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# LORENTZIAN $\beta$-KENMOTSU MANIFOLDS ADMITTING SEMI-SYMMETRIC NON-METRIC CONNECTION

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Abstract. The objective of the present paper is to study some results on Lorentzian $\beta$-Kenmotsu manifolds admitting semi-symmetric non-metric connection.

## 1. Introduction

As a generalization of both Sasakian and Kenmotsu manifolds, Oubina [**14**] introduced the notion of trans-Sasakian manifolds, which are closely related to the locally conformal Kahler manifolds. Trans-Sasakian manifolds of types $(0,0)$, $(0,\beta)$ and $(\alpha,0)$ are, respectively, called the cosymplectic, $\alpha$-Sasakian, and $\beta$-Kenmotsu manifolds, with $\alpha,\beta$ being scalar functions. In particular, if $\alpha=0$, $\beta=1$; $\alpha=1$, $\beta=0$; then a trans-Sasakian manifold will be Kenmotsu and Sasakian manifolds, respectively.

A linear connection $\tilde{\nabla}$ defined on $(M^n,g)$ is said to be semi-symmetric [**11**] if its torsion tensor $T$ is of the form

$$T(X_1,X_2)=u(X_2)X_1-u(X_1)X_2, \tag{1.1}$$

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where $u$ is a 1-form and $\rho$ is a vector field defined by

$$u(X_1) = g(X_1, \rho), \tag{1.2}$$

for all vector fields $X_1, X_2 \in \chi(M)$, where $\chi(M)$ is the set of all differentiable vector fields on $M$.

Hayden [**12**] introduced semi-symmetric linear connections on a Riemannian manifold. Let $M$ be an $n$-dimensional Riemannian manifold of class $\mathbb{C}^\infty$ endowed with the Riemannian metric $g$ and $\nabla$ be the Levi-Civita connection on $(M^n, g)$. A semi-symmetric connection $\tilde{\nabla}$ is said to be a semi-symmetric metric connection if it satisfies

$$\tilde{\nabla} g = 0. \tag{1.3}$$

A relation between the semi-symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection $\nabla$ of $(M, g)$ was given by Yano [**30**],

$$\tilde{\nabla}_{X_1} X_2 = \nabla_{X_1} X_2 = u(X_2) X_1 - g(X_1, X_2), \tag{1.4}$$

where $u(X_2) = g(X_2, \rho)$.

A semi-symmetric connection $\tilde{\nabla}$ is said to be a semi-symmetric non-metric connection if it satisfies

$$\tilde{\nabla} g \neq 0. \tag{1.5}$$

In 1992, Agashe and Chafle [**1**] defined and studied a semi-symmetric non-metric connection on Riemannian manifold whose torsion tensor $T$ satisfies

$$T(X_1, X_2) = u(X_2) X_1 - u(X_1) X_2$$

and

$$(\tilde{\nabla}_{X_1} g)(X_2, X_3) = u(X_2) g(X_1, X_3) - u(X_3) g(X_1, X_2).$$

They proved that the projective curvature tensor of the manifold with respect to these two connections are equal to each other.

In this paper, we consider semi-symmetric non-metric connection due to Barua and Mukhopadhyay [**8**] on Kenmotsu manifolds. A relation between semi-symmetric non-metric connection $\tilde{\nabla}$ and the Levi-Civita connection $\nabla$ has been obtained by Barua and Mukhopadhyay [4] and is given by

$$(\tilde{\nabla}_{X_1} g)(X_2, X_3) = 2u(X_1) g(X_2, X_3) - u(X_2) g(X_1, X_3) - u(X_3) g(X_1, X_2).$$

Since $\tilde{\nabla}_g f = 0$, it is an another type of semi-symmetric non-metric connection, but the authors preferred the name semi-symmetric semimetric connection.

The semi-symmetric non-metric connections were further studied by several authors such as De and Biswas [**9**], De and Kamilya [**10**], Liang [**13**], Ozen et al. [**15**], Smaranda and Andonie [**29**], and many others. Some related developments can be found in [2-4, 16-28].

We consider $\eta$-Ricci solitons on Lorentzian $\beta$-Kenmotsu manifold admitting semi-symmetric non-metric connection and the $\eta$-Ricci soliton is given by

$$L_\xi g(X_1, X_2) + 2S(X_1, X_2) + 2\lambda g(X_1, X_2) + 2\mu\eta(X_1)\eta(X_2) = 0, \tag{1.6}$$

for any vector field $X_1, X_2 \in M$, where $\xi$ is a contravariant vector field and $L_\xi$ denotes the Lie derivative operator on $\xi$, $S$ is the Ricci tensor and $\lambda$ is a constant.

## 2. Preliminaries

A differentiable manifold $M$ of dimension $(2n + 1)$ is called a Lorentzian $\beta$-Kenmotsu (See [**5, 6**]) if it admits a $(1, 1)$-tensor field $\phi$, a contravariant vector field $\xi$, a covariant vector field $\eta$ and Lorentzian metric $g$ which satisfies the following :

$$\eta(\xi) = -1, \phi(\xi) = 0, \eta o\phi = 0, \tag{2.1}$$

$$\phi^2 X_1 = X_1 + \eta(X_1)\xi, g(X_1, \xi) = \eta(X_1), \tag{2.2}$$

$$g(\phi X_1, \phi X_2) = g(X_1, X_2) + \eta(X_1)\eta(X_2). \tag{2.3}$$

Also, Lorentzian $\beta$-Kenmotsu manifold $(M^{2n+1}, g)$ satisfies:

$$\nabla_{X_1}\xi = \beta(X_1 - \eta(X_1)\xi), \tag{2.4}$$

$$(\nabla_{X_1}\eta)X_2 = \beta(g(X_1, X_2) - \eta(X_1)\eta(X_2)), \tag{2.5}$$

$$(\nabla_{X_1}\phi)X_2 = \beta(g(\phi X_1, X_2) - \eta(X_2)\phi X_1), \tag{2.6}$$

where $\nabla$ denotes the operator of covariant differentiation with respect to the Lorentzian metric $g$. Further, on Lorentzian $\beta$-Kenmotsu manifold $(M^{2n+1}, g)$ the following relations hold:

$$\eta(R(X_1, X_2)X_3) = \beta^2(g(X_1, X_3)\eta(X_2) - g(X_2, X_3)\eta(X_1)), \tag{2.7}$$

$$R(\xi, X_1)X_2 = \beta^2(\eta(X_2)X_1 - g(X_1, X_2)\xi), \tag{2.8}$$

$$R(X_1, X_2)\xi = \beta^2(\eta(X_1)X_2 - \eta(X_2)X_1), \tag{2.9}$$

$$S(X_1, \xi) = -2n\beta^2\eta(X_1), \tag{2.10}$$

$$Q\xi = -2n\beta^2\xi, \tag{2.11}$$

$$S(\xi, \xi) = -2n\beta^2. \tag{2.12}$$

From [**8**], the relation between semi-symmetric non-metric connection $\tilde{\nabla}$ and the Levi-Civita connection $\nabla$ is given by:

$$\tilde{\nabla}_{X_1}X_2 = \nabla_{X_1}X_2 - \eta(X_1)X_2 + g(X_1, X_2)\xi. \tag{2.13}$$

Further on using (2.13), we get:

$$\begin{aligned}(\tilde{\nabla}_{X_1}g)(X_2, X_3) &= \tilde{\nabla}_{X_1}g(X_2, X_3) - g(\tilde{\nabla}_{X_1}X_2, X_3) - g(X_2, \tilde{\nabla}_{X_1}X_3)\\ &= 2\eta(X_1)g(X_2, X_3) - \eta(X_2)g(X_1, X_3) - \eta(X_3)g(X_1, X_2)\\ &\neq 0.\end{aligned} \tag{2.14}$$

We now define the curvature tensor $\tilde{R}$ of $M$ with respect to semi-symmetric non-metric connection $\tilde{\nabla}$ is:

$$\tilde{R}(X_1,X_2)X_3 = \tilde{\nabla}_{X_1}\tilde{\nabla}_{X_2}X_3 - \tilde{\nabla}_{X_2}\tilde{\nabla}_{X_1}X_3 - \tilde{\nabla}_{[X_1,X_2]}X_3, \tag{2.15}$$

where $X_1, X_2, X_3 \in \chi(\mathrm{M})$, the set of all differential vector field on $M$.

In the view of equation (2.13) and (2.15) the curvature tensor $\tilde{R}$ of $M$ is given by ([**7**]):

$$\begin{aligned}\tilde{R}(X_1,X_2)X_3 =& R(X_1,X_2)X_3 - (\nabla_{X_1}\eta)(X_2)X_3 \\ &+ (\nabla_{X_2}\eta)(X_1)X_3 - 2\eta(X_2)g(X_1,X_3)\xi \\ &+ 2\eta(X_1)g(X_2,X_3)\xi + g(X_2,X_3)\nabla_{X_1}\xi - g(X_1,X_3)\nabla_{X_2}\xi.\end{aligned} \tag{2.16}$$

Using (2.4) and (2.5) in (2.16), we get

$$\begin{aligned}\tilde{R}(X_1,X_2)X_3 =& R(X_1,X_2)X_3 - 2\eta(X_2)g(X_1,X_3)\xi + 2\eta(X_1)g(X_2,X_3)\xi \\ &+ \beta g(X_2,X_3)X_1 - \beta\eta(X_1)g(X_2,X_3)\xi - \beta g(X_1,X_3)X_2 \\ &+ \beta\eta(X_2)g(X_1,X_3)\xi.\end{aligned} \tag{2.17}$$

Interchanging $X_1 = X_2$ and $X_2 = X_1$, equation (2.16) reduces to

$$\begin{aligned}\tilde{R}(X_2,X_1)X_3 =& R(X_2,X_1)X_3 - (\nabla_{X_2}\eta)(X_1)X_3 \\ &+ (\nabla_{X_1}\eta)(X_2)X_3 - 2\eta(X_1)g(X_2,X_3)\xi\xi \\ &+ 2\eta(X_2)g(X_1,X_3) + g(X_1,X_3)\nabla_{X_2}\xi - g(X_2,X_3)\nabla_{X_1}\xi.\end{aligned} \tag{2.18}$$

Again using (2.4) and (2.5) in (2.18), we get

$$\begin{aligned}\tilde{R}(X_2,X_1)X_3 =& R(X_2,X_1)X_3 - 2\eta(X_1)g(X_2,X_3)\xi + 2\eta(X_2)g(X_1,X_3)\xi \\ &+ \beta g(X_1,X_3)X_2 - \beta\eta(X_2)g(X_1,X_3)\xi - \beta g(X_2,X_3)X_1 \\ &+ \beta\eta(X_1)g(X_2,X_3)\xi.\end{aligned} \tag{2.19}$$

In view of (2.17) and (2.19), we have

$$\tilde{R}(X_1,X_2)X_3 = -\tilde{R}(X_2,X_1)X_3, \tag{2.20}$$

$$\tilde{R}(X_1,X_2)X_3 + \tilde{R}(X_2,X_3)X_1 + \tilde{R}(X_3,X_1)X_2 = 0. \tag{2.21}$$

Equation (2.21) is called as the first Bianchi identity with respect to semi-symmetric non-metric connection $\tilde{\nabla}$. Substituting $X_1 = \xi$ in (2.17) and using (2.1), (2.2) and (2.8), we obtain

$$\tilde{R}(\xi,X_2)X_3 = A_1\eta(X_3)X_2 + A_2 g(X_2,X_3)\xi + A_3\eta(X_2)\eta(X_3), \tag{2.22}$$

where

$$A_1 = \beta(\beta-1),\ A_2 = \beta(2-\beta)-2,\ \text{and}\ A_3 = \beta - 2.$$

Also, substituting $X_3 = \xi$ in (2.17) and using (2.1), (2.2) and (2.9), we get:

$$\tilde{R}(X_1,X_2)\xi = A_4(\eta(X_1)X_2 - \eta(X_2)X_1), \tag{2.23}$$

where

$$A_4 = \beta(\beta - 1).$$

Taking inner product for (2.17) with $\xi$ and using (2.1), (2.2) and (2.7), we get

$$\eta(\tilde{R}(X_1, X_2)X_3) = A_5(\eta(X_1)g(X_2, X_3) - \eta(X_2)g(X_1, X_3)), \tag{2.24}$$

where

$$A_5 = (2\beta - 2 - \beta^2).$$

Now taking an inner product for (2.17) with $Y_1$, using (2.1) and (2.2), we get

$$\begin{aligned} \tilde{R}(X_1, X_2, X_3, Y_1) =& R(X_1, X_2, X_3, Y_1) - 2\eta(X_2)g(X_1, X_3)\eta(Y_1) \\ &+ 2\eta(X_1)g(X_2, X_3) + \eta(Y_1)\beta g(X_2, X_3)g(X_1, Y_1) \\ &- \beta\eta(X_1)g(X_2, X_3)\eta(Y_1) - \beta g(X_1, X_3)g(X_2, Y_1) \\ &+ \beta\eta(X_2)g(X_1, X_3)\eta(Y_1), \end{aligned} \tag{2.25}$$

where $Y_1 \in \chi(M)$.
From (2.25), we obtain

$$\tilde{R}(X_1, X_2, X_3, Y_1) = -\tilde{R}(X_1, X_2, Y_1, X_3). \tag{2.26}$$

Let $\{e_1, e_2, \cdots, e_{2n+1}\}$ be a local orthonormal basis of vector fields in $M$. Then by putting $X_1 = Y_1 = e_i$ in (2.25), we obtain

$$\tilde{S}(X_2, X_3) = S(X_2, X_3) + A_6\eta(X_2)\eta(X_3) - A_7 g(X_2, X_3), \tag{2.27}$$

where

$$A_6 = \beta - 2 \text{ and } A_7 = 2 - (2n + 1)\beta.$$

Putting $X_3 = \xi$ in (2.27) and using (2.1), (2.2) and (2.10), we obtain

$$\tilde{S}(X_2, \xi) = A_8\eta(X_2), \tag{2.28}$$

where

$$A_8 = -2\beta.$$

By putting $X_2 = X_3 = e_i$ in (2.27) we obtain

$$\tilde{r} = r - (\beta - 2) + (2n + 1)^2\beta - 2(2n + 1). \tag{2.29}$$

## 3. Covariant constant admitting semi-symmetric non-metric connection

This section deals with a type of semi-symmetric non-metric connection on a Lorentzian $\beta$-Kenmotsu manifold. We consider a type of semi-symmetric non-metric connection which was introduced by Barua and Mukhopadhyay [**8**].

A Lorentzian $\beta$-Kenmotsu manifold $M$ with respect to the Levi-Civita connection said to be recurrent if its curvature tensor $R$ satisfies the condition

$$(\nabla_{Y_1} R)(X_1, X_2)X_3 = A(Y_1)R(X_1, X_2)X_3, \tag{3.1}$$

where $A$ is a non-zero 1-form and $X_1, X_2, X_3, Y_1 \in \chi(M)$. We now have

$$\begin{aligned} (\tilde{\nabla}_{Y_1} R)(X_1, X_2)X_3 =& \tilde{\nabla}_{Y_1} R(X_1, X_2)X_3 - R(\tilde{\nabla}_{Y_1} X_1, X_2)X_3 \\ &- R(X_1, \tilde{\nabla}_{Y_1} X_2)X_3 - R(X_1, X_2)\tilde{\nabla}_{Y_1} X_3. \end{aligned} \tag{3.2}$$

In the view of(2.1), (2.8), (2.9) and (2.13), (3.2) reduces to

$$(\tilde{\nabla}_{Y_1} R)(X_1, X_2)X_3 = \nabla_{Y_1} R(X_1, X_2)X_3 + g(Y_1, R(X_1, X_2)X_3)\xi - R(\nabla_{Y_1} X_1, X_2)X_3 - \beta^2 g(Y_1, X_1)\,[\eta(X_3)X_2 - g(X_2, X_3)\xi] - R(X_1, \nabla_{Y_1} X_2)X_3 - 2\eta(Y_1)R(X_1, X_2)X_3 - \beta^2 g(Y_1, X_2)\,[g(X_1, X_3) - \eta(X_3)X_1] - R(X_1, X_2)\nabla_{Y_1} X_3 - \beta^2 g(Y_1, X_3)\,[\eta(X_1)X_2 - \eta(X_2)X_1]\,. \tag{3.3}$$

Suppose $\tilde{\nabla}_{Y_1} R(X_1, X_2)X_3 = 0$. Then we have

$$\nabla_{Y_1} R(X_1, X_2)X_3 + g(Y_1, R(X_1, X_2)X_3)\xi - R(\nabla_{Y_1} X_1, X_2)X_3 - \beta^2 g(Y_1, X_1)\,[\eta(X_3)X_2 - g(X_2, X_3)\xi] - R(X_1, \nabla_{Y_1} X_2)X_3 - 2\eta(Y_1)R(X_1, X_2)X_3 - \beta^2 g(Y_1, X_2)\,[g(X_1, X_3) - \eta(X_3)X_1] - R(X_1, X_2)\nabla_{Y_1} X_3 - \beta^2 g(Y_1, X_3)\,[\eta(X_1)X_2 - \eta(X_2)X_1] = 0. \tag{3.4}$$

In the view of equation (3.1), the equation (3.4) reduces to

$$A(Y_1)R(X_1, X_2)X_3 + g(Y_1, R(X_1, X_2)X_3)\xi - R(\nabla_{Y_1} X_1, X_2)X_3 - \beta^2 g(Y_1, X_1)\,[\eta(X_3)X_2 - g(X_2, X_3)\xi] - R(X_1, \nabla_{Y_1} X_2)X_3 - 2\eta(Y_1)R(X_1, X_2)X_3 - \beta^2 g(Y_1, X_2)\,[g(X_1, X_3) - \eta(X_3)X_1] - R(X_1, X_2)\nabla_{Y_1} X_3 - \beta^2 g(Y_1, X_3)\,[\eta(X_1)X_2 - \eta(X_2)X_1] = 0. \tag{3.5}$$

Taking inner product with $\xi$ and using (2.1), (2.2), (2.7), we obtain

$$\beta^2 A(Y_1)g(X_1, X_3)\eta(X_2) - \beta^2 A(Y_1)g(X_2, X_3)\eta(X_1) + 2\beta^2 g(X_1, X_3)\eta(Y_1)\eta(X_2) - 2\beta^2 g(X_2, X_3)\eta(Y_1)\eta(X_1) - g(Y_1, R(X_1, X_2)X_3) - \beta^2 g(Y_1, X_1)\eta(X_2)\eta(X_3) - \beta^2 g(Y_1, X_1)g(X_2, X_3) + \beta^2 g(Y_1, X_2)g(X_1, X_3) + \beta^2\eta(X_1)\eta(X_3)g(Y_1, X_2) = 0. \tag{3.6}$$

Let $\{e_1, e_2, \cdots, e_{2n+1}\}$ be a local orthonormal basis of vector fields in $M$. Then by putting $X_1 = Y_1 = e_i$ in (3.6), we get

$$\beta^2 A(e_i)\,[g(e_i, X_3)\eta(X_2) - g(X_2, X_3)\eta(e_i)] + 2\beta^2\eta(X_2)\eta(X_3) + 2\beta^2 g(X_2, X_3) - S(X_2, X_3) - \beta^2(2n+1)\,[\eta(X_2)\eta(X_3) + g(X_2, X_3)] + \beta^2 g(X_2, X_3) + \beta^2\eta(X_2)\eta(X_3) = 0. \tag{3.7}$$

The first term on the RHS of (3.7) vanishes by putting $X_3 = \xi$ and (3.7) reduces to

$$S(X_2, X_3) = G_1\,[g(X_2, X_3) + \eta(X_2)\eta(X_3)]\,, \tag{3.8}$$

where

$$G_1 = 2\beta^2(1-n).$$

The above discussion leads to the following theorem:

THEOREM 3.1. *In a Lorentzian $\beta$-Kenmotsu manifold, if the curvature tensor is covariant constant with respect to the semi-symmetric non-metric connection and the manifold is recurrent with respect to the Levi-Civita connection, then the manifold is an $\eta$-Einstein manifold.*

Let $\xi^{\perp}$ denotes the $(2n+1)$-dimensional distribution orthogonal to $\xi$ in a Lorentzian $\beta$-Kenmotsu manifold admitting a semi-symmetric non-metric connection whose curvature tensor vanishes. Then for any $X_1 \in \xi^{\perp}$, $g(X_1, \xi) = 0$ or $\eta(X_1) = 0$. Now we shall find the sectional curvature $'R$ of the plane determined by the vectors $X_1, X_2 \in \xi^{\perp}$. Putting $\tilde{R} = 0$ and $X_3 = X_2$, $Y_1 = X_1$ in (2.25), we have

$$R(X_1, X_2, X_2, X_1) = -\beta\left[g(X_1, X_2)g(X_2, X_1) - g(X_1, X_1)g(X_2, X_2)\right]. \tag{3.9}$$

Then

$$'R(X_1, X_2) = \frac{R(X_1, X_2, X_2, X_1)}{g(X_1, X_2)^2 - g(X_1, X_1)g(X_2, X_2)} = -\beta.$$

Hence, we can state the following theorem:

THEOREM 3.2. *In a Lorentzian $\beta$-Kenmotsu manifold, if the curvature tensor of the semi-symmetric non-metric connection vanishes, then the sectional curvature of the plane determined by the two vectors $X_1, X_2 \in \xi^{\perp}$ is $-\beta$.*

## 4. Semi-symmetric Lorentzian β-Kenmotsu manifold admitting semi-symmetric non-symmetric connection

We suppose that the manifold under consideration is semi-symmetric with respect to the semi-symmetric non-metric connection $\tilde{\nabla}$. Then

$$(\tilde{R}(X_1, X_2).\tilde{R})(Y_1, Y_2)X_3 = 0. \tag{4.1}$$

Then equation (4.1) can be written as

$$\begin{aligned}\tilde{R}(X_1, X_2)\tilde{R}(Y_1, Y_2)X_3 - \tilde{R}(\tilde{R}(X_1, X_2)Y_1, Y_2)X_3 \\ -\tilde{R}(Y_1, \tilde{R}(X_1, X_2)Y_2)X_3 - \tilde{R}(Y_1, Y_2)\tilde{R}(X_1, X_2)X_3 = 0.\end{aligned} \tag{4.2}$$

Replacing $X_1 = \xi$, then equation (4.2) reduces to

$$\begin{aligned}\tilde{R}(\xi, X_2)\tilde{R}(Y_1, Y_2)X_3 - \tilde{R}(\tilde{R}(\xi, X_2)Y_1, Y_2)X_3 - \tilde{R}(Y_1, \tilde{R}(\xi, X_2)Y_2)X_3 \\ -\tilde{R}(Y_1, Y_2)\tilde{R}(\xi, X_2)X_3 = 0.\end{aligned} \tag{4.3}$$

In view of (2.1), (2.22) and (2.24), equation (4.3) reduces to

$$
\begin{aligned}
&A_1A_5\eta(Y_1)g(Y_2,X_3)X_2 - A_1A_5\eta(Y_2)g(Y_1,X_3)X_2 \\
&+A_2g(\tilde{R}(Y_1,Y_2)X_3,X_2)\xi + A_3A_5\eta(Y_1)\eta(X_2)g(Y_2,X_3)\xi \\
&-A_3A_5\eta(Y_2)\eta(X_2)g(Y_1,X_3)\xi - A_1\eta(Y_1)\tilde{R}(Y_2,X_2)X_3 \\
&-A_2g(Y_1,X_2)\tilde{R}(\xi,Y_2)X_3 - A_3\eta(Y_1)\eta(X_2)\tilde{R}(\xi,Y_2)X_3 \\
&-A_1\eta(Y_2)\tilde{R}(Y_1,X_2)X_3 - A_2g(Y_2,X_2)\tilde{R}(\xi,Y_1)X_3 \\
&-A_3\eta(Y_2)\eta(X_2)\tilde{R}(\xi,Y_1)X_3 - A_1\eta(X_3)\tilde{R}(Y_2,Y_1)X_2 \\
&-A_2g(X_2,X_3)\tilde{R}(Y_1,Y_2)\xi - A_3\eta(X_3)\eta(X_2)\tilde{R}(Y_1,Y_2)\xi = 0.
\end{aligned}
\tag{4.4}
$$

Taking inner product with $\xi$ and using (2.1), (2.2) and (2.24), we have

$$
\begin{aligned}
&A_2g(\tilde{R}(Y_1,Y_2)X_3,X_2) - A_2A_5g(Y_2,X_3)g(Y_1,X_2) \\
&- (A_2A_5 + A_1A_5\eta(Y_2)\eta(X_3)g(Y_2,X_2)) \\
&+A_2A_5g(Y_2,X_2)g(Y_1,X_3) \\
&+ (A_1A_5 + A_2A_5\eta(Y_1)\eta(X_3)g(Y_2,X_2)) = 0.
\end{aligned}
\tag{4.5}
$$

Taking $X_2 = Y_1 = e_i$ and using (2.1), (2.2) and (2.27), we have

$$S(Y_2,X_3) = G_2g(Y_2,X_3) + G_3\eta(Y_2)\eta(X_3), \tag{4.6}$$

where

$$G_2 = A_7 + 2nA_5 \text{ and } G_3 = \frac{(2n+1)A_5(A_1+A_2) - A_2A_6 - A_5(A_1+A_2)}{A_2}.$$

Hence, we have the following result:

THEOREM 4.1. *A semi-symmetric Lorentzian $\beta$-Kenmotsu manifold admitting semi-symmetric non-symmetric connection of dimension $(M^{2n+1},g)$ is an $\eta$-Einstein manifold.*

## 5. $\phi$-recurrent Lorentzian $\beta$-Kenmotsu manifold admitting semi-symmetric non-metric connection

A Lorentzian $\beta$-Kenmotsu manifold admitting semi-symmetric non-metric connection is said to be a $\phi$-recurrent manifold if there exists a nowhere vanishing unique 1-form $A$ such that

$$\phi^2((\nabla_{Y_3}\tilde{R})(X_1,X_2)X_3) = A(Y_3)\tilde{R}(X_1,X_2)X_3, \tag{5.1}$$

for all vector fields $X_1,X_2,X_3,Y_3$, where $A$ is 1-form defined by $A(X_1) = g(X_1,\rho)$ and $\rho$ is a vector field associated with 1-form $A$. By virtue of (2.2) and (5.1), we obtain

$$(\nabla_{Y_3}\tilde{R})(X_1,X_2)X_3 + \eta((\nabla_{Y_3}\tilde{R})(X_1,X_2)X_3)\xi = A(Y_3)\tilde{R}(X_1,X_2)X_3. \tag{5.2}$$

Contracting (5.2) with $Y_1$, we have

$$g((\nabla_{Y_3}\tilde{R})(X_1,X_2)X_3,Y_1)+\eta((\nabla_{Y_3}\tilde{R})(X_1,X_2)X_3)\eta(Y_1)\xi = A(Y_3)g(\tilde{R}(X_1,X_2)X_3,Y_1). \tag{5.3}$$

Let $\{e_i \mid i=1,2,3,\cdots,n\}$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X_1=Y_1=e_i$ and taking summation over $i$, $1\leqslant i\leqslant n$ in (5.3), we get:

$$(\nabla_{Y_3}\tilde{S})(X_2,X_3)+\sum_{i=1}^{n}\eta((\nabla_{Y_3}\tilde{R})(e_i,X_2)X_3)\eta(e_i)=A(Y_3)\tilde{S}(X_2,X_3). \tag{5.4}$$

Substituting $X_3=\xi$ in (5.4), the second term in the L.H.S becomes zero and (5.4) reduces to

$$(\nabla_{Y_3}\tilde{S})(X_2,X_3)=A(Y_3)\tilde{S}(X_2,X_3). \tag{5.5}$$

Substituting $X_3=\xi$ in (5.5) and using (2.28), we obtain

$$(\nabla_{Y_3}\tilde{S})(X_2,\xi)=A_8A(Y_3)\eta(X_2). \tag{5.6}$$

We have

$$(\nabla_{Y_3}\tilde{S})(X_2,\xi)=\nabla_{Y_3}\tilde{S}(X_2,\xi)-\tilde{S}(\nabla_{Y_3}X_2,\xi)-\tilde{S}(X_2,\nabla_{Y_3}\xi). \tag{5.7}$$

In the view of (2.4) and (2.28) in (5.7), we obtain

$$(\nabla_{Y_3}\tilde{S})(X_2,\xi)=A_8\left((\nabla_{Y_3}\eta)X_2+\beta\eta(Y_3)\eta(X_2)\right)-\beta\tilde{S}(X_2,Y_3). \tag{5.8}$$

Using (2.5), we get

$$(\nabla_{Y_3}\tilde{S})(X_2,\xi)=A_8\beta\left(g(Y_3,X_2)\right)-\beta\tilde{S}(Y_3,X_2). \tag{5.9}$$

In view of (5.6) and (5.9), replacing $X_2=\phi X_2$, we have

$$S(Y_3,\phi X_2)=2A_8g(Y_3,\phi X_2). \tag{5.10}$$

Substituting $X_2=\phi X_2$ in (5.10) and using (2.2) and (2.10), we obtain

$$S(Y_3,X_2)=A_9g(Y_3,X_2)+A_{10}\eta(Y_3)\eta(X_2), \tag{5.11}$$

where

$$A_9=2A_8 \text{ and } A_{10}=2\left(n\beta^2+A_8\right).$$

Hence, we can state the following theorem:

THEOREM 5.1. *A $\phi$-recurrent Lorentzian $\beta$-kenmotsu manifold admitting semi-symmetric non-metric connection of dimension $(M^{2n+1},g)$ is an $\eta$-Einstein manifold.*

Further more, the Bianchi identity is given by:

$$A(Y_3)\eta(R(X_1,X_2)X_3)+A(X_1)\eta(R(X_2,Y_3)X_3) +A(X_2)\eta(R(Y_3,X_1)X_3)=0. \tag{5.12}$$

In the view of (2.24) and (5.12), we have

$$\begin{aligned}A_5[&A(Y_3)(\eta(X_1)g(X_2,X_3)-\eta(X_2)g(X_1,X_3))+\\ &A(X_1)(\eta(X_2)g(Y_3,X_3)-\eta(Y_3)g(X_2,X_3))+\\ &A(X_2)(\eta(Y_3)g(X_1,X_3)-\eta(X_1)g(Y_3,X_3))]=0,\end{aligned} \tag{5.13}$$

where

$$A_5\neq 0.$$

Therefore, equation (5.13) reduces to

$$\begin{aligned}A_5[&A(Y_3)(\eta(X_1)g(X_2,X_3)-\eta(X_2)g(X_1,X_3))+\\ &A(X_1)(\eta(X_2)g(Y_3,X_3)-\eta(Y_3)g(X_2,X_3))+\\ &A(X_2)(\eta(Y_3)g(X_1,X_3)-\eta(X_1)g(Y_3,X_3))]=0,\end{aligned} \tag{5.14}$$

Putting $X_2=X_3=e_i$ in (5.14) and taking summation over $i$, $1\leqslant i\leqslant 2n+1$, we obtain

$$A(Y_3)\eta(X_1)=A(X_1)\eta(Y_3), \tag{5.15}$$

for any vector field $X_1$ and $Y_3$.
Putting $X_1=\xi$ in above equation, we obtain

$$A(Y_3)=-\eta(\rho)\eta(Y_3), \tag{5.16}$$

for any vector field $Y_3$, where $A(\xi)=g(\xi,\rho)=\eta(\rho)$, $\rho$ being the vector field associated to the 1-form $A$. Hence, from (5.15) and (5.16), we can state the following theorem:

THEOREM 5.2. *In a $\phi$-recurrent Lorentzian $\beta$-Kenmotsu manifold admitting semi-symmetric non-metric connection, the characteristic vector field $\xi$ and the vector field $\rho$ associated to the 1-form $A$ are co-directional and the 1-form $A$ is given by* (5.16).

We have,

$$\begin{aligned}(\nabla_{Y_3}\tilde{R})(X_1,X_2)\xi=&\nabla_{Y_3}\tilde{R}(X_1,X_2)\xi-\tilde{R}(\nabla_{Y_3}X_1,X_2)\xi\\ &-\tilde{R}(X_1,\nabla_{Y_3}X_2)\xi-\tilde{R}(X_1,X_2)\nabla_{Y_3}\xi.\end{aligned} \tag{5.17}$$

In the view of (2.4) and (2.23), (5.17) reduces to

$$(\nabla_{Y_3}\tilde{R})(X_1,X_2)\xi=A_4\beta\left(g(Y_3,X_1)X_2-g(Y_3,X_2)X_1\right)-\beta\tilde{R}(X_1,X_2)Y_3, \tag{5.18}$$

Taking inner product with $\xi$ and using (2.2) and (2.24), equation (5.18) reduces to

$$\eta\left((\nabla_{Y_3}\tilde{R})(X_1,X_2)\xi\right)=\beta(A_4+A_5)\left(g(Y_3,X_1)\eta(X_2)-g(Y_3,X_2)\eta(X_1)\right). \tag{5.19}$$

Considering (5.2) and putting $Z=\xi$, we get

$$(\nabla_{Y_3}\tilde{R})(X_1,X_2)\xi+\eta\left((\nabla_{Y_3}\tilde{R})(X_1,X_2)\xi\right)\xi=A(Y_3)\tilde{R}(X_1,X_2)\xi. \tag{5.20}$$

In the view of (5.18) and (5.19), (5.20) reduces to

$$A_4\beta\left(g(Y_3,X_1)X_2-g(Y_3,X_2)X_1\right)+(\beta(A_4+A_5)g(Y_3,X_1)\eta(X_2) -g(Y_3,X_2)\eta(X_1))\,\xi-\beta\tilde{R}(X_1,X_2)Y_3=A(Y_3)\tilde{R}(X_1,X_2)\xi. \tag{5.21}$$

Taking inner product with $\xi$ and using (2.23), equation (5.21) reduces to

$$A_5\beta\left(g(Y_3,X_2)X_1-g(Y_3,X_1)X_2\right)+\tilde{R}(X_1,X_2)Y_3= -A_4A(Y_3)(\eta(X_1)X_2-\eta(X_2)X_1). \tag{5.22}$$

Hence, if $X_1$ and $X_2$ are orthogonal to $\xi$, then (5.22) reduces to

$$\beta A_5\left(g(Y_3,X_1)X_2-g(Y_3,X_2)X_1\right)=\tilde{R}(X_1,X_2)Y_3. \tag{5.23}$$

Thus, we have the following theorem:

THEOREM 5.3. *A locally $\phi$-recurrent Lorentzian $\beta$-Kenmotsu manifold admitting semi-symmetric non-metric connection is a manifold $(M^{2n+1},g)$ $(n>1)$ of constant curvature i.e., a hyperbolic space.*

## 6. $\eta$-Ricci soliton on Lorentzian $\beta$-Kenmotsu manifold admitting semi-symmetric non-metric connection

Let $(M^{2n+1},g)$ be a Lorentzian $\beta$-Kenmotsu manifold admitting semi-symmetric non-metric connection. Then the $\eta$-Ricci soliton is given by

$$(L_\xi g)(X_1,X_2)+2\tilde{S}(X_1,X_2)+2\lambda g(X_1,X_2)+2\mu\eta(X_1)\eta(X_2)=0, \tag{6.1}$$

for all $X_1$, $X_2\in\chi(M)$, where $L_\xi$ is Lie derivative operator along the vector field $\xi$, $S$ is the Ricci tensor field for the metric $g$, and $\lambda$, $\mu$ are constants.

Considering the term on the left hand side of (6.1), we have

$$(L_\xi g)(X_1,X_2)=g(\nabla_{X_1}\xi,X_2)+g(\nabla_{X_2}\xi,X_1). \tag{6.2}$$

Using (2.4), (6.2) reduces to,

$$(L_\xi g)(X_1,X_2)=2\beta\left(g(X_1,X_2)-\eta(X_1)\eta(X_2)\right). \tag{6.3}$$

In the view of (6.3) and (6.1), we get

$$\tilde{S}(X_1,X_2)=-(\lambda+\beta)g(X_1,X_2)-(\mu-\beta)\eta(X_1)\eta(X_2). \tag{6.4}$$

Using (2.27), (6.4) reduces to

$$S(X_1,X_2)=A_{11}g(X_1,X_2)+A_{12}\eta(X_1)\eta(X_2), \tag{6.5}$$

where

$$A_{11}=(A_7-\lambda-\beta)\ \text{and}\ A_{12}=(\beta-\mu-A_6).$$

Hence, we can state the following theorem:

THEOREM 6.1. *An $\eta$-Ricci soliton of Lorentzian $\beta$-Kenmotsu manifold admitting semi-symmetric non-metric connection is an $\eta$-Einstein manifold.*

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