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A NOVEL AVERAGE PARAMETER FOR NETWORK ANALYSIS

Vecdi Aytaç and Zeynep Nihan Berberler

Abstract. Graph theory has turned out to be one of the most significant mathematical methods for studying and analyzing network design. For various systems and contexts, networks may be essential frameworks. Networks have become extremely popular as they are now common ground for multidisciplinary research from chemistry to social sciences. In applied mathematics, graphs are used on the differential equations and partial differential equations. Since, under certain cases, the average parameters are more efficient than other similar measures based on the worst-case condition, in this paper, a novel average parameter is defined and studied for connected graphs.

1. Introduction

The function of mathematics in numerous fields is indispensable. Graph theory is one of the more important fields in applied mathematics, especially when it comes to structural models. These structural configurations of different artifacts or technologies contribute to new developments and changes in the current environment for development in these fields.
A significant number of data analyzing problems can be inevitably modeled as networks. Therefore, recently the increasing significance of big data analytics and image processing also contributed to an increasing interest in graph-based topics, especially among mathematicians who study partial differential equations. These kinds of problems have a deep background in computer science, combinatorics, and especially graph theory.
Graphs are used in many diverse fields, including operations research, chemistry,

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economics, computer science, engineering, and applied mathematics [**15, 13, 1, 5, 9**]. In applied mathematics, graphs are used on the differential equations and partial differential equations. The differential equations are always defined on the edge set of a graph. One needs some connection and boundary conditions to determine a solution uniquely. Due to the restriction of the graph, the connection conditions at the vertices become an important component to solve the differential equations. Differential equations on networks are a relatively new branch existing over 20 years in the theory of differential equations.

Networks are fundamental frameworks and are present in numerous applications and environments. More and more everyday life networks are computer networks, telecommunication networks, railways, and highways. In a communication network, it is important to use network vulnerability measurements to direct network designers in selecting the best network architecture. These vulnerability measurements can have a significant effect on difficult network optimization problems. Furthermore, the fundamental component of a distributed system is the interconnection network [**15, 16**].

The network architecture is critical as it defines how data is transferred between processors. This is why a communication network is conceptualized as a graph in order to quantify its weakness. Stations correspond to vertices, and connections between stations correspond to edges. A communication network's weakness, i.e., vulnerability, indicates how powerful the network is before any centers or link lines are made inoperable. As networks begin to lose the link between centers, there will inevitably be a loss of performance. Different metrics for measuring network robustness have been given in the literature [**3, 4, 2, 6, 10, 18**]. Also, a number of theoretical graph parameters were used for deriving network vulnerability measurement formulas. The vulnerability of a graph concerns graph analysis where some elements of it are affected. For example, the vertices or edges of a graph are damaged. The graph vulnerability measures usually are invariants that measure the changes in the network characteristics of removing one or more of its elements. Then the connectivity, toughness, bondage, reinforcement, domination number, total domination, etc., have been suggested to measure the networks vulnerability [**3, 4, 2, 6, 10, 18**]. Connectivity is probably the most famous and best-practice measure about how strong a graph is linked. The smallest quantity of nodes in the set whose removal leads to an unconnected graph is known as the connectivity. New vulnerability parameters on average concepts have recently been added to the literature, such as the average lower domination, the average lower independence, the average lower bondage, the average connectivity, and the average lower connectivity [**3, 4, 2, 10, 18, 19, 14**].

2. Definitions and notation

In this paper, all graphs considered are simple, finite, and undirected. For notation and graph-theoretical terminology not defined here, we follow [**17**]. Let $G = (V(G), E(G))$ be a graph, in which $V(G)$ and $E(G)$ denote the vertex and edge sets of G, respectively. For two vertices u and v in $V(G)$ there is an edge

$e = uv$ if u and v are adjacent. The distance $d_G(u, v)$ between u and v is the minimum number of edges of $(u - v)$-path in G. The diameter $diam(G)$ of G is the maximum distance between any two vertices in G. For any vertex v in $V(G)$, the open neighborhood of v in G is $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and the open neighborhood of v in G is $N_G[v] = N_G(v) \cup \{v\}$. The degree of vertex v in G is the size of its open neighborhood and denoted by $\deg_G(v)$ whereas a leaf is a vertex of degree one, support vertex is the adjacent vertex of a leaf. For a real number x, the greatest (respectively, least) integer less (respectively, greater) than or equal to x is denoted by $\lfloor x \rfloor$ (respectively, $\lceil x \rceil$).

Under certain cases, the average parameters are more efficient than other similar measures based on the worst-case condition. Thus, inspired by the various average parameters mentioned above, combining the disjunctive total domination number concept with the average parameter idea allows us to obtain a new graph parameter called average disjunctive total domination number, $\gamma_{av}^{dt}(G)$. The concept of average disjunctive total domination number in graphs is a concept closely related to the problem of finding large disjunctive total domination sets in graphs.

The domination [**17**] is one of the most widely studied topics in graph theory. The idea of domination is based on the set of vertices that are near all the vertices of a graph. A set D is a dominating set of G if every vertex of G except vertices of S is adjacent to at least one vertex of D. Then Cockayne et al. [**6**] introduced total domination to extend the domination problem, including redundancy. A set D is a total dominating set of G if every vertex in G is adjacent to at least one vertex of D. The domination number $\gamma(G)$ (respectively, total domination number $\gamma_t(G)$) is the minimum cardinality over all dominating (respectively, total dominating) sets of G. A set $S \subseteq V$ is a k-dominating set if every vertex $v \in V \backslash S$ satisfies $\deg_S(v) \geqslant k$. The k-domination number $\gamma_k(G)$ is the minimum cardinality among all k-dominating sets [**17**]. The domination number is the 1-domination number. Recently, Henning [**11, 12**] introduced disjunctive total domination as a relaxation of domination and total domination.

Let S be a subset of the set of $V(G)$. For all $v_i \in S$ if there is at least one neighbor in S or at least two vertices in S, such that their distance to v_i is two, we call S as disjunctive total dominating set of G. It is understood that v_i is disjunctively totally dominated with vertices of S when a v_i vertex meets any of these requirements. Also, it is understood that v_i is totally dominated with vertices of S when a v_i vertex meets the first requirement and v_i is disjunctively dominated of S when a v_i vertex meets the second requirement. The disjunctive total domination number, $\gamma^{dt}(G)$ is the minimum cardinality of a DTD-set in G. A DTD-set which gives the value $\gamma^{dt}(G)$ is called $\gamma^{dt}(G)$-set [**7, 8, 11, 12**].

Definition 2.1. *For $v \in V(G)$, we define the disjunctive total domination number of G relative to v, denoted $\gamma_v^{dt}(G)$, as the minimum cardinality of a DTD-set in G that contains v. A minimum cardinality of a DTD-set in that contains v which gives the value $\gamma_v^{dt}(G)$ is called $\gamma_v^{dt}(G)$-set. If the order of G is n, then that the* ***average disjunctive total domination number*** *of G, denoted$\gamma_{av}^{dt}(G)$, is defined to be $\gamma_{av}^{dt}(G) = \frac{1}{n} \sum\limits_{v \in V(G)} \gamma_v^{dt}(G)$, where $\gamma_v^{dt}(G)$is the minimum cardinality* *of a DTD-set that contains v. Here, the value of $\gamma_{av}^{dt}(G)$ does not have to be an integer.*

It's likely that the average disjunctive total domination number is more sensitive to a graph's vulnerability than the other vulnerability parameters. As an example, we consider two graphs having the same number of vertices and edges illustrated in Figure 1.

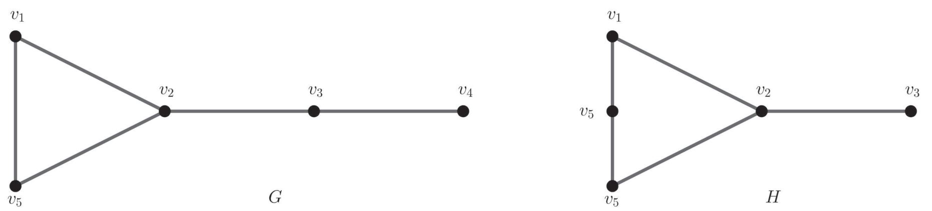


FIGURE 1. Graphs G and H.

The set $S = \{v_2, v_3\}$ constitutes the $\gamma(G)$-set, $\gamma_t(G)$-set and $\gamma^{dt}(G)$-set of G. Hence $\gamma(G) = \gamma_t(G) = \gamma^{dt}(G) = 2$. The set $S = \{v_1, v_2\}$ constitutes the $\gamma(H)$-set, $\gamma_t(H)$-set and $\gamma^{dt}(H)$-set of H. Hence $\gamma(H) = \gamma_t(H) = \gamma^{dt}(H) = 2$. Furthermore, it is easy to see that $\kappa(G) = \kappa(H) = 1$. So, how can the graphs G and H be distinguished? In this case, calculating the average disjunctive total domination number of graphs G and H can be effective.
The following table shows the minimum cardinality of a DTD-set of G that contains v.

TABLE 1. For $\forall v \in V(G)$, $\gamma_v^{dt}(G) - set$ of G.

$v \in V(G)$	$\gamma_v^{dt}(G) -$ set	$\gamma_v^{dt}(G)$
v_1	$\{v_1, v_2, v_3\}$	3
v_2	$\{v_2, v_3\}$	2
v_3	$\{v_3, v_2\}$	2
v_4	$\{v_4, v_3, v_2\}$	3
v_5	$\{v_5, v_2, v_3\}$	3

Hence, we have $\gamma_{av}^{dt}(G) = \frac{1}{5} \sum_{v \in V(G)} \gamma_v^{dt}(G) = \frac{1}{5}(3 + 2 + 2 + 3 + 3) = \frac{13}{5}$.
The following table shows the minimum cardinality of a DTD-set of H that contains v.

TABLE 2. For $\forall v \in V(H)$, $\gamma_v^{dt}(H) - set$ of H.

$v \in V(H)$	$\gamma_v^{dt}(H)-$ set	$\gamma_v^{dt}(H)$
v_1	$\{v_1, v_2\}$	2
v_2	$\{v_2, v_1\}$	2
v_3	$\{v_3, v_2, v_1\}$	3
v_4	$\{v_4, v_2\}$	2
v_5	$\{v_5, v_2, v_1\}$	3

Hence, we have $\gamma_{av}^{dt}(H) = \frac{1}{5} \sum\limits_{v \in V(G)} \gamma_v^{dt}(H) = \frac{1}{5}(2+2+3+2+3) = \frac{12}{5}$.
Thus, the average lower domination number may be a better parameter than the connectivity, domination, total domination and disjunctive total domination number in order distinguish these two graphs G and H. Both of these graphs have connectivity, domination, total domination and disjunctive total domination number, but the second one would be a more reliable communication network than the first. This is reflected by the average disjunctive total domination number since $\gamma_{av}^{dt}(G) = 13/5$ and $\gamma_{av}^{dt}(H) = 12/5$, $\gamma_{av}^{dt}(G) > \gamma_{av}^{dt}(H)$. In other words, H is more tough than the graph G, we can tell.
In this study, some bounds of the average disjunctive total domination number are obtained. Further, some results of the average disjunctive total domination number for some special graphs are also calculated.

LEMMA 2.1. [**11**] *If a graph G has a support vertex v with exactly one neighbor w that is not a leaf, then there is a $\gamma^{dt}(G)$-set that contains v. Further if* $\deg(w) = 2$*, then there is a $\gamma^{dt}(G)$-set that contains both v and w.*

PROPOSITION 2.1. [**11**] *For $n \geqslant 3$, $\gamma^{dt}(C_n) = 2n/5$ if $n \equiv 0\,(\mathrm{mod} 5)$ and $\gamma^{dt}(C_n) = \lceil 2(n+1)/5 \rceil$ otherwise.*

PROPOSITION 2.2. [**11**] *For $n \geqslant 3$, $\gamma^{dt}(P_n) = \lceil 2(n+1)/5 \rceil + 1$ if $n \equiv 1\,(\mathrm{mod}\ 5)$ and $\gamma^{dt}(P_n) = \lceil 2(n+1)/5 \rceil$ otherwise.*

OBSERVATION 2.1. [**20**] For a connected graph G on at least two vertices, $\gamma^{dt}(G) = 2$ if $diam(G) \in \{1, 2\}$.

3. Main results

In this section, an upper bounds of the average disjunctive total domination number are given. Furthermore, the average disjunctive total domination numbers of the well-known graph classes such as P_n, C_n, K_n, W_n, S_n and $K_{r,s}$.

THEOREM 3.1. *Let G be a graph of order n. Then,*

$$\gamma_{av}^{dt}(G) \leqslant \gamma^{dt}(G) + 1 - \frac{\gamma^{dt}(G)}{n},$$

and the equality holds if and only if G has a unique $\gamma^{dt}(G)$-set.

Proof. If the graph G has a unique $\gamma^{dt}(G)$-set, then for every vertex v of $\gamma^{dt}(G)$-set, it is clear that $\gamma_v^{dt}(G) = \gamma^{dt}(G)$. For the vertex u, it holds that $\gamma_u^{dt}(G) = \gamma^{dt}(G) + 1$. Thus, we have that

$$\gamma_{av}^{dt}(G) = \frac{1}{n}\sum\nolimits_{v\in V(G)} \gamma_v^{dt}(G) = \frac{1}{n}(\gamma^{dt}(G)\gamma^{dt}(G) + (n - \gamma^{dt}(G))(\gamma^{dt}(G) + 1))$$

$$\gamma_{av}^{dt}(G) = \gamma^{dt}(G) + 1 - \frac{\gamma^{dt}(G)}{n}.$$

If the graph G has more than one $\gamma^{dt}(G)$-set, then $\gamma_{av}^{dt}(G) \leqslant \gamma^{dt}(G)+1-\frac{\gamma^{dt}(G)}{n}$. □

By Theorem 3.1, the following corollary is immediate.

Corollary 3.1. *For any connected graph G, $\gamma_{av}^{dt}(G) < \gamma^{dt}(G) + 1$.*

Theorem 3.2. *Let G be a connected graph of order n. If G has a vertex with degree $n - 1$, then*

$$\gamma_{av}^{dt}(G) = \gamma^{dt}(G) = 2.$$

Proof. Let S be a $\gamma^{dt}(G)$-set of G. If $\deg(v_1) = n-1$, then it should be $\{v_1\} \in S$. Then, whole vertices in the set $V(G) - \{v_1\}$ of G are totally dominated by the set S. For the vertex v_1, it is sufficient to add one of the vertices v_i $i \in \{2, 3, \dots, n\}$ to the set S. Hence, each vertex of G is disjunctive dominated by the set S, yielding $\gamma^{dt}(G) = 2 = |S|$. The set S is not unique. There exist $n - 1$ distinct sets of S. These are the sets $S_1 = \{v_1, v_2\}, S_2 = \{v_1, v_3\}, \dots, S_{n-1} = \{v_1, v_n\}$. Therefore, the $\gamma_v^{dt}(G)$-set for $\forall v \in V(G)$ is the set of those $n - 1$ distinct sets. Then, we have that $\gamma_v^{dt}(G) = 2$ for $\forall v \in V(G)$. By the definition of average disjunctive domination number, we conclude that $\gamma_{av}^{dt}(G) = \frac{1}{n}(2n) = 2$. □

The following result is immediate by Theorem 3.2.

Corollary 3.2. *If $G \cong K_n$ $(n \geqslant 3)$, W_n $(n \geqslant 4)$, S_n $(n \geqslant 3)$, then $\gamma_{av}^{dt}(G) = \gamma^{dt}(G) = 2$.*

By Corollary 3.1 and Theorem 3.2, a lower and an upper bound for the average disjunctive domination of any graph G is given by the following corollary.

Corollary 3.3. *For any connected graph G, $2 \leqslant \gamma_{av}^{dt}(G) < \gamma^{dt}(G) + 1$.*

Theorem 3.3. *Let C_n $(n \geqslant 3)$ be a cycle of order n. Then, $\gamma_{av}^{dt}(C_n) = \gamma^{dt}(C_n)$.*

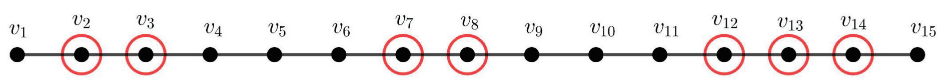
Proof. Let $\{v_1, v_2, \dots, v_n\}$ be the set of vertices of C_n and let S be the $\gamma^{dt}(G)$-set of C_n. Since C_n is a vertex-transitive graph, the set S of C_n is not unique. Therefore, there exist a $\gamma^{dt}(G)$-set with cardinality $|S|$ for $\forall v \in V(C_n)$ yielding $\gamma_v^{dt}(C_n) = \gamma^{dt}(C_n)$ for $\forall v \in V(C_n)$. Then, the average disjunctive domination of C_n is $\gamma_{av}^{dt}(C_n) = \frac{1}{n}(n(\gamma^{dt}(C_n))) = \gamma^{dt}(C_n)$. □

THEOREM 3.4. *Let P_n $(n \geqslant 3)$ be a path of order n. Then,*

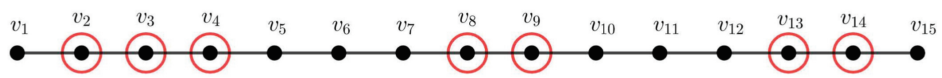
$$\gamma_{av}^{dt}(P_n) = \begin{cases} \dfrac{2n+7}{5}, & \text{if } n \equiv 0\,(\text{mod}5)\,; \\ \dfrac{2n+8}{5}, & \text{if } n \equiv 1\,(\text{mod}5)\,; \\ \dfrac{2n^2+7n-2}{5n}, & \text{if } n \equiv 2\,(\text{mod}5)\,; \\ \dfrac{2n^2+6n-6}{5n}, & \text{if } n \equiv 3\,(\text{mod}5)\,; \\ \dfrac{2n^2+5n-2}{5n}, & \text{if } n \equiv 4\,(\text{mod}5)\,. \end{cases}$$

PROOF. Let $\{v_1, v_2, \ldots, v_n\}$ be the set of vertices of P_n and let S be the $\gamma^{dt}(G)$-set of P_n. If $v \in S$, then $\gamma_v^{dt}(P_n) = \gamma^{dt}(P_n)$; if $v \notin S$, then $\gamma_v^{dt}(P_n) = \gamma^{dt}(P_n) + 1$. In order to compute the average disjunctive domination number of P_n, there exist five cases depending on the number of vertices of P_n.

Case 1. If $n \equiv 0(\text{mod}5)$, then by Proposition 2.2, we have that $\gamma^{dt}(P_n) = \frac{2n+5}{5}$. Let $S_1 = \bigcup\limits_{i=0}^{\frac{n}{5}-1} \{v_{5i+2}, v_{5i+3}\} \cup \{v_{n-1}\}$ or $S_2 = \bigcup\limits_{i=0}^{\frac{n}{5}-1} \{v_{5i+3}, v_{5i+4}\} \cup \{v_2\}$. The vertices in the $S_1$and S_2 set correspond to the marked vertices in (a) and (b), respectively, in Figure 2 for clarity.



(a) S_1-set for P_{15}



(b) S_2-set for P_{15}

FIGURE 2. $S_1 - set$ and $S_2 - set$ for P_{15}

Then both the sets S_1 and S_2 are the $\gamma^{dt}(G)$-sets. Let $V^* = V - (S_1 \cup S_2)$. If $S_1 \cap S_2 = \bigcup\limits_{i=0}^{\frac{n}{5}-1} \{v_{5i+3}\} \cup \{v_2, v_{n-1}\}$, then $|S_1 \cap S_2| = \frac{n}{5} + 2$. $|S_1 \cup S_2| = |S_1| + |S_2| - |S_1 \cap S_2| = (\frac{2n}{5} + 1) + (\frac{2n}{5} + 1) - (\frac{n}{5} + 2) = \frac{3n}{5}$. If $v \in S_1 \cup S_2$, then $\gamma_v^{dt}(P_n) = \gamma^{dt}(P_n) = \frac{2n+5}{5}$. If $v \notin S_1 \cup S_2$ (that is $v \in V^*$), then $\gamma_v^{dt}(P_n) = \gamma^{dt}(P_n) + 1 = \frac{2n+5}{5} + 1 = \frac{2n+10}{5}$. Since we have that $|S_1 \cup S_2| = \frac{3n}{5}$ and $|V^*| = n - \frac{3n}{5} = \frac{2n}{5}$,

$$\gamma_{av}^{dt}(P_n) = \frac{1}{n}(|S_1 \cup S_2|\,|S| + |V^*|\,(|S|+1))$$
$$= \frac{1}{n}((\frac{3n}{5})(\frac{2n+5}{5}) + (\frac{2n}{5})(\frac{2n+10}{5}))$$
$$= \frac{2n+7}{5}.$$

Case 2. If $n \equiv 1(\mathrm{mod}5)$, then by Proposition 2.2, we have that $\gamma^{dt}(P_n) = \frac{2n+8}{5}$. Let $S = \bigcup_{i=0}^{\frac{n-1}{5}-1} \{v_{5i+2}, v_{5i+3}\} \cup \{v_{n-2}, v_{n-1}\}$. The set S is a $\gamma^{dt}(G)$-set of P_n. By this way, it is possible to find the $\gamma^{dt}(G)$-set including $\forall v \in V(P_n)$. Hence, we obtain $\gamma_v^{dt}(P_n) = \gamma^{dt}(P_n)$ $\forall v \in V(P_n)$. As a result,

$$\gamma_{av}^{dt}(P_n) = \frac{1}{n}(n\,\gamma^{dt}(P_n)) = \gamma^{dt}(P_n) = \frac{2n+8}{5}.$$

Case 3. If $n \equiv 2(\mathrm{mod}5)$, then by Proposition 2.2, we have that $\gamma^{dt}(P_n) = \frac{2n+6}{5}$. The vertices in the set $V^* = \bigcup_{i=0}^{\frac{n-2}{5}-1} \{v_{5i+4}\}$ do not belong to any $\gamma^{dt}(G)$-set of P_n. There exists a set S as a $\gamma^{dt}(G)$-set of P_n including the other vertices of the graph. Therefore, $\gamma_v^{dt}(P_n) = |S| + 1 = \frac{2n+6}{5} + 1 = \frac{2n+11}{5}$ $\forall v \in V^*$ whereas $\gamma_v^{dt}(P_n) = |S| = \frac{2n+6}{5}$ $\forall v \in (V - V^*)$. Then we receive that

$$\gamma_{av}^{dt}(P_n) = \frac{1}{n}\left[|V^*|\,(|S|+1) + (n - |V^*|)\,|S|\right]$$
$$= \frac{1}{n}\left[(\frac{n-2}{5})(\frac{2n+11}{5}) + (\frac{4n+2}{5})(\frac{2n+6}{5})\right]$$
$$= \frac{2n^2+7n-2}{5n}.$$

Case 4. If $n \equiv 3(\mathrm{mod}5)$, then by Proposition 2.2, we have that $\gamma^{dt}(P_n) = \frac{2n+4}{5}$. The vertices in the set $V^* = \bigcup_{i=0}^{\frac{n-2}{5}-1} \{v_{5i+4}\}$ do not belong to any $\gamma^{dt}(G)$-set of P_n. There exists a set S as a $\gamma^{dt}(G)$-set of P_n including the other vertices of the graph. Therefore, $\gamma_v^{dt}(P_n) = |S| + 1 = \frac{2n+6}{5} + 1 = \frac{2n+11}{5}$ $\forall v \in V^*$ whereas $\gamma_v^{dt}(P_n) = |S| = \frac{2n+6}{5}$ $\forall v \in (V - V^*)$. Then we receive that

$$\gamma_{av}^{dt}(P_n) = \frac{1}{n}\left[|V^*|\,(|S|+1) + (n - |V^*|)\,|S|\right]$$
$$= \frac{1}{n}\left[2(\frac{n-3}{5})(\frac{2n+9}{5}) + (\frac{3n+6}{5})(\frac{2n+4}{5})\right]$$
$$= \frac{2n^2+6n-6}{5n}.$$

Case 5. If $n \equiv 4(\mathrm{mod}5)$, then by Proposition 2.2, we have that $\gamma^{dt}(P_n) = \frac{2n+2}{5}$. For $n \equiv 4(\mathrm{mod}5)$, the $\gamma^{dt}(G)$-set S of P_n is unique, that is,

$$S = \bigcup_{i=0}^{\frac{n+1}{5}-1} \{v_{5i+2}, v_{5i+3}\}.$$

Being $V^* = V - S$, if $v \in V^*$, then $\gamma_v^{dt}(P_n) = \gamma^{dt}(P_n) + 1$; otherwise $\gamma_v^{dt}(P_n) = \gamma^{dt}(P_n)$. Then we have

$$\begin{aligned}\gamma_{av}^{dt}(P_n) &= \frac{1}{n}\left[|V^*|\,(|S|+1) + (n - |V^*|)\,|S|\right] \\ &= \frac{1}{n}\left[(\frac{n-3}{5})(\frac{2n+7}{5}) + (\frac{4n+3}{5})(\frac{2n+2}{5})\right] \\ &= \frac{2n^2+5n-2}{5n}.\end{aligned}$$

By Cases 1-5, the proof of the theorem holds.

□

THEOREM 3.5. *Let $K_{r,s}$ $(r, s \geqslant 2)$ be a complete bipartite graph of order $r+s$. Then,*

$$\gamma_{av}^{dt}(K_{r,s}) = \gamma^{dt}(K_{r,s}) = 2.$$

PROOF. By Obsevation 2.1, we have that $\gamma^{dt}(K_{r,s}) = 2$. The vertex set of $K_{r,s}$ can be partitioned into two sets as $V(K_{r,s}) = V_1 \cup V_2$. Let $V_1 = \{v_1, \ldots, v_r\}$ and $V_2 = \{u_1, \ldots, u_s\}$. By the definition of average lower disjunctive domination, there exists a set $S_1 = \{v_k, u_i\}$ that is a $\gamma^{dt}(G)$-set for $\forall v_k \in V_1$, and , there exists a set for $\exists v_j \in V_1$ $S_2 = \{u_t, v_j\}$ that is a $\gamma^{dt}(G)$-set for $\exists u_i \in V_2$ and $\forall u_t \in V_2$. Then we receive that

$$\gamma_{av}^{dt}(K_{r,s}) = \frac{(r+s)(2)}{r+s} = 2 = \gamma^{dt}(K_{r,s}).$$

□

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Vecdi Aytaç, Department of Computer Engineering, Faculty of Engineering, Ege University, 35100 Izmir, Turkey

Email address: vecdi.aytac@ege.edu.tr

Zeynep Nihan Berberler, Department of Computer Sciences, Faculty of Science, Dokuz Eylül University, 35160 Izmir, Turkey

Email address: zeynep.berberler@deu.edu.tr