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A STUDY OF IDEALS OF $\Gamma-$SEMIRING WITH INVOLUTION

Murali Krishna Rao Marapureddy

Abstract. In this paper, we introduce the notion of involution in Γ-semirings. We define bi-ideal, quasi ideal, interior ideal, bi-quasi interior ideal, and bi-interior ideals of Γ-semirings with involution and study their properties.

1. INTRODUCTION

The algebraic structures play a prominent role in mathematics with wide range of applications. Generalization of ideals of algebraic structures and ordered algebraic structure plays a very remarkable role and also necessary for further advance studies and applications of various algebraic structures. Many mathematicians proved important results and charecterization of algebraic structures by using the concept and the properties of generalization of ideals in algebraic structures.

During 1950-1980, the concepts of bi-ideals, quasi ideals and interior ideals were studied by many mathematicians and during 1950-2019, the applications of these ideals only studied by mathematicians. Between 1980 and 2016 there have been no new generalization of these ideals of algebraic structures. Then the author [20-27] introduced and studied weak interior ideals, bi-interior ideals, bi quasi ideals, quasi interior ideals, and tri ideals. Tri quasi ideals and bi quasi interior ideals of $\Gamma-$Semirings, Semirings, $\Gamma-$Semigroups and Semigroups as a generalization of bi-ideal, quasi ideal and interior ideal of algebraic structures and charecterized regular algebraic structures as well as simple algebraic structures using these ideals.

In 1995, M.Murali Krishna Rao [15-19] introduced the notion of $\Gamma-$Semiring as a generalization of $\Gamma-$ring, ternary semiring and semiring. As a generalization of ring, the notion of a Γ-ring was introduced by Nobusawa [31] in 1964. In 1981 Sen

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[32] introduced the notion of a $\Gamma-$semigroup as a generalization of semigroup. The notion of a ternary algebraic system was introduced by Lehmer [13] in 1932. Dutta & Sardar[2] introduced the notion of operator semirings of $\Gamma-$semiring. In 1971. Lister [14] introduced ternary ring. The set of all negative integers Z is not a semiring with respect to usual addition and multiplication but Z forms a $\Gamma-$semiring where $\Gamma = Z$. The important reason for the development of Γ-semiring is a generalization of results of rings, Γ-rings, semirings, semigroups and ternary semirings. The notion of a semiring was introduced by Vandiver [35] in 1934, but semirings had appeared in earlier studies on the theory of ideals of rings. A universal algebra $(S, +, \cdot)$ is called a semiring if and only if $(S, +), (S, \cdot)$ are semigroups which are connected by distributive laws, *i.e.*, $a(b+c) = ab+ac,\ (a+b)c = ac+bc,$ for all $a, b, c \in S$. The theory of rings and theory of semigroups have considerable impact on the development of theory of semirings. Semirings play an important role in studying matrices and determinants. Semirings are useful in the areas of theoretical computer science as well as in the solution of graph theory, optimization theory, in particular for studying automata, coding theory and formal languages. Semiring theory has many applications in other branches of mathematics.

We know that the notion of a one sided ideal of any algebraic structure is a generalization of an ideal. The quasi ideals are generalization of left ideal and right ideal whereas the bi-ideals are generalization of quasi ideals. In 1952, the concept of bi-ideals was introduced by Good and Hughes [3] for semigroups. The notion of bi-ideals in rings and semigroups were introduced by Lajos and Szasz [10,11]. Bi-ideal is a special case of (m-n) ideal. In 1976, the concept of interior ideals was introduced by Lajos [12] for semigroups. Steinfeld [34] first introduced the notion of quasi ideals for semigroups and then for rings. Iseki and Izuka [5,6,7,8] introduced the concept of quasi ideal for a semiring. Quasi ideals bi-ideals in $\Gamma-$semirings studied by Jagtap and Pawar [9]. Henriksen [4] and Shabir et al. [33] studied ideals in semirings. Murali Krishna Rao et al. [27,28] studied ideals in $\Gamma-$semirings.

2. Preliminaries

In this section, we will recall some of the fundamental concepts and definitions, which are necessary for this paper.

Definition 2.1. [1] *A set S together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$ respectively will be called semiring provided*

(i) *addition is a commutative operation.*
(ii) *multiplication distributes over addition both from the left and from the right.*
(iii) *there exists $0 \in S$ such that $x + 0 = x$ and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in S$.*

Definition 2.2. *Let $(M, +)$ and $(\Gamma, +)$ be commutative semigroups. Then we call M a Γ-semiring, if there exists a mapping $M \times \Gamma \times M \to M$ (images of (x, α, y) will be denoted by $x\alpha y, x, y \in M, \alpha \in \Gamma$) such that it satisfies the following axioms for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$*

(i) $x\alpha(y+z) = x\alpha y + x\alpha z$
(ii) $(x+y)\alpha z = x\alpha z + y\alpha z$
(iii) $x(\alpha+\beta)y = x\alpha y + x\beta y$
(iv) $x\alpha(y\beta z) = (x\alpha y)\beta z$.

Definition 2.3. *A* Γ*-semiring* M *is said to be commutative* Γ*-semiring if* $x\alpha y = y\alpha x$, *for all* $x, y \in M$ *and* $\alpha \in \Gamma$.

Definition 2.4. *Let* M *be a* Γ*-semiring. An element* $1 \in M$ *is said to be unity if for each* $x \in M$ *there exists* $\alpha \in \Gamma$ *such that* $x\alpha 1 = 1\alpha x = x$.

Definition 2.5. *In a* Γ*-semiring* M *with unity* 1, *an element* $a \in M$ *is said to be left invertible (right invertible) if there exist* $b \in M, \alpha \in \Gamma$ *such that* $b\alpha a = 1(a\alpha b = 1)$.

Definition 2.6. *In a* Γ*-semiring* M *with unity* 1, *an element* $a \in M$ *is said to be invertible if there exist* $b \in M, \alpha \in \Gamma$ *such that* $a\alpha b = b\alpha a = 1$.

Definition 2.7. *A* Γ*-semiring* M *is said to have zero element if there exists an element* $0 \in M$ *such that* $0 + x = x$ *and* $0\alpha x = x\alpha 0 = 0$, *for all* $x \in M, \alpha \in \Gamma$.

Definition 2.8. *An element* a *in a* Γ*-semiring* M *is said to be idempotent if there exists* $\alpha \in \Gamma$ *such that* $a = a\alpha a$.

Definition 2.9. *Every element of* M, *is an idempotent of* M *then* M *is said to be idempotent* Γ*-semiring* M.

Definition 2.10. *A* Γ*-semiring* M *is called a division* Γ*-semiring if for each non-zero element of* M *has multiplication inverse.*

Definition 2.11. *A non-empty subset* A *of a* Γ*-semiring* M *is called*

(i) *a* Γ*-subsemiring of* M *if* $(A, +)$ *is a subsemigroup of* $(M, +)$ *and* $A\Gamma A \subseteq A$.
(ii) *a quasi ideal of* M *if* A *is a* Γ*-subsemiring of* M *and* $A\Gamma M \cap M\Gamma A \subseteq A$.
(iii) *a bi-ideal of* M *if* A *is a* Γ*-subsemiring of* M *and* $A\Gamma M\Gamma A \subseteq A$.
(iv) *an interior ideal of* M *if* A *is a* Γ*-subsemiring of* M *and* $M\Gamma A\Gamma M \subseteq A$.
(v) *a left (right) ideal of* M *if* A *is a* Γ*-subsemiring of* M *and* $M\Gamma A \subseteq A(A\Gamma M \subseteq A)$.
(vi) *an ideal if* A *is a* Γ*-subsemiring of* $M, A\Gamma M \subseteq A$ *and* $M\Gamma A \subseteq A$.
(vii) *a* $k-$*ideal if* A *is a* Γ*-subsemiring of* $M, A\Gamma M \subseteq A, M\Gamma A \subseteq A$ *and* $x \in M,\ x + y \in A, y \in A$ *then* $x \in A$.
(viii) *a bi-interior ideal of* M *if* A *is a* Γ*-subsemiring of* M *and* $M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B$.
(ix) *a left bi-quasi ideal (right bi-quasi ideal) of* M *if* A *is a subsemigroup of* $(M, +)$ *and* $M\Gamma A \cap A\Gamma M\Gamma A \subseteq A$ $(A\Gamma M \cap A\Gamma M\Gamma A \subseteq A)$.
(x) *a bi-quasi ideal of* M *if* B *is a* $\Gamma-$*subsemiring of* M *and* B*is a left bi-quasi ideal and a right bi-quasi ideal of* M.
(xi) *a left quasi-interior ideal (right quasi-interior ideal) of* M *if* A *is a* Γ*-subsemiring of* M *and* $M\Gamma A\Gamma M\Gamma A \subseteq A$ $(A\Gamma M\Gamma A\Gamma M \subseteq A)$.

(xii) *a quasi-interior of M if B is a $\Gamma-$subsemiring of M and $B$is a left quasi-interior ideal and a right quasi-interior ideal of M.*
(xiii) *a bi-quasi-interior ideal of M if A is a Γ-subsemiring of M and $B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B$.*
(xiv) *a left tri- ideal (right tri- ideal) of M if A is a Γ-subsemiring of M and $A\Gamma M\Gamma A\Gamma A \subseteq A$ ($A\Gamma A\Gamma M\Gamma A \subseteq A$).*
(xv) *a tri- ideal of M if A is a Γ-subsemiring of M and $A\Gamma M\Gamma A\Gamma A \subseteq A$ and $A\Gamma A\Gamma M\Gamma A \subseteq A$.*
(xvi) *a left(right) weak-interior ideal of M if B is a $\Gamma-$subsemiring of M and $M\Gamma B\Gamma B \subseteq B$($B\Gamma B\Gamma M \subseteq B$).*
(xvii) *a weak-interior ideal of M if B is a $\Gamma-$subsemiring of M and $B$is a left weak-interior ideal and a right weak-interior ideal)of M.*

3. Ideals of Γ-semirings with involution

In this section, we introduce the notion of Γ-semiring with involution and ideals of Γ-semiring with involution.

Definition 3.1. *Let M be a Γ-semiring then $(M, *)$ is called a Γ-semiring with involution, provided that $*$ is an involution satisfying the identities*

(i) $(x + y)^* = x^* + y^*$.
(ii) $(x\alpha y)^* = y^*\alpha x^*$.
(iii) $1^* = 0$.
(iv) $0^* = 1$, *for all* $x, y \in M, \alpha \in \Gamma$.

Definition 3.2. *Let $(M, *)$ be a Γ-semiring with involution.Then A^* is said to be quasi ideal of $(M, *)$, if $(M\Gamma A^*) \cap (A^*\Gamma M) \subseteq A^*$.*

Definition 3.3. *Let $(M, *)$ be a Γ-semiring with involution. Then A^* is said to be left(right) ideal of $(M, *)$, if $M\Gamma A^* \subseteq A^*(A^*\Gamma M) \subseteq A^*)$.*

Definition 3.4. *Let $(M, *)$ be a Γ-semiring with involution. Then B^* is said to be interior ideal of $(M, *)$, if $M\Gamma B^*\Gamma M \subseteq B^*$.*

Definition 3.5. *Let $(M, *)$ be a Γ-semiring with involution. Then B^* is said to be bi-ideal of $(M, *)$, if $B^*\Gamma M\Gamma B^* \subseteq B^*$.*

Definition 3.6. *Let $(M, *)$ be a Γ-semiring with involution. Then B^* is said to be bi-interior ideal of $(M, *)$, if $B^*\Gamma M\Gamma B^* \cap M\Gamma B^*\Gamma M \subseteq B^*$.*

Definition 3.7. *Let $(M, *)$ be a Γ-semiring with involution. Then B^* is said to be left(right) tri-ideal of $(M, *)$, if $B^*\Gamma M\Gamma B^*\Gamma B^* \subseteq B^*(B^*\Gamma B^*\Gamma M\Gamma B^* \subseteq B^*)$.*

Definition 3.8. *Let $(M, *)$ be a Γ-semiring with involution. Then B^* is said to be bi-quasi interior ideal of $(M, *)$, if $B^*\Gamma M\Gamma B^*\Gamma M\Gamma B^* \subseteq B^*$.*

Theorem 3.1. *Let M be a Γ-semiring with involution. Then A^* is a right ideal of the Γ-semiring with involution $(M, *)$, if A is a left ideal of M.*

PROOF. Let A be a left ideal of M. Then $M\Gamma A \subseteq A$

$$\begin{aligned} &M\Gamma A \subseteq A \\ \Rightarrow &(M\Gamma A)^* \subseteq A^* \\ \Rightarrow &A^*\Gamma M^* \subseteq A^* \\ \Rightarrow &A^*\Gamma M \subseteq A^*. \end{aligned}$$

Hence A^* is a right ideal of M. □

THEOREM 3.2. *Let M be a Γ-semiring with involution $*$. Then*
(i) $(x\Gamma M\Gamma y)^* = y^*\Gamma M\Gamma x^*$
(ii) $(M\Gamma x\Gamma M)^* = M\Gamma x^*\Gamma M$, *for all $x, y \in M$.*

THEOREM 3.3. *Let M be a Γ-semiring with involution if A is a quasi ideal of M. Then A^* is a quasi ideal of $(M, *)$.*

PROOF. Suppose A is a quasi ideal of M. Then

$$\begin{aligned} &(A\Gamma M) \cap (M\Gamma A) \subseteq A \\ \Rightarrow &[(A\Gamma M) \cap (M\Gamma A)]^* \subseteq A^* \\ \Rightarrow &(A\Gamma M)^* \cap (M\Gamma A)^* \subseteq A^* \\ \Rightarrow &(M^*\Gamma A^*) \cap (A^*\Gamma M^*) \subseteq A^* \\ \Rightarrow &(M\Gamma A^*) \cap (A^*\Gamma M) \subseteq A^*. \end{aligned}$$

Hence A^* is a quasi ideal of the $(M, *)$. □

THEOREM 3.4. *Let M be a Γ-semiring . Then A^* is a left ideal of $(M, *)$, for any right ideal A of M.*

PROOF. Let A be a right ideal of the Γ-semiring M. Then $A\Gamma M \subseteq A$ and $M^* = M$.

$$\begin{aligned} M\Gamma A^* &= M^*\Gamma A^* \\ &= (A\Gamma M)^* \\ &\subseteq A^*. \end{aligned}$$

Thus A^* is a left ideal of the Γ-semiring M with involution. □

THEOREM 3.5. *Let M be a Γ-semiring with involution. If A is a bi-quasi ideal of M then A^* is a bi-quasi ideal of M.*

PROOF. Let A be a bi-quasi ideal of M. Then

$$\begin{aligned} &A\Gamma M \cap A\Gamma M\Gamma A \subseteq A \\ \Rightarrow &(A\Gamma M \cap A\Gamma M\Gamma A)^* \subseteq A^* \\ \Rightarrow &A^*\Gamma M^*\Gamma A^* \cap M^*\Gamma A^* \subseteq A^* \\ \Rightarrow &A^*\Gamma M\Gamma A^* \cap M\Gamma A^* \subseteq A^*. \end{aligned}$$

Hence A^* is a bi-quasi ideal of M. □

THEOREM 3.6. *Let M be a Γ-semiring with involution. Then $A = A^*\Gamma A^*$, for all Γ-subsemirings of M if and only if M is a regular Γ-semiring.*

PROOF. Let A be a left ideal of M with involution. Then
$A = A^*\Gamma A^* \subseteq A^* \subseteq A$ Therefore $A^* = A$.
Let A be a right ideal and B be a left ideal of M. Then $A = A^*A$ and $B = B^*$.

$$A^*\Gamma B^* = A \cap B$$
$$A\Gamma B \subseteq A\Gamma M \subseteq A$$
$$A\Gamma B \subseteq M\Gamma B \subseteq B.$$

Therefore $A\Gamma B \subseteq A \cap B$. Now

$$\begin{aligned} A^*\Gamma B^* &= A \cap B \\ A\Gamma B &= (A\Gamma B)\Gamma(A\Gamma B) \\ &\subseteq (A \cap B)\Gamma(A \cap B) \\ A\Gamma B &\subseteq A \cap B \\ B\Gamma A &\subseteq A \cap B. \end{aligned}$$

Now

$$\begin{aligned} A^*\Gamma B^* &= A \cap B \\ A \cap B &= (A \cap B)\Gamma(A \cap B) \\ &\subseteq A\Gamma B. \end{aligned}$$

Therefore $A\Gamma B = A \cap B$.
Hence M is a regular Γ-semiring. □

THEOREM 3.7. *Let M be a Γ-semiring with involution $*$. If $\{A_i \mid i \in I\}$ is a family of left ideals of Γ-semiring then the $\cap A_i \neq \phi$ is a left ideals of Γ-semiring M.*

PROOF. Let $\{A_i \mid i \in I\}$ is a family of left ideals of Γ-semiring M. Then

$$\begin{aligned} M\Gamma \cap A_i^* &\subseteq M\Gamma A_i^*, \quad \text{for all } i \\ &\subseteq A_i^*, \quad \text{for all } i \\ \Rightarrow \ M\Gamma \cap A_i^* &\subseteq \cap A_i \end{aligned}$$

□

THEOREM 3.8. *Let M be a Γ-semiring with involution$*$. Then A^* is a bi-interior ideal of M if A is a bi-interior ideal of M.*

PROOF. Suppose A is a bi-interior ideal of M. Then

$$\begin{aligned} &A\Gamma M\Gamma A \cap M\Gamma A\Gamma M \subseteq A \\ &(A\Gamma M\Gamma A \cap M\Gamma A\Gamma M)^* \subseteq A^* \\ \Rightarrow\ &A^*\Gamma M\Gamma A^* \cap M\Gamma A^*\Gamma M \subseteq A^*. \end{aligned}$$

Hence A^* is a bi-interior ideal of M. □

Corollary 3.1. *Let M be a Γ-semiring with involution$*$. If A is a left ideal and right ideal of M then A^* is an ideal of M with involution.*

Theorem 3.9. *Let M be a Γ-semiring with involution if A is a quasi ideal of M. Then A^* is a quasi ideal of M.*

Proof. Suppose A is a quasi ideal of M. Then

$$\begin{aligned} &(A\Gamma M) \cap (M\Gamma A) \subseteq A \\ &\Rightarrow [(A\Gamma M) \cap (M\Gamma A)] \subseteq A^* \\ &\Rightarrow (A\Gamma M)^* \cap (M\Gamma A)^* \subseteq A^* \\ &\Rightarrow (M^*\Gamma A^*) \cap (A^*\Gamma M^*) \subseteq A^* \\ &\Rightarrow (M\Gamma A^*) \cap (A^*\Gamma M) \subseteq A^*. \end{aligned}$$

Hence A^* is a quasi ideal of M. □

Theorem 3.10. *Let M be a Γ-semiring with involution $*$. Then A^* is a left (right) ideal for any right ideal A of M.*

Proof. Let A be a right ideal of the Γ-semiring M. Then $A\Gamma M \subseteq A$ and $M^* = M$.

$$\begin{aligned} M\Gamma A^* &= M^*\Gamma A^* \\ &= (A\Gamma M)^* \\ &\subseteq A^*. \end{aligned}$$

Thus A^* is a left ideal of the Γ-semiring M with involution. □

Theorem 3.11. *Let M be a Γ-semiring with involution $*$. If B is an interior ideal of M, then B^* is an interior ideal of M.*

Proof. Let B be a interior ideal of the $\Gamma-$semiring M.
Then $M\Gamma B\Gamma M \subseteq B$ and $M^* = M$
$\Rightarrow M^*\Gamma B^*\Gamma M^* = (M\Gamma B\Gamma M)^* \subseteq B^*$.
Hence B^* is an interior ideal of M □

Theorem 3.12. *Let M be a $\Gamma-$semiring with involution $*$. If B is a bi-ideal of M, then B^* is a bi-ideal of M.*

Proof. Let B be a bi-ideal of M.
Then $B\Gamma M\Gamma B \subseteq B$ and $M^* = M$.
$\Rightarrow B^*\Gamma M^*\Gamma B^* = (B\Gamma M\Gamma B)^* \subseteq B^*$.
Hence B^* is a bi-ideal of M. □

Theorem 3.13. *Let M be a Γ-semiring with involution. If A is a bi quasi interior ideal of M then A^* is a bi quasi interior ideal of M.*

PROOF. Let A be a bi quasi interior ideal of the Γ- semiring with involution$M$. Then

$$\begin{aligned}&A\Gamma M\Gamma A\Gamma M\Gamma A\subseteq A\\ \Rightarrow&(A\Gamma M\Gamma A\Gamma M\Gamma A)^*\subseteq A^*\\ \Rightarrow&\ A^*\Gamma M^*\Gamma A^*\Gamma M^*\Gamma A^*\subseteq A^*\\ \Rightarrow&\ A^*\Gamma M\Gamma A^*\Gamma M\Gamma A^*\subseteq A^*.\end{aligned}$$

Hence A^* is a bi-quasi interior ideal of M. □

THEOREM 3.14. *Let M be a Γ- semiring with involution $*$. If $\{A_i/i\in I\}$ is a family of bi-interior ideals of Γ-semiring M then $\cap A_i^*\neq\phi$ is a bi-interior ideal of a Γ- semiring M.*

PROOF. Let $\{A_i/i\in I\}$ be a family of bi-interior ideals of Γ-semiring M with involution.

$$\begin{aligned}&A_i\Gamma M\Gamma A_i\cap M\Gamma A_iM\subseteq A_i\\ \Rightarrow&A_i^*\Gamma M\Gamma A_i^*\cap M\Gamma A_i^*\Gamma M\subseteq A_i^*\\ \Rightarrow&(\cap A_i^*\Gamma M\Gamma\cap A_i^*)\bigcap(M\Gamma\cap A_i^*\Gamma M)\\ &\subseteq A_i^*\Gamma M\cap A_i^*\bigcap(M\Gamma A_i^*\Gamma M)\subseteq A_i\\ \Rightarrow&\ (\cap A_i^*\Gamma M\Gamma\cap A_i^*)\bigcap(M\Gamma\cap A_i^*\Gamma M)\subseteq\cap A_i^*.\end{aligned}$$

Hence $\cap A_i^*$ is a bi-interior ideal of a Γ- semiring M with involution. □

THEOREM 3.15. *Let M be a Γ-semiring with involution. Then $A=A\Gamma A$, for all left ideals and right deals of M, if and only if M is a regular Γ-semiring.*

PROOF. Let A be a left ideal of M with involution. Then

$$\begin{aligned}&A=A\Gamma A\\ \Rightarrow&\ (A\Gamma A)^*=A^*\subseteq A\\ \Rightarrow&\ A^*\Gamma A^*=A\Gamma A\\ \Rightarrow&\ A^*\subseteq A\Gamma A)\subseteq A.\end{aligned}$$

Therefore $A^*=A$.

Let A be a right ideal and B be a left ideal of M. Then $A=A^*$ and $B=B^*$.

$$\begin{aligned}A\Gamma B\subseteq A\Gamma M\subseteq A\\ A\Gamma B\subseteq M\Gamma B\subseteq B.\end{aligned}$$

Therefore $A\Gamma B\subseteq A\cap B$.

Now

$$\begin{aligned}A^*\Gamma B^* &\subseteq A^* \cap B^*\\ A\Gamma B &= (A\Gamma B)\Gamma(A\Gamma B)\\ &\subseteq (A\cap B)\Gamma(A\cap B)\\ A\Gamma B &\subseteq A\Gamma B\\ B\Gamma A &\subseteq A\Gamma B.\end{aligned}$$

Now

$$\begin{aligned}A\cap B &= (A\cap B)\cap(A\cap B)\\ &\subseteq A\Gamma B.\end{aligned}$$

Therefore $A\Gamma B = A\cap B$.
Hence M is a regular $\Gamma-$semiring. □

THEOREM 3.16. *Let M be a Γ-semiring with involution $*$. If $\{A_i \mid i\in I\}$ is a family of right ideals of Γ-semiring, then $\cap A_i^* \neq \phi$ is a left ideals of Γ-semiring M.*

PROOF. Let $\{A_i \mid i\in I\}$ is a family of right ideals of Γ-semiring M. Then

$$\begin{aligned}M\Gamma\cap A_i^* &\subseteq M\Gamma A_i^*, \quad \text{for all } i\\ &\subseteq A_i^*, \quad \text{for all } i\\ \Rightarrow\ M\Gamma\cap A_i^* &\subseteq \cap A_i^*.\end{aligned}$$

Hence $\cap A_i^*$ is a left ideal of M. □

COROLLARY 3.2. *Let M be a Γ-semiring with involution $*$. If $\{A_i \mid i\in I\}$ is a family of left ideals of Γ-semiring then $\cap A_i^* \neq \phi$ is a right ideal of Γ-semiring M.*

COROLLARY 3.3. *Let M be a Γ-semiring with involution $*$. If $\{A_i \mid i\in I\}$ is a family of ideals of Γ-semiring then the $\cap A_i^* \neq \phi$ is an ideal of Γ-semiring M*

THEOREM 3.17. *Let M be a Γ-semiring with involution$*$. Then A^* is a bi-interior ideal of M, if A is a bi-interior ideal of M.*

PROOF. Suppose A is a bi-interior ideal of M. Then

$$\begin{aligned}&A\Gamma M\Gamma A\cap M\Gamma A\Gamma M \subseteq A\\ &(A\Gamma M\Gamma A\cap M\Gamma A\Gamma M)^* \subseteq A^*\\ \Rightarrow\ &A^*\Gamma M\Gamma A^*\cap M\Gamma A^*\Gamma M \subseteq A^*.\end{aligned}$$

Hence A^* is a bi-interior ideal of M. □

THEOREM 3.18. *Let M be a Γ-semiring with involution$*$. Then A^* is a right tri-ideal of M, if A is a left tri-ideal of M.*

Proof. Let A be a left tri-ideal of M. Then

$$
\begin{aligned}
&A\Gamma M\Gamma A\Gamma A \subseteq A\\
&\Rightarrow (A\Gamma M\Gamma A\Gamma A)^* \subseteq A^*\\
&\Rightarrow A^*\Gamma A^*\Gamma M^*\Gamma A^* \subseteq A^*\\
&\Rightarrow A^*\Gamma A^*\Gamma M\Gamma A^* \subseteq A^*
\end{aligned}
$$

Hence A^* is a right tri-ideal of M. □

Corollary 3.4. *Let M be a Γ-semiring with involution$*$. Then A^* is a left tri-ideal of M, if A is a right tri-ideal of M.*

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Murali Krishna Rao Marapureddy, Department of Mathematics, Sankethika Institute of Tech. and Management, Visakhapatnam - 530 041, A.P.,India.
Email address: mmarapureddy@gmail.com