypersemigroup.*

γ	a	b	c	d
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
c	$\{a\}$	$\{a\}$	$\{a, b\}$	$\{a, b\}$
d	$\{a\}$	$\{a\}$	$\{a, b\}$	$\{a\}$

Define a Pythagorean fuzzy set $P = (\varphi, \varrho)$ as

$$\varphi(x) = \begin{cases} 0.5 & if \quad x = a, b \\ 0.3 & if \quad x = c, d \end{cases}$$

$$\varrho(x) = \begin{cases} 0.7 & if \quad x = a, b \\ 1 & if \quad x = c, d \end{cases}$$

By calculation we can note that P is a Pythagorean left(right) $\Gamma-$hyperideal of $\Gamma-$hypersemigroup.

Theorem 2.1. *If $\{P_i = (\varphi_i, \varrho_i)\}_{i \in N}$ are Pythagorean fuzzy left(right) $\Gamma-$hyperideals of Y then $\bigcap\limits_{i \in N} P_i = (\bigcap\limits_{i \in N} \varphi_i, \bigcap\limits_{i \in N} \varrho_i)$ is also a Pythagorean fuzzy left(right) $\Gamma-$hyperideal of Y.*

Proof. Consider for $z \in Y$

$$\begin{aligned} \bigcap_{i \in N} \varphi_i(d) &= \varphi_1(d) \wedge \varphi_2(d) \wedge .... \wedge \varphi_n(d) \\ &\leqslant \inf_{z \in c\gamma d} \varphi_1(z) \wedge \inf_{z \in c\gamma d} \varphi_2(z) \wedge ... \wedge \inf_{z \in c\gamma d} \varphi_n(z) \\ &\leqslant \inf_{z \in c\gamma d} \{\varphi_1(z) \wedge \varphi_2(z) \wedge ... \wedge \varphi_n(z)\} \\ &\leqslant \inf_{z \in c\gamma d} \left\{ \bigcap_{i \in N} \varphi_i(z) \right\} \forall\ c, d \in Y. \end{aligned}$$

Now

$$\begin{aligned}\bigcup_{i\in N}\varrho_i(d) &= \varrho_1(d)\vee\varrho_2(d)\vee....\vee\varrho_n(d)\\ &\geqslant \sup_{z\in c\gamma d}\varrho_1(z)\vee\sup_{z\in c\gamma d}\varrho_2(z)\vee...\vee\sup_{z\in c\gamma d}\varrho_n(z)\\ &\geqslant \sup_{z\in c\gamma d}\{\varrho_1(z)\vee\varrho_2(z)\vee...\vee\varrho_n(z)\}\\ &\geqslant \sup_{z\in c\gamma d}\left\{\bigcup_{i\in N}\varrho_i(z)\right\}\ \forall\ c,d\in Y.\end{aligned}$$

Thus $\bigcap_{i\in N} P_i$ is a Pythagorean fuzzy left $\Gamma-$hyperideal. Similarly we can prove that $\bigcap_{i\in N} P_i$ is a Pythagorean fuzzy right $\Gamma-$hyperideal. □

THEOREM 2.2. *If $\{P_i = (\varphi_i, \varrho_i)\}_{i\in N}$ are Pythagorean fuzzy left(right) $\Gamma-$hyperideals of Y then $\bigcup_{i\in N} P_i = (\bigcup_{i\in N}\varphi_i, \bigcup_{i\in N}\varrho_i)$ is also a Pythagorean fuzzy left(right) $\Gamma-$hyperideal of Y.*

PROOF. The proof is similar as in Theorem 2.1 □

DEFINITION 2.6. *If $P = (\varphi, \varrho)$ is a Pythagorean fuzzy set of Y then the image of P is defined by $IMG(P) = IMG(\varphi)\cup IMG(\varrho)$ where*
$IMG(\varphi) = \{\varphi(z) : z \in Y\}$ and $IMG(\varrho) = \{\varrho(z) : z\in Y\}$.

DEFINITION 2.7. *Let $P = (\varphi, \varrho)$ be a Pythagorean fuzzy set of Y and let $t \in IMG(P)$ then the sets*
$\varphi^t = \{z \in Y : \varphi(z) \geqslant t\}$ and
$\varrho^t = \{z\in Y : \varrho(z)\leqslant t\}$
are called $t-$ level cut of φ and $t-$ level cut of ϱ respectively.

THEOREM 2.3. *If $P = (\varphi, \varrho)$ is a Pythagorean fuzzy $\Gamma-$hyperideal of Y and for $t\in IMG(P)$ then $P^t = (\varphi^t, \varrho^t)$ is a $\Gamma-$hyperideal of Y.*

PROOF. Let $c, d, z \in Y$ such that $c \in \varphi^t$ then

$$\varphi(c)\geqslant t \tag{2.2}$$

Since $\varphi(c) \leqslant \inf_{z\in c\gamma d}\varphi(z)$
$\Longrightarrow \inf_{z\in c\gamma d}\varphi(z)\geqslant t$ by (2.2)
$\Longrightarrow \varphi(z)\geqslant t$ for $z\in c\gamma d$
$\Longrightarrow c\gamma d\in\varphi^t$.
Thus φ^t is a right $\Gamma-$hyperideal.

Now, let $c, d, z \in Y$ such that $c\in\varrho^t$ then

$$\varrho(c)\leqslant t \tag{2.3}$$

$\Longrightarrow \varrho(c) \geqslant \sup\limits_{z\in c\gamma d} \varrho(z)$
$\Longrightarrow \sup\limits_{z\in c\gamma d} \varrho(z) \leqslant t$ by (2.3)
$\Longrightarrow \varrho(z) \leqslant t$ for $z \in c\gamma d$
$\Longrightarrow c\gamma d \in \varrho^t$.
Thus ϱ^t is a right $\Gamma-$hyperideal. Similarly we can prove that P^t is a left $\Gamma-$hyperideal of Y. □

THEOREM 2.4. *If $P = (\varphi, \varrho)$ of Y is a Pythagorean $\Gamma-$hyperideal then so $\Box P = (\varphi, \hat{\varphi})$ where $\hat{\varphi} = 1 - \varphi$.*

PROOF. Since φ is a Pythagorean fuzzy $\Gamma-$hyperideal. Now to show that $\hat{\varphi}$ is a Pythagorean fuzzy $\Gamma-$hyperideal. Consider for $z, c, d \in Y$,

$$\begin{aligned}\hat{\varphi}(c) &= 1 - \varphi(c)\\ &\geqslant 1 - \sup_{z\in c\gamma d} \varphi(z)\\ &\geqslant \sup_{z\in c\gamma d} (1 - \varphi(z))\\ &\geqslant \sup_{z\in c\gamma d} \hat{\varphi}(z)\end{aligned}$$

Thus $\Box P$ is a Pythagorean fuzzy $\Gamma-$hyperideal of $Y^\star$. □

3. Pythagorean fuzzy bi Γ−hyperideals

In this section we discuss Pythagorean fuzzy $\Gamma-$subsemihypergroup and Pythagorean fuzzy bi $\Gamma-$hyperideals in Y.

DEFINITION 3.1. *A Pythagorean fuzzy set $P = (\varphi, \varrho)$ of Y is said to be Pythagorean fuzzy $\Gamma-$subsemihypergroup if for $z \in Y$ we have*
(i) $\varphi(c) \wedge \varphi(d) \leqslant \inf\limits_{z\in c\gamma d} \varphi(z)$
(ii) $\varrho(c) \vee \varrho(d) \geqslant \sup\limits_{z\in c\gamma d} \varrho(z)$ *for all* $c, d \in Y$ *and* $\gamma \in \Gamma$.

EXAMPLE 3.1. *Let us consider a hypersemigroup $Y = \{a, b, c, d\}$ with the hyperoperation $\gamma \in \Gamma$:*

γ	a	b	c	d
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a, b\}$	$\{a, c\}$	$\{a\}$
c	$\{a\}$	$\{a\}$	$a\}$	$\{a\}$
d	$\{a\}$	$\{a, d\}$	$\{a\}$	$\{a\}$

Define a Pythagorean fuzzy set $P = (\varphi, \varrho)$ as
$P = \{a/(1, 0.3), b/(0.2, 1), c/(0.2, 0.9), d/(0.7, 0.6)\}$. *We have*
$\varphi(b) \wedge \varphi(d) = 0.2 < 1 = \inf\limits_{a\in b\gamma d} \varphi(a)\ \forall\ b, d \in Y$,
$\varrho(c) \vee \varrho(d) = 0.9 > 0.3 = \sup\limits_{a\in c\gamma d} \varrho(a)\ \forall\ c, d \in Y$,
Similarly (i) and (ii) of Definition 3.1 holds for all $a, b, c, d \in Y$. Hence P is a Pythagorean fuzzy $\Gamma-$subsemihypergroup.

THEOREM 3.1. *If $\{P_i = (\varphi_i, \varrho_i)\}_{i\in N}$ are Pythagorean fuzzy $\Gamma-$ subsemihypergroup of Y then $\bigcap\limits_{i\in N} P_i = (\bigcap\limits_{i\in N} \varphi_i, \bigcap\limits_{i\in N} \varrho_i)$ is also a Pythagorean fuzzy $\Gamma-$subsemihypergroup of Y.*

PROOF. Consider for $z \in Y$

$$\begin{aligned}\bigcap_{i\in N} \varphi_i(c) \wedge \bigcap_{i\in N} \varphi_i(d) &= \{\varphi_1(c) \wedge \varphi_2(c) \wedge .... \wedge \varphi_n(c)\} \wedge \{\varphi_1(d) \wedge \varphi_2(d) \wedge .... \wedge \varphi_n(d)\} \\ &= \{\varphi_1(c) \wedge \varphi_1(d)\} \wedge \{\varphi_2(c) \wedge \varphi_2(d)\} \wedge ... \wedge \{\varphi_n(c) \wedge \varphi_n(d)\} \\ &\leqslant \inf_{z\in c\gamma d} \varphi_1(z) \wedge \inf_{z\in c\gamma d} \varphi_2(z) \wedge ... \wedge \inf_{z\in c\gamma d} \varphi_n(z) \\ &\leqslant \inf_{z\in c\gamma d} \{\varphi_1(z) \wedge \varphi_2(z) \wedge ... \wedge \varphi_n(z)\} \\ &\leqslant \inf_{z\in c\gamma d} \left\{\bigcap_{i\in N} \varphi_i(z)\right\} \forall\ c, d \in Y.\end{aligned}$$

$$\begin{aligned}\bigcap_{i\in N} \varrho_i(c) \vee \bigcap_{i\in N} \varrho_i(d) &= \{\varrho_1(c) \vee \varrho_2(c) \vee .... \vee \varrho_n(c)\} \vee \{\varrho_1(d) \vee \varrho_2(d) \vee .... \vee \varrho_n(d)\} \\ &= \{\varrho_1(c) \vee \varrho_1(d)\} \vee \{\varrho_2(c) \vee \varrho_2(d)\} \vee ... \vee \{\varrho_n(c) \vee \varrho_n(d)\} \\ &\geqslant \sup_{z\in c\gamma d} \varrho_1(z) \vee \sup_{z\in c\gamma d} \varrho_2(z) \vee ... \vee \sup_{z\in c\gamma d} \varrho_n(z) \\ &\geqslant \sup_{z\in c\gamma d} \{\varrho_1(z) \vee \varrho_2(z) \vee ... \vee \varrho_n(z)\} \\ &\geqslant \sup_{z\in c\gamma d} \left\{\bigcap_{i\in N} \varrho_i(z)\right\} \forall\ c, d \in Y.\end{aligned}$$

Hence $\bigcap\limits_{i\in N} P_i$ is a $\Gamma-$subsemihypergroup in $\Gamma-$hypersemigroup.

□

THEOREM 3.2. *Let $P = (\varphi, \varrho)$ be a Pythagorean fuzzy set of Y. For $t \in IMG(P)$, the $t-$level cut of $P(P^t)$ is a $\Gamma-$subsemihypergroup then P is a Pythagorean fuzzy $\Gamma-$subsemihypergroup of Y.*

PROOF. Let φ^t be a $\Gamma-$subsemihypergroup of Y then for $c, d \in \varphi^t$ we have

$$c\gamma d \in \varphi^t \tag{3.1}$$

Suppose if $\varphi(c) \wedge \varphi(d) \nleqslant \inf\limits_{z\in c\gamma d} \varphi(z)$ then we have

$\varphi(c) \wedge \varphi(d) > \inf\limits_{z\in c\gamma d} \varphi(z)$.

Then there exists some $t_0 \in IMG(P)$ such that

$\varphi(c) \wedge \varphi(d) > t_0 > \inf\limits_{z\in c\gamma d} \varphi(z)$. This implies that

$\varphi(c) \wedge \varphi(d) > t_0$ and $\inf\limits_{z\in c\gamma d} \varphi(z) < t_0$.

$\Longrightarrow \varphi(c) \wedge \varphi(d) > t_0$ and $\varphi(z) < t_0$ for $z \in c\gamma d$

$\Longrightarrow c\gamma d \notin \varphi^t$ and either $c \in \varphi^t$ or $d \in \varphi^t$.

Which is a contradiction to Equation (3.1). Hence $\varphi(c) \wedge \varphi(d) \leqslant \inf\limits_{z \in c\gamma d} \varphi(z)$.

Now, Since ϱ^t is a $\Gamma-$subsemihypergroup then for $c, d \in \varrho^t$ we have

$$c\gamma d \in \varrho^t \tag{3.2}$$

Suppose if $\varrho(c) \vee \varrho(d) \ngeqslant \sup\limits_{z \in c\gamma d} \varrho(z)$ then we have

$\varrho(c) \vee \varrho(d) < \sup\limits_{z \in c\gamma d} \varrho(z)$.

Then there exists some $t_1 \in IMG(P)$ such that

$\varrho(c) \vee \varrho(d) < t_1 < \sup\limits_{z \in c\gamma d} \varrho(z)$. This implies that

$\varrho(c) \vee \varrho(d) < t_1$ and $\sup\limits_{z \in c\gamma d} \varrho(z) > t_1$

$\Longrightarrow \varrho(c) \vee \varrho(d) < t_1$ and $\varrho(z) > t_1$ for $z \in c\gamma d$

$\Longrightarrow c\gamma d \notin \varrho^t$ and either $c \in \varrho^t$ or $d \in \varrho^t$.

Which is a contradiction to Equation (3.2). Hence $\varrho(c) \vee \varrho(d) \geqslant \sup\limits_{z \in c\gamma d} \varrho(z)$.

□

Definition 3.2. *A Pythagorean fuzzy $\Gamma-$subsemihypergroup $P = (\varphi, \varrho)$ of Y is said to be Pythagorean fuzzy bi $\Gamma-$hyperideal if for $z \in Y$ we have*

(i) $\varphi(c) \wedge \varphi(d) \leqslant \inf\limits_{z \in c\gamma m\eta d} \varphi(z)$

(ii) $\varrho(c) \vee \varrho(d) \geqslant \sup\limits_{z \in c\gamma m\eta d} \varrho(z)$ *for all $c, d, m \in Y$ and $\gamma, \eta \in \Gamma$.*

Theorem 3.3. *If $\{P_i = (\varphi_i, \varrho_i)\}_{i \in N}$ are Pythagorean fuzzy bi $\Gamma-$hyperideals of Y then $\bigcap\limits_{i \in N} P_i = (\bigcap\limits_{i \in N} \varphi_i, \bigcap\limits_{i \in N} \varrho_i)$ is also a Pythagorean fuzzy bi $\Gamma-$hyperideal of Y.*

Proof. Straightforward. □

Theorem 3.4. *Let $P = (\varphi, \varrho)$ be a Pythagorean fuzzy set of Y. For $t \in IMG(P)$, the $t-$level cut of P is a bi $\Gamma-$hyperideal of Y then P is a Pythagorean fuzzy bi $\Gamma-$hyperideal of Y.*

Proof. Let P be a Pythagorean fuzzy set of Y. From Theorem 3.2 P is a Pythagorean fuzzy $\Gamma-$subsemihypergroup.

Since φ^t is a bi $\Gamma-$hyperideal then for $c, d \in Y$, $\gamma, \eta \in \Gamma$ and $m \in \varphi^t$ we have

$$c\gamma m\eta d \in \varphi^t \tag{3.3}$$

Suppose if φ is not a Pythagorean fuzzy bi $\Gamma-$hyperideal, we have

$\varphi(c) \wedge \varphi(d) \nleqslant \inf\limits_{z \in c\gamma m\eta d} \varphi(z)$

$\Longrightarrow \varphi(c) \wedge \varphi(d) \geqslant \inf\limits_{z \in c\gamma m\eta d} \varphi(z)$.

Then there exists some $t_0 \in IMG(P)$ such that

$\varphi(c) \wedge \varphi(d) > t_0 > \inf\limits_{z \in c\gamma m\eta d} \varphi(z)$.

This implies that $\varphi(c) \wedge \varphi(d) > t_0$ and $\inf\limits_{z \in c\gamma m\eta d} \varphi(z) < t_0$.

i.e., $\varphi(c) > t_0$ or $\varphi(d) > t_0$ and $\varphi(z) < t_0$ for $z \in c\gamma m\gamma d$
$\Longrightarrow c\gamma m\eta d \notin \varphi^t$ and $z \in \varphi^t$.
Which is a contradiction to Equation(3.3). Hence $\varphi(c) \wedge \varphi(c) \leqslant \inf\limits_{z\in c\gamma m\eta d} \varphi(z)$.

Now, since ϱ^t is a bi $\Gamma-$hyperideal then for $c, d \in Y$ and $m \in \varrho^t$ we have

$$c\gamma m\eta d \in \varrho^t \tag{3.4}$$

Suppose if ϱ is not a Pythagorean fuzzy bi $\Gamma-$hyperideal, we have
$\varrho(c) \vee \varrho(d) \ngeqslant \sup\limits_{z\in c\gamma m\eta d} \varrho(z)$

$\Longrightarrow \varrho(c) \vee \varrho(d) < \sup\limits_{z\in c\gamma m\eta d} \varrho(z)$.

Then there exists some $t_1 \in IMG(P)$ such that
$\varrho(c) \vee \varrho(d) < t_1 < \sup\limits_{z\in c\gamma\eta d} \varrho(z)$.
This implies that $\varrho(c) \vee \varrho(d) < t_1$ and $\sup\limits_{z\in c\gamma m\eta d} \varrho(z) > t_1$.
i.e., $\varrho(c) < t_1$ or $\varrho(d) < t_1$ and $\varrho(z) > t_1$ for $z \in c\gamma m\eta d$
$\Longrightarrow c\gamma m\eta d \notin \varrho^t$ and $z \in \varrho^t$.
Which is a contradiction to Equation (3.4). Hence $\varrho(c) \vee \varrho(d) \geqslant \sup\limits_{z\in c\gamma m\eta d} \varrho(z)$.

□

4. Relationship between Pythagorean fuzzy $\Gamma-$hyperideal, Pythagorean fuzzy $\Gamma-$subsemihypergroup and Pythagorean fuzzy bi $\Gamma-$hyperideal

In this section we discuss the relationship between Pythagorean fuzzy $\Gamma-$hyperideal, Pythagorean fuzzy $\Gamma-$subsemihypergroup and Pythagorean fuzzy bi $\Gamma-$hyperideal.

Lemma 4.1. *Every Pythagorean fuzzy* $\Gamma-$*hyperideal of* Y *is a Pythagorean fuzzy* $\Gamma-$ *subsemihypergroup.*

Proof. Let $P = (\varphi, \varrho)$ be a Pythagorean fuzzy $\Gamma-$hyperideal. For $c, d, z \in Y$ we have

$$\varphi(c) \wedge \varphi(d) \leqslant \inf_{z\in c\gamma d} \varphi(z) \leqslant \inf_{z\in c\gamma d} \varphi(z). \tag{4.1}$$

and

$$\varrho(c) \vee \varrho(b) \geqslant \sup_{z\in c\gamma d} \varrho(z) \geqslant \sup_{z\in c\gamma d} \varrho(z). \tag{4.2}$$

From Equations (4.1) and (4.2) we have P is a Pythagorean fuzzy $\Gamma-$subsemihypergroup. Therefore every Pythagorean fuzzy $\Gamma-$hyperideal is a Pythagorean fuzzy $\Gamma-$subsemihypergroup. □

The converse part of the above Lemma is not true. It has been proved by the following Example.

Example 4.1. *Let* $Y = \{a, b, c, d, e, f, e\}$ *and* $\Gamma = \{\gamma\}$, *then* Y *is a* $\Gamma-$*hypersemigroup.*

γ	a	b	c	d	e	f	g
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{b\}$	$\{b,c\}$	$\{d\}$	$\{d,e\}$	$\{f\}$	$\{f,g\}$
c	$\{a\}$	$\{c\}$	$\{c\}$	$\{e\}$	$\{e\}$	$\{g\}$	$\{g\}$
d	$\{a\}$	$\{d\}$	$\{d,e\}$	$\{d\}$	$\{d,e\}$	$\{d\}$	$\{d,e\}$
e	$\{a\}$	$\{e\}$	$\{e\}$	$\{e\}$	$\{e\}$	$\{e\}$	$\{e\}$
f	$\{a\}$	$\{f\}$	$\{f,g\}$	$\{d\}$	$\{d,e\}$	$\{f\}$	$\{f,g\}$
g	$\{a\}$	$\{g\}$	$\{g\}$	$\{e\}$	$\{e\}$	$\{g\}$	$\{g\}$

Define a Pythagorean fuzzy set $P=(\varphi,\varrho)$ *as*
P=$\{a/(0.1,1)$,$b/(0.1,1)$,$c/(0.2,0.7)$,$d/(0.5,0.6)$,$e/(0.5,0.6)$,$f/(0.9,0.3)$,$e/(0.9,0.3)\}$.
P satisfies the conditions of (i) and (ii) in Definition 3.2. Hence we can say that P is a Pythagorean fuzzy $\Gamma-$subsemihypergroup. But P is not a Pythagorean fuzzy $\Gamma-$hyperideal of Y. Since
$\varphi(d)=0.5\nleqslant 0.1=\inf\limits_{a\in a\gamma d}\varphi(a)\ \forall\ a\in Y$
φ is not a Pythagorean fuzzy left $\Gamma-$hyperideal.
$\varrho(c)=0.7\ngeqslant 1=\sup\limits_{a\in a\gamma c}\varrho(a)\ \forall\ a\in Y$
ϱ is not a Pythagorean fuzzy left $\Gamma-$hyperideal.
$\varphi(d)=0.5\nleqslant 0.1=\inf\limits_{a\in d\gamma a}\{\varphi(a)\}\ \forall\ a\in Y$
φ is not a Pythagorean fuzzy right $\Gamma-$hyperideal.
$\varrho(f)=0.3\ngeqslant 1=\sup\limits_{a\in f\gamma a}\{\varrho(a)\}\ \forall\ a\in Y$
ϱ is not a Pythagorean fuzzy right $\Gamma-$hyperideal.

LEMMA 4.2. *Every Pythagorean fuzzy $\Gamma-$hyperideal of Y is a Pythagorean fuzzy bi $\Gamma-$hyperideal of Y.*

PROOF. Let $P=(\varphi,\varrho)$ be a Pythagorean fuzzy $\Gamma-$hyperideal. For $c,d,m,z\in Y$ we have

$$\inf_{z\in c\gamma m\gamma d}\varphi(z)=\inf_{z\in c\gamma(m\gamma d)}\varphi(z)$$
$$\geqslant\varphi(c)\qquad(\varphi\ is\ a\ Pythagorean\ fuzzy\ right\ \Gamma-\ \textit{hyperideal}.)$$

$$\inf_{z\in c\gamma m\gamma d}\varphi(z)=\inf_{z\in(c\gamma m)\gamma d}\varphi(z)$$
$$\geqslant\varphi(d)\qquad(\varphi\ is\ a\ Pythagorean\ fuzzy\ left\ \Gamma-\ \textit{hyperideal}.)$$

Hence $\varphi(c)\wedge\varphi(d)\leqslant\inf\limits_{z\in c\gamma m\gamma d}\varphi(z)$ Now,

$$\sup_{z\in c\gamma m\gamma d}\varrho(z)=\sup_{z\in c\gamma(m\gamma d)}\varrho(z)$$
$$\leqslant\varrho(c)\qquad(\varrho\ is\ a\ Pythagorean\ fuzzy\ right\ \Gamma-\ \textit{hyperideal}.)$$

$$\sup_{z\in c\gamma m\gamma d}\varrho(z)=\sup_{z\in(c\gamma m)\gamma d}\varrho(z)$$
$$\leqslant\varrho(d)\qquad(\varrho\ is\ a\ Pythagorean\ fuzzy\ left\ \Gamma-\ \textit{hyperideal}.)$$

Hence $\varrho(c) \vee \varrho(d) \geqslant \sup\limits_{z \in c\gamma m\gamma d} \varrho(z)$.

Therefore P is a Pythagorean fuzzy bi $\Gamma-$hyperideal.

□

The following Example shows that every Pythagorean fuzzy bi $\Gamma-$hyperideal of Y need not be a Pythagorean fuzzy $\Gamma-$hyperideal of Y.

EXAMPLE 4.2. *Let $Y = \{a, b, c, d\}$ and $\Gamma = \{\gamma\}$ be two sets, then Y is a $\Gamma-$hypersemigroup.*

γ	a	b	c	d
a	$\{b, d\}$	$\{c, d\}$	$\{d\}$	$\{d\}$
b	$\{c, d\}$	$\{b\}$	$\{d\}$	$\{d\}$
c	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$

The Pythagorean fuzzy set P is defined by
$P = \{a/(0.3, 1), b/(0.5, 0.7), c/(0.3, 1), d/(0.9, 0.3)\}$.
By routine calculation we can say that P is a Pythagorean fuzzy $\Gamma-$hyperideal of Y. But it is not a Pythagorean fuzzy $\Gamma-$hyperideal, since
$\varphi(b) = 0.5 \nleqslant \inf\limits_{\{c,d\} \subseteq a\gamma b} = 0.3$. *Hence $\varphi(b) \nleqslant \inf\limits_{\{c,d\} \subseteq a\gamma b}$ for all $a, b \in Y$.*

5. Inverse image of Pythagorean fuzzy set

In this section we define inverse image of Pythagorean fuzzy set and study some properties.

DEFINITION 5.1. *Let U and V be any two $\Gamma-$hypersemigroups. By a homomorphism we mean a mapping $\Phi : U \to V$ satisfying the identity $\Phi(x\gamma y) = \Phi(x)\Phi(\gamma)\Phi(y)$ for all $x, y \in U$ and $\gamma \in \Gamma$.*

DEFINITION 5.2. *Let Φ be a mapping from a $\Gamma-$hypersemigroup U to a $\Gamma-$hypersemigroup V and $P = (\varphi, \varrho)$ be a Pythagorean fuzzy set in V. Then the inverse image of P , $\Phi^{-1}(P) = (\Phi^{-1}(\varphi), \Phi^{-1}(\varrho))$ is a Pythagorean fuzzy set in U and is defined by*
(i) $\Phi^{-1}(\varphi)(x) = \varphi(\Phi(x))$ and
(ii) $\Phi^{-1}(\varrho)(x) = \varrho(\Phi(x))$ for all $x \in V$.

THEOREM 5.1. *Let U and V be any two $\Gamma-$hypersemigroups and $\Phi : U \to V$ be an onto homomorphism of $\Gamma-$hypersemigroups. If $P = (\varphi, \varrho)$ is a Pythagorean fuzzy $\Gamma-$subsemihypergroup of V then $\Phi^{-1}(P) = (\Phi^{-1}(\varphi), \Phi^{-1}(\varrho))$ is also a Pythagorean fuzzy $\Gamma-$subsemihypergroup of U .*

PROOF. $P = (\varphi, \varrho)$ is a Pythagorean fuzzy $\Gamma-$subsemihypergroup of V. Let $c, d \in V$ and $\gamma \in \Gamma$.

$$\begin{aligned} \inf_{z \in c\gamma y} \Phi^{-1}(\varphi)(z) &= \inf_{\Phi(z) \in \Phi(c)\Phi(\gamma)\Phi(d)} \varphi(\Phi(z)) \\ &\geqslant \varphi(\Phi(c)) \wedge \varphi(\Phi(d)) \\ &\geqslant \Phi^{-1}(\varphi)(c) \wedge \Phi^{-1}(\varphi)(d) \end{aligned}$$

$$\begin{aligned}\sup_{z\in c\gamma y}\Phi^{-1}(\varrho)(z) &= \sup_{\Phi(z)\in\Phi(c)\Phi(\gamma)\Phi(d)}\varrho(\Phi(z))\\ &\leqslant \varrho(\Phi(c))\vee\varrho(\Phi(d))\\ &\leqslant \Phi^{-1}(\varrho)(c)\vee\Phi^{-1}(\varrho)(d)\end{aligned}$$

Thus $\Phi^{-1}(P)$ is also a Pythagorean fuzzy $\Gamma-$subsemihypergroup of U. □

THEOREM 5.2. *Let U and V be any two $\Gamma-$hypersemigroups and $\Phi : U \to V$ be an onto homomorphism of $\Gamma-$hypersemigroups. If $P = (\varphi, \varrho)$ is a Pythagorean fuzzy bi $\Gamma-$hyperideal of V then $\Phi^{-1}(P)$ is also a Pythagorean fuzzy bi $\Gamma-$hyperideal of U.*

PROOF. Since by Theorem 5.1 we know that the inverse image of Pythagorean fuzzy $\Gamma-$subsemihypergroup is also a Pythagorean fuzzy $\Gamma-$subsemihypergroup in $\Gamma-$hypersemigroup. Let $P = (\varphi, \varrho)$ be a Pythagorean fuzzy bi $\Gamma-$hyperideal of V, $a, b \in V$ and $\gamma \in \Gamma$.

$$\begin{aligned}\inf_{z\in c\gamma m\eta d}\Phi^{-1}(\varphi)(z) &= \inf_{\Phi(z)\in\Phi(c)\Phi(\gamma)\Phi(m)\Phi(\eta)\Phi(d)}\varphi(\Phi(z))\\ &\geqslant \varphi(\Phi(c))\wedge\varphi(\Phi(d))\\ &\geqslant \Phi^{-1}(\varphi)(c)\wedge\Phi^{-1}(\varphi)(d)\end{aligned}$$

$$\begin{aligned}\sup_{z\in c\gamma m\eta d}\Phi^{-1}(\varrho)(z) &= \sup_{\Phi(z)\in\Phi(c)\Phi(\gamma)\Phi(m)\Phi(\eta)\Phi(d)}\varrho(\Phi(z))\\ &\leqslant \varrho(\Phi(c))\vee\varrho(\Phi(d))\\ &\leqslant \Phi^{-1}(\varrho)(c)\vee\Phi^{-1}(\varrho)(d)\end{aligned}$$

Thus $\Phi^{-1}(P)$ is also a Pythagorean fuzzy bi $\Gamma-$hyperideal of Y. □

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