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# ON SOME DERIVATIVES OF $k$-ORESME POLYNOMIALS

**Serpil Halıcı and Zehra Betül Gür**

Abstract. In this study, we examined some properties of recently defined polynomial sequence called $k$-Oresme polynomials. We studied first order derivatives of this sequence and deduced some formulas. Moreover, we formulated the higher order derivative of $k$-Oresme polynomials and observed a relation between the formula which is newly given and Pascal's triangle.

## 1. Introduction

In [5], A.F. Horadam defined the well-known number sequence called Horadam numbers denoted by $W_n = W_n(W_0, W_1; p, q)$ by second order linear homogeneous reccurence relation

$$W_{n+2} = pW_{n+1} - qW_n \tag{1.1}$$

for integers $p, q$ and $n \geqslant 0$. By choosing $W_0, W_1$, $p$ and $q$ properly, some special number sequences can be obtained such as Fibonacci numbers $F_n = W_n(0, 1; 1, -1)$ and Pell numbers $P_n = W_n(0, 1; 2, -1)$. Nicole Oresme extended the values of $p$ and $q$ to be rational numbers and defined a new number sequence called Oresme numbers [7]. From (1.1), we can obtain the Oresme numbers as

$$\{O_n\} = \{W_n(0, \frac{1}{2}; 1, \frac{1}{4})\}$$

and terms of this sequence as

$$\left\{ 0, \frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \frac{4}{16}, \frac{3}{8}, \frac{5}{32}, \frac{6}{64}, \frac{7}{128}, \dots \right\}.$$

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The reccurence relation of Oresme numbers is written as

$$O_n = O_{n-1} - \frac{1}{4}O_{n-2}$$

with $O_0 = 0, O_1 = \frac{1}{2}$. The following properties are derived by the authors in [5,7].

$$\sum_{i=0}^{\infty} O_i = 2, \tag{1.2}$$

$$\sum_{i=0}^{n-1} O_i = 4\left(\frac{1}{2} - O_{n+1}\right), \tag{1.3}$$

$$\sum_{i=0}^{n-1} (-1)^i O_i = \frac{4}{9}\left(-\frac{1}{2} + (-1)^n (O_{n+1} - 2O_n)\right), \tag{1.4}$$

$$\sum_{i=0}^{n-1} O_{2i+1} = \frac{1}{9}\left(10 + 5O_{2n-1} - 16O_{2n}\right). \tag{1.5}$$

In [1], for $k > 2$, a generalization of the Oresme numbers called $k$-Oresme numbers is presented and defined by

$$O_n^{(k)} = O_{n-1}^{(k)} - \frac{1}{k^2}O_{n-2}^{(k)}, \quad n \geqslant 2 \tag{1.6}$$

where $O_0^{(k)} = 0,\ O_1^{(k)} = \frac{1}{k^2}$. It can be clearly seen that $O_n^{(k)} = W_n(0, \frac{1}{k}; 1, \frac{1}{k^2})$ and $O_n^{(2)} = O_n$. Using standard techniques to solve reccurence relation, Binet formula can be deduced and written as

$$O_n^{(k)} = \frac{1}{\sqrt{k^2-4}}\left[\left(\frac{k+\sqrt{k^2-4}}{2k}\right)^n - \left(\frac{k-\sqrt{k^2-4}}{2k}\right)^n\right]$$

with $k^2 - 4 > 0$. Several properties and identities of $k$-Oresme numbers are given by authors (see [1,9,10,12]).

In [8], Horadam polynomial denoted by $h_n(x) = h_n(x, a, b; p, q)$ is given by the following second-order linear recurrence relation:

$$h_n(x) = pxh_{n-1}(x) + qh_{n-2}(x), \qquad n \geqslant 3$$

with initial conditions $h_1(x) = a$ and $h_2(x) = bx$.

The characteristic equation of the relation below

$$t^2 - pxt - q = 0$$

has the roots

$$\alpha = \frac{px + \sqrt{p^2x^2 + 4q}}{2}, \qquad \beta = \frac{px - \sqrt{p^2x^2 + 4q}}{2}.$$

Some properties of the Horadam polynomials can be found in [6,8,11].

In some particular cases, several special polynomials are obtained from the Horadam polynomial. Some of these polynomials are Fibonacci, Lucas, Pell-Lucas,

Jacobsthal, Jacobsthal-Lucas, Chebyshev polynomials studied in [2,3,11,13].
If $a = b = p = q = 1$ in $h_n(x, a, b; p, q)$, Fibonacci polynomials are obtained as

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x); \qquad F_1(x) = 1, F_2(x) = 1.$$

In [8], Horzum and Kocer gave an explicit formula for the Fibonacci polynomials as

$$F_{n+1}(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} (x)^{n-2k}.$$

For a real number $x \neq 0$, Oresme polynomials are defined by the following relation [1].

$$O_{n+1}(x) = \begin{cases} \dfrac{1}{x} & n = 0, 1, \\ O_n(x) - \dfrac{1}{x^2} O_{n-1}(x) & n \geqslant 2. \end{cases}$$

The relation between Oresme polynomials and Fibonacci numbers can be given by $O_n(3) = F_3/3^n$ in [10].
In [4], Halıcı et al. defined the n-th $k$-Oresme polynomial by the following reccurence relation by initial conditions $O_0^{(k)}(x) = 0,\ O_1^{(k)}(x) = \frac{1}{kx}$.

$$O_{n+2}^{(k)}(x) = O_{n+1}^{(k)}(x) - \frac{1}{k^2x^2} O_n^{(k)}(x), \tag{1.7}$$

where $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Solving the auxiliary equation (1.7), the Binet Formula can be written as

$$O_n^{(k)}(x) = \frac{1}{\sqrt{k^2x^2 - 4}} \left[ \left( \frac{kx + \sqrt{k^2x^2 - 4}}{2kx} \right)^n - \left( \frac{kx - \sqrt{k^2x^2 - 4}}{2kx} \right)^n \right]. \tag{1.8}$$

The generating function of the $k$-Oresme polynomials is given as

$$f(z) = \sum_{i \geqslant 0} O_i^{(k)}(x) z^i = \frac{\dfrac{z}{kx}}{1 - z + \dfrac{z^2}{k^2x^2}}, \tag{1.9}$$

for a real number $z$. Authors also gave the reccurence relation of the derivative sequence of $k$-Oresme polynomials by

$$\frac{d}{dx} O_{n+1}^{(k)}(x) = (O_{n+1}^{(k)})'(x) = (O_n^{(k)})'(x) + \frac{2}{k^2x^3} O_{n-1}^{(k)}(x) - \frac{1}{k^2x^2} (O_{n-1}^{(k)})'(x). \tag{1.10}$$

Note that, by taking $x = 1$ and $kx = 2$ in (1.7) respectively, one can get $k$-Oresme numbers and Oresme numbers.
One can consider that this study is a continuation of the study in [4]. Some special identities and properties of the new polynomial sequence defined in [4] are given and derivative sequence is examined in the following sections.

## 2. Some results

In the following theorem, some sums of $k$-Oresme polynomials are given.

THEOREM 2.1. *For $n \geqslant 0$, we have the followings.*

$$\sum_{i=0}^{n} O_i^{(k)}(x) = k^2x^2\left(\frac{1}{kx} - O_{n+2}^{(k)}(x)\right), \tag{2.1}$$

$$\sum_{i=0}^{n}(-1)^i O_i^{(k)}(x) = \frac{k^2x^2}{2k^2x^2+1}\left(-\frac{1}{kx} + (-1)^{n+1}\left(O_{n+2}^{(k)}(x) - 2O_{n+1}^{(k)}(x)\right)\right). \tag{2.2}$$

PROOF. It is clear that the equation (2.1) satisfies for $n = 0$. Suppose for induction that the equation is true for all $m \leqslant n$. Using (1.7), we get

$$\sum_{i=0}^{n+1} O_i^{(k)}(x) = \sum_{i=0}^{n} O_i^{(k)}(x) + O_{n+1}^{(k)}(x) = k^2x^2\left(\frac{1}{kx} - O_{n+2}^{(k)}(x)\right)$$

which implies

$$\sum_{i=0}^{n+1} O_i^{(k)}(x) = k^2x^2\left(\frac{1}{kx} - O_{n+2}^{(k)}(x) + \frac{O_{n+1}^{(k)}(x)}{k^2x^2}\right) = k^2x^2\left(\frac{1}{kx} - O_{n+3}^{(k)}(x)\right).$$

Similarly, the equation (2.2) can be shown by induction. □

With the help of the previous theorem, we have the following results.

COROLLARY 2.1. *For $n \geqslant 0$, we have,*

$$\sum_{i=0}^{n} O_{2i+1}^{(k)}(x) = \frac{k^2x^2}{2k^2x^2+1}\left(\frac{k^2x^2+1}{kx} + \frac{k^2x^2+1}{k^2x^2}O_{2n+1}^{(k)}(x) - k^2x^2 O_{2n+2}^{(k)}(x)\right) \tag{2.3}$$

*and*

$$\sum_{i=0}^{n} O_{2i}^{(k)}(x) = \frac{k^2x^2}{2k^2x^2+1}\left(kx - (k^2x^2+1)O_{2n+2}^{(k)}(x) + O_{2n+1}^{(k)}(x)\right). \tag{2.4}$$

PROOF. By observing that

$$\sum_{i=0}^{n} O_{2i}^{(k)}(x) = \frac{1}{2}\left(\sum_{i=0}^{2n+1} O_i^{(k)}(x) + \sum_{i=0}^{2n+1}(-1)^i O_i^{(k)}(x)\right)$$

and

$$\sum_{i=0}^{n} O_{2i+1}^{(k)}(x) = \frac{1}{2}\left(\sum_{i=0}^{2n+1} O_i^{(k)}(x) - \sum_{i=0}^{2n+1}(-1)^i O_i^{(k)}(x)\right),$$

the equations can be obtained clearly. □

It is easy to see from (1.9) that

$$\lim_{z\to 1} f(z) = \sum_{i\geqslant 0} O_i^{(k)}(x) = kx. \tag{2.5}$$

Furthermore, by using the equation (2.1) we can write

$$kx = \sum_{i\geqslant 0} O_i^{(k)}(x) = \lim_{n\to\infty} \sum_{i=0}^{n-1} O_i^{(k)}(x) = k^2x^2\left(\frac{1}{kx} - \lim_{n\to\infty} O_{n+1}^{(k)}(x)\right).$$

Thus we obtain

$$\lim_{n\to\infty} O_n^{(k)}(x) = 0. \tag{2.6}$$

Notice that, setting $kx = 2$ in (2.1), (2.2), (2.3) and (2.5), we obtain (1.3), (1.4), (1.5) and (1.2) respectively. For example, taking $n = 3$ in (2.1), we find

$$\sum_{i=0}^{3} O_i^{(k)}(x) = k^2x^2\left(\frac{1}{kx} - O_5^{(k)}(x)\right) = \frac{3k^2x^2-1}{k^3x^3}.$$

Setting $kx = 2$ in the last equation, we can observe that

$$\sum_{i=0}^{3} O_i = 4\left(\frac{1}{2} - O_5\right) = \frac{11}{8}$$

which satisfies the equation (1.3).
In the next theorem, we give a product formula for $k$-Oresme polynomials.

Theorem 2.2. *For $n \geqslant 1$, we have*

$$O_n^{(k)}(x) = \prod_{1\leqslant m\leqslant n-1}\left(kx - 2\cos\frac{m\pi}{n}\right). \tag{2.7}$$

Proof. Observe that the degree of the polynomial $k^nx^nO_n^{(k)}(x)$ is $n-1$. By using the Binet formula, we express $O_n^{(k)}(x)$ in terms of hyperbolic functions to deduce the zeros. Take $kx = 2\cosh z$, where $z = u + iv$. Then by (1.8), we have

$$O_n^{(k)}(x) = \frac{1}{(kx)^n}\frac{r_1^n - r_2^n}{r_1 - r_2} = \frac{1}{(kx)^n}\frac{e^{nz} - e^{-nz}}{e^z - e^{-z}} = \frac{1}{(kx)^n}\frac{\sinh nz}{\sinh z},$$

where $r_1, r_2 = (kx \pm \sqrt{k^2x^2-4})/2kx$. Notice that $O_n^{(k)}(x) = 0$ only if $\sinh nz = 0$ which implies $e^{2nz} = 1$, i.e., $u = 0$. Therefore, $\sinh nz = i\sin nz = i\sin nv = 0$ when $v = \frac{m\pi}{n}$ for an integer $m$. As a result, we find

$$kx = 2\cosh\frac{im\pi}{n} = 2\cos\frac{m\pi}{n}.$$

Thus, we have

$$(kx)^nO_n^{(k)}(x) = \prod_{1\leqslant m\leqslant n-1}\left(kx - 2\cos\frac{m\pi}{n}\right).$$

□

As an example of this theorem, for $n = 3$, one can observe that

$$\frac{1}{(kx)^3}\prod_{1\leqslant m\leqslant 2}\left(kx - 2\cos\frac{m\pi}{3}\right) = \frac{1}{(kx)^3}\left(kx - 2\cos\frac{\pi}{3}\right)\left(kx - 2\cos\frac{2\pi}{3}\right)$$

$$= \frac{k^2x^2-1}{(kx)^3}$$

which equals to $O_3^{(k)}(x)$.

## 3. Derivatives

In this section, we study the derivatives sequence of $k$-Oresme polynomials. We give generating function of this sequence and some sums related to $k$-Oresme polynomials and their derivatives. An important identity which gives a relation between polynomial sequence and their derivative sequence is given. In the end of this section, we generalize and formulate the higher order derivatives of $k$-Oresme polynomials. Throughout this study, we denote the derivatives of $k$-Oresme polynomials by $(O_n^{(k)})'(x)$.
In the following theorem, we give sums of derivatives by using the results given in Theorem 2.1.

Theorem 3.1. *For $n \geqslant 0$, we have*

$$\sum_{i=0}^{n}(O_i^{(k)})'(x) = k^2x^2\left(\frac{1}{kx^2} - \frac{2}{x}O_{n+2}^{(k)}(x) - (O_{n+2}^{(k)})'(x)\right) \tag{3.1}$$

*and*

$$\sum_{i=0}^{n}(-1)^i(O_i^{(k)})'(x) = \frac{k(2k^2x^2-1)}{(2k^2x^2+1)^2} \tag{3.2}$$

$$+\frac{k^2x(-1)^{n+1}}{(2k^2x^2+1)^2}\left(x(2k^2x^2+1)\left((O_{n+2}^{(k)})'(x) - 2(O_{n+1}^{(k)})'(x)\right) + 2\left(O_{n+2}^{(k)}(x) - 2O_{n+1}^{(k)}(x)\right)\right).$$

Proof. By differentiating with respect to $x$ (2.1) and (2.2) respectively, (3.1) and (3.2) can be obtained clearly. □

Theorem 3.2. *Generating function of $(O_n^{(k)})'(x)$ is of the form*

$$g(z) = -k\frac{f(z)^2}{z}\left(1 - z - \frac{z^2}{k^2x^2}\right), \tag{3.3}$$

*where $z \in \mathbb{R}$.*

Proof. Observing that

$$\sum_{i\geqslant 0}(O_i^{(k)})'(x)z^i = \frac{d}{dx}\sum_{i\geqslant 0}(O_i^{(k)})'(x)z^i$$

and using (1.9), we have

$$\sum_{i\geqslant 0}(O_i^{(k)})'(x)z^i = -\frac{kz}{k^2x^2}\frac{\left(1 - z - \frac{z^2}{k^2x^2}\right)}{\left(1 - z - \frac{z^2}{k^2x^2}\right)^2}.$$

By the definition of $f(z)$, we complete the proof. □

PROPOSITION 3.1. *For $n \geqslant 2$, we have*

$$(O_n^{(k)})'(x) = \frac{k^2x^2\left(2n-k^2x^2\right)O_n^{(k)}(x) - 2nO_{n-2}^{(k)}(x)}{k^2x^3(k^2x^2-4)}. \tag{3.4}$$

PROOF. Recall that, the roots of the auxiliary equation (1.7) are

$$\alpha(x), \beta(x) = \frac{kx \pm \sqrt{k^2x^2-4}}{2kx}$$

and $\Delta(x) = \sqrt{k^2x^2-4}$. Then by differentiating, we find

$$\alpha'(x), \beta'(x) = \pm\frac{2}{kx^2\sqrt{k^2x^2-4}}.$$

and

$$\Delta'(x) = \frac{kx^2}{\Delta(x)}.$$

By Binet Forumla, we can write

$$(O_n^{(k)})'(x) = \frac{d}{dx}\left(\frac{\alpha(x)^n - \beta(x)^n}{\Delta(x)}\right) = \frac{\left((\alpha(x)^n)' - (\beta(x)^n)'\right)\Delta(x) - kx^2O_n^{(k)}(x)}{\Delta(x)^2}$$

$$= \frac{\left(n\alpha(x)^{n-1}\alpha'(x) - n\beta(x)^{n-1}\beta'(x)\right)\Delta(x) - kx^2O_n^{(k)}(x)}{\Delta(x)^2}$$

$$= \frac{\frac{2n}{x\Delta(x)}\left(\alpha(x)^{n-1} + \beta(x)^{n-1}\right)\left(\alpha(x) - \beta(x)\right) - kx^2O_n^{(k)}(x)}{\Delta(x)^2}$$

$$= \frac{\frac{2n}{x\Delta(x)}\left(\alpha(x)^n - \beta(x)^n - \beta(x)\alpha(x)^{n-1} + \alpha(x)\beta(x)^{n-1}\right) - kx^2O_n^{(k)}(x)}{\Delta(x)^2}.$$

Using the fact that $\alpha(x)\beta(x) = 1/k^2x^2$ , we have

$$(O_n^{(k)})'(x) = \frac{O_n^{(k)}(x)\left(\frac{2n}{x} - kx^2\right) - \frac{2n}{k^2x^3}O_{n-2}^{(k)}(x)}{k^2x^2-4}.$$

Finally, some elementary operations complete the proof. □

As an example of this result, by taking $n = 2$ in (3.4) we can find

$$\frac{k^2x^2\left(4-k^2x^2\right)O_2^{(k)}(x) - 4O_0^{(k)}(x)}{k^2x^3(k^2x^2-4)} = -\frac{1}{kx^2} = (O_2^{(k)})'(x).$$

THEOREM 3.3. . *For $n \geqslant 2$, we have the followings.*

$$\sum_{j=1}^{n-1} O_j^{(k)}(x)O_{n-j}^{(k)}(x) = \frac{1}{k}(O_n^{(k)})'(x) + \frac{n}{kx}O_n^{(k)}(x), \tag{3.5}$$

$$\sum_{j=1}^{\lfloor\frac{n-1}{2}\rfloor} \frac{n-2j-1}{(kx)^{2j+1}}O_{n-2j-1}^{(k)}(x) = \frac{1}{k}(O_n^{(k)})'(x) + \frac{n}{kx}O_n^{(k)}(x). \tag{3.6}$$

Proof. Obviously for $n = 2$, the equation (3.5) is true. Assume for induction that equation satifies $\forall m \leqslant n$. By (1.7) and (1.10), we obtain

$$\frac{1}{k}(O_{n+1}^{(k)})'(x) + \frac{n+1}{kx}O_{n+1}^{(k)}(x) = \frac{1}{kx}O_n^{(k)}(x) + \frac{1}{k}(O_n^{(k)})'(x) + \frac{n}{kx}O_n^{(k)}(x) - \frac{1}{k^2x^2}\left(\frac{1}{kx}(O_{n-1}^{(k)})'(x) + \frac{n-1}{kx}O_{n-1}^{(k)}(x)\right)$$

which equals to

$$\frac{1}{kx}O_n^{(k)}(x) + \sum_{j=1}^{n-1} O_j^{(k)}(x)O_{n-j}^{(k)}(x) - \frac{1}{k^2x^2}\sum_{j=1}^{n-2} O_j^{(k)}(x)O_{n-j+1}^{(k)}(x). \tag{3.7}$$

Using some operations on sums and the relation (1.7), (3.7) equals to

$$\frac{1}{kx}O_n^{(k)}(x) + O_1^{(k)}(x)O_{n-1}^{(k)}(x) + \sum_{j=1}^{n-2} O_j^{(k)}(x)O_{n-j+1}^{(k)}(x).$$

Notice that

$$\sum_{j=1}^{n-2} O_j^{(k)}(x)O_{n-j-1}^{(k)}(x) = \sum_{j=1}^{n} O_j^{(k)}(x)O_{n-j-1}^{(k)}(x) - O_n^{(k)}(x)O_1^{(k)}(x) - O_{n-1}^{(k)}(x)O_2^{(k)}(x).$$

Using this fact and $O_1^{(k)}(x) = O_2^{(k)}(x) = \frac{1}{kx}$, we complete the proof. Other equation can be shown similarly by setting $n = 2m$ and using induction. □

Second derivatives of $k$-Oresme polynomials can be obtained by differentiating the formula (1.10) as follows :

$$(O_{n+1}^{(k)})''(x) = (O_n^{(k)})''(x) - \frac{6}{k^2x^4}O_{n-1}^{(k)}(x) + \frac{4}{k^2x^3}(O_{n-1}^{(k)})'(x) - \frac{1}{k^2x^2}(O_{n-1}^{(k)})''(x).$$

Similarly we get third and 4-th derivatives as

$$(O_{n+1}^{(k)})'''(x) = (O_n^{(k)})'''(x) + \frac{24}{k^2x^5}O_{n-1}^{(k)}(x) - \frac{18}{k^2x^4}(O_{n-1}^{(k)})'(x) + \frac{6}{k^2x^3}(O_{n-1}^{(k)})''(x) - \frac{(O_{n-1}^{(k)})'''(x)}{k^2x^2}$$

and

$$(O_{n+1}^{(k)})^{(4)}(x) = (O_n^{(k)})^{(4)}(x) - \frac{120}{k^2x^6}O_{n-1}^{(k)}(x) + \frac{96}{k^2x^5}(O_{n-1}^{(k)})'(x) - \frac{36}{k^2x^4}(O_{n-1}^{(k)})''(x) + \frac{8}{k^2x^3}(O_{n-1}^{(k)})'''(x) - \frac{1}{k^2x^2}(O_{n-1}^{(k)})^{(4)}(x).$$

By continuing this process, it can be observed that the r-th derivative of $k$-Oresme polynomial is

$$(O_{n+1}^{(k)})^{(r)}(x) = (O_n^{(k)})^{(r)}(x) + \frac{1}{k^2x^2}\sum_{i=0}^{r}(-1)^{i+1}\frac{c_{r,i+1}}{x^i}(O_{n-1}^{(k)})^{(r-i)}(x), \tag{3.8}$$

with initial conditions $(O_0^{(k)})^{(r)}(x) = 0, (O_1^{(k)})^{(r)}(x) = \frac{(-1)^r r!}{kx^{r+1}}$. Notice that the coefficients $\{c_{i,j}\}$ are the entries of the modified Pascal triangle below.

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 2 & & \\
 & & 1 & 4 & 6 & & \\
 & 1 & 6 & 18 & 24 & & \\
 & 1 & 8 & 36 & 96 & 120 & \\
1 & 10 & 60 & 240 & 600 & 720 & \\
1 & 12 & 90 & 480 & 1800 & 4320 & 5040 \\
 & & & . & & & \\
 & & & . & & & \\
 & & & . & & &
\end{array}$$

Here $c_{i,j}$ is the $i-th$ row and $j-th$ column element of triangle. Notice that this triangle is constructed by multiplying rows of classical Pascals triangle by $1, 2, 6, 24, 120, 720, \ldots$ which is the set $\{n!\}_{n=0,1,\ldots}$. For example, $4-th$ row $1, 8, 36, 96, 120$ is $(1.1), (4.2), (6.6), (4.24), (1.120)$, where $1, 4, 6, 4, 1$ is the $4-th$ row of the classical Pascal's triangle.

It can be observe that all results given in this paper satisfy for the Oresme polynomials by setting $k = 1$ (see [1]).

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Serpil Halici, Department of Mathematics, Pamukkale University, Denizli, Turkey
*Email address*: shalici@pau.edu.tr

Zehra Betül Gür, Department of Mathematics, Pamukkale University, Denizli, Turkey
*Email address*: gurbetul35@gmail.com