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ON A VARIATIONAL PROBLEM IN p-CALCULUS

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Abstract. This study focuses on to bring together a new type of quantum calculus, namely p-calculus, and variational calculus. We give necessary optimality conditions for p-variational problem. Sufficient optimality conditions are also given.

1. Introduction

The quantum calculus is an old and one of the interesting fields of mathematics, sometimes called calculus without limits. It is well-known that it changes the classical derivative by a quantum difference operator. In recent years, many papers in several fields of mathematics, such as orthogonal polynomials, analytic number theory, geometric function theory, combinatorics etc., have been produced by using concepts of quantum calculus. [**2, 12, 13, 15**].

In the beginning of the twentieth century, Jackson introduced the q-calculus with following notation

$$\frac{f(t)-f(qt)}{(1-q)t},$$

where q is a fixed number different from 1, $t \neq 0$ and f is a real function. We refer to reader to [**14**] for basic concepts of quantum calculus and [**9**] for history of q-calculus.

In [**16**], authors produced a new type of quantum calculus with the following expression

$$\frac{f(t^p)-f(t)}{t^p-t},$$

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and above notation is definition of the p-derivative, where p stands for power. Moreover, in p-calculus, some features of functions and Steffensen inequality [**17, 18**] and pq-calculus which a generalization of p-calculus were given in [**10**].

The calculus of variation is one of the classical subjects in mathematics and it establishes the relation between other fields of mathematics such as differential equations, geometry, and physics. Also it has important applications in mechanics, engineering, economics, biology and electrical engineering. The calculus of variation copes with identifying extrema and, in this regard, one can say that it is a branch of optimization. Because of its importance, studies based in quantum calculus are occurred. In this sense, we refer [**1, 3, 6, 7, 8, 11**] to readers to for details in calculus of variations based in different quantum operators.

The main objective of this paper is to provide necessary and sufficient optimality conditions for the p-variational problem for a and b are fixed real number

$$\text{(P)}\qquad \begin{gathered} \mathcal{L}[y] = \int_a^b L\Big(t, y(t^p), D_p[y](t)\Big) d_p t \longrightarrow \mathit{extremize}, \\ y \in \mathcal{Y}\left([a,b]_p, \mathbb{R}\right),\ y(a) = \alpha,\ y(b) = \beta. \end{gathered}$$

In this Problem (P), *extremize* indicates that *minimize* or *maximize*. Problem (P) with detailed will be given in Section 3. Furthermore, for I is an interval of $\mathbb{R}$ containing 1; $a, b \in I$, $a < b$, Lagrangian L has the desired the following hypotheses:

- (H1) For any $t \in I$, a function which belongs to $C^1(\mathbb{R}^2, \mathbb{R})$ maps (u, v) to $L(t, u, v)$,
- (H2) For any admissible function y, a function with continuity at the point 1 maps t to $L\big(t, y(t^p), D_p[y](t)\big)$,
- (H3) For all admissible functions y and $i = 0, 1$, functions which exists in $\mathcal{Y}\big([a,b]_p, \mathbb{R}\big)$ map t to $\partial_{i+2} L\big(t, y(t^p), D_p[y](t)\big)$,
  where ∂_j stands for the first order partial derivative with respect to $j-$th argument.

This paper consists of 3 sections. Section 1 (this section) is devoted to introduction, In Section 2, p-calculus, the new type quantum calculus, with some necessary definitions and theorems, is introduced from Ref.[**16**]. In Section 3, we discuss some results about p-variational calculus.

2. Preliminaries

Let $p \in (0, 1)$ be and consider interval $J = [0, \infty)$. We will denote by J_p the set $J_p := \{x^p : x \in J\}$. All over the paper, we accept that function $f(x)$ is defined on J.

We will give only the needed definitions and fundamental results on p-calculus to construct the main findings of the study.

Definition 2.1. [**16**] p-derivative of an arbitrary function $f(x)$ is denoted by

$$D_p f(x) = \frac{f(x^p) - f(x)}{x^p - x}, \text{ if } x \neq 0, 1,$$

and

$$D_pf(0)=\lim_{x\to 0^+}D_pf(x),\quad D_pf(1)=\lim_{x\to 1}D_pf(x).$$

COROLLARY 2.1. [**16**] *If $f(x)$ is classically differentiable, then by taking limits (letting $p\to 1$) $D_pf(x)$ tends to $f'(x)$, and also suppose $f'(x)$ is defined in a neighborhood of $x=0$, $x=1$ and suppose $f'(x)$ has the property that continuity at $x=0$, $x=1$, then we have*

$$D_pf(0)=f'_+(0),\ D_pf(1)=f'(1).$$

DEFINITION 2.2. [**16**] The nth order p-derivative of function $f(x)$ can be denoted by

$$\left(D_p^0f\right)(x)=f(x),\ \left(D_p^nf\right)(x)=D_p\left(D_p^{n-1}f\right)(x),\ n\in\mathbb{N}.$$

We note that the p-derivative has the following properties [**16**].

THEOREM 2.1. *Suppose that α, $\beta\in\mathbb{R}$ and $x\in J_p$ and also suppose that $f(x)$ and $g(x)$ is p-differentiable on J. Then*

1. $D_pf\equiv 0$, *then f is a fixed number. Conversely, $D_pc\equiv 0$ for any c. (Constant rule)*
2. $D_p(\alpha\ f+\beta\ g)(x)=\alpha\ D_pf(x)+\beta\ D_pg(x)$. *(Linearity rule)*
3. $D_p(f(x)\ g(x))=f(x)\ D_pg(x)+g(x^p)\ D_pf(x)$. *(Product rule)*
4. $D_p\left(\dfrac{f}{g}\right)(x)=\dfrac{g(x)D_pf(x)-f(x)D_pg(x)}{g(x)g(x^p)}$. *(Quotient rule)*

DEFINITION 2.3. [**16**] Suppose that $D_pF(x)=f(x)$. Then $F(x)$ is called a p-antiderivative of $f(x)$ and it is expressed by

$$F(x)=\int f(x)d_px.$$

The $F(x)$ which p-integral of $f(x)$ can be shown as with series

$$\sum_{j=0}^{\infty}(x^{p^j}-x^{p^{j+1}})f(x^{p^j}).$$

In [**16**], authors give definitions of definite p-integral with the following three cases. In each cases, a and b are taken as a real numbers.

Case 1. Assume that $1<a<b$. Let function f exist on $(1,b]$. Then, we notice that for any $j\in\{0,1,2,\ldots\}$, b^{p^j} is in $(1,b]$.

DEFINITION 2.4. The p-integral of a function $f(x)$ on $(1,b]$ is denoted as

$$\int_1^b f(x)d_px=\lim_{n\to\infty}\sum_{j=0}^{N}(b^{p^j}-b^{p^{j+1}})f(b^{p^j})=\sum_{j=0}^{\infty}(b^{p^j}-b^{p^{j+1}})f(b^{p^j}),$$

and

$$\int_a^b f(x)d_px:=\int_1^b f(x)d_px-\int_1^a f(x)d_px.$$

Case 2. Suppose that $0 < b < 1$. Let function f exist on $[b, 1)$. Then, we notice that for any $j \in \{0, 1, 2, \ldots\}$, b^{p^j} is in $[b, 1)$ and $b^{p^j} < b^{p^{j+1}}$.

Definition 2.5. The p-integral of a function $f(x)$ on $[b, 1)$ is denoted by

$$\int_b^1 f(x) d_p x = \lim_{n \to \infty} \sum_{j=0}^{N} (b^{p^{j+1}} - b^{p^j}) f(b^{p^j}) = \sum_{j=0}^{\infty} (b^{p^{j+1}} - b^{p^j}) f(b^{p^j}).$$

The p-integrals defined above are also given by

$$\int_1^b f(x) d_p x = I_{p^+} f(b), \qquad \int_b^1 f(x) d_p x = I_{p^-} f(b).$$

Case 3 Assume that $0 < a < b < 1$. Let f exist on $(0, b]$. Then, we noticed that for any $j \in \{0, 1, 2, \ldots\}$, $b^{p^{-j}}$ is in $(0, b]$ and $b^{p^{-j-1}} < b^{p^{-j}}$.

Definition 2.6. The p-integral of a function $f(x)$ on $(0, b](b < 1)$ is denoted by

$$\begin{aligned} I_p f(b) = \int_0^b f(x) d_p x &= \lim_{n \to \infty} \sum_{j=0}^{N} (b^{p^{-j}} - b^{p^{-j-1}}) f(b^{p^{-j-1}}) \\ &= \sum_{j=0}^{\infty} (b^{p^{-j}} - b^{p^{-j-1}}) f(b^{p^{-j-1}}), \end{aligned}$$

and

$$\int_a^b f(x) d_p x := \int_0^b f(x) d_p x - \int_0^a f(x) d_p x.$$

Remark 2.1. [**16**] If $p \in (0, 1)$, then for any $j \in \{0, \pm 1, \pm 2, \pm 3, \ldots\}$, we have $p^{p^j} \in (0, 1)$, $p^{p^j} < p^{p^{j+1}}$ and

$$\int_0^1 f(x) d_p x = \sum_{j=-\infty}^{\infty} \int_{p^{p^j}}^{p^{p^{j+1}}} f(x) d_p x = \sum_{j=-\infty}^{\infty} (p^{p^{j+1}} - p^{p^j}) f(p^{p^j}). \tag{2.1}$$

By using Definition 2.4, Definition 2.5 and Definition 2.6, we give a more general formula:

Corollary 2.2 (Cf. Corollary 4.12, [**16**]). *Suppose* $0 \leqslant a < 1 < b$. *The p-integral of a function* $f(x)$ *on* $[a, b]$ *is given by*

$$\int_a^b f(t) d_p t = \int_0^b f(t) d_p t - \int_0^a f(t) d_p t, \tag{2.2}$$

where

$$\int_0^x f(t)d_pt = \sum_{j=0}^{\infty}(x^{p^{-j}} - x^{p^{-j-1}})f(x^{p^{-j-1}}), \text{ if } 0 \leqslant x < 1, \tag{2.3}$$

and

$$\begin{aligned}\int_0^x f(t)d_pt &= \int_0^1 f(t)d_pt + \int_1^x f(t)d_pt \\ &= \sum_{j=-\infty}^{\infty} (p^{p^{j+1}} - p^{p^j})f(p^{p^j}) \\ &\quad + \sum_{j=0}^{\infty}(x^{p^{j+1}} - x^{p^j})f(x^{p^j}) \text{ if } x > 1.\end{aligned} \tag{2.4}$$

If the series in (2.3), (2.4) *individually converge at $x = a$ and $x = b$, then we call f p-integrable on* $[a, b]$.

Definition 2.7. The nth order p-integral of a function f can be denoted by

$$\left(I_p^0 f\right)(x) = f(x), \quad \left(I_p^n f\right)(x) = I_p\left(I_p^{n-1} f\right)(x), \ n \in \mathbb{N}.$$

Following lemmas are given to obtain fundamental theorem of p-calculus by authors in [**16**]. In each lemmas, it is assumed that $p \in (0, 1)$.

Lemma 2.1. *Suppose that x belongs to interval $(1, \infty)$. In this case, $D_p\ I_{p^+} f(x) = f(x)$, and in additionally suppose that function f has the property that continuity at $x = 1$, then $I_{p^+}\ D_p f(x) = f(x) - f(1)$.*

Lemma 2.2. *Suppose that x belongs to interval $(0, 1)$. In this case, $D_p\ I_{p^-} f(x) = -f(x)$, and in additionally suppose that function f has the property that continuity at $x = 1$, then $I_{p^-}\ D_p f(x) = f(1) - f(x)$.*

Lemma 2.3. *Suppose that x belongs to interval $(0, 1)$ and $I_p f(x)$ is given as $I_p f(x) = \int_0^x f(s)d_ps$. In this case, $D_p\ I_p f(x) = f(x)$, and in additionally suppose that function f has the property that continuity at $x = 0$, then $I_p\ D_p f(x) = f(x) - f(0)$.*

Theorem 2.2 (**Fundamental theorem of p-calculus**,[**16**]). *Assume that $p \in (0, 1)$. If the function $F(x)$ which is an antiderivative of $f(x)$ has the property that continuity at $x = 0$, $x = 1$, then for all a, b real numbers which satisfy the condition $0 \leqslant a < b \leqslant \infty$, we obtain*

$$\int_a^b f(x)d_px = F(b) - F(a).$$

Corollary 2.3. *Assume that $f(x)$ has the property that continuity at $x=0$, $x=1$, then we obtain*

$$\int_a^b D_p f(x) d_p x = f(b) - f(a).$$

Corollary 2.4. *Assume that $f(x)$ and $g(x)$ have the property that continuity at $x=0$, $x=1$, then we obtain*

$$\int_a^b f(x) D_p g(x) d_p x = f(b)g(b) - f(a)g(a) - \int_a^b g(x^p) D_p f(x) d_p x. \tag{2.5}$$

This formula is called p-integration by parts.

The p-integral has the same fundamental properties as the Riemann integral.

Theorem 2.3. *For any two p-integrable functions $f, g : J \to \mathbb{R}$, for $a, b, c \in J$ and $\alpha, \beta \in \mathbb{R}$. Then*

(1) $\int_a^a f(t) d_p t = 0;$

(2) $\int_a^b f(t) d_p t = -\int_b^a f(t) d_p t;$

(3) $\int_a^b f(t) d_p t = \int_a^c f(t) d_p t + \int_c^b f(t) d_p t;$

(4) $\int_a^b (\alpha f + \beta g)(t) d_p t = \alpha \int_a^b f(t) d_p t + \beta \int_a^b g(t) d_p t.$

Proof. By using definitions of definite p-integral, the proof is clear. □

In the following, for a given $t \in J$, we denote

$$[t]_p = \{t^{p^j} : j \in N_0\} \cup \{0, 1\},$$

and

$$[a, b]_p = [a]_p \cup [b]_p.$$

Because of different definitions of p-integral depending on interval, from now on, it is required that $0 < a < 1 < b$ for $a, b \in J$.

Proposition 2.1. *For any two p-integrable functions f, g on J, and $a, b \in J$ such that $0 < a < 1 < b$ suppose that f, g satisfying $|f(t)| \leqslant g(t)$, $t \in [a, b]_p$, then for $x, y \in [a, b]_p$, $x < 1 < y$, one can obtain*

(1)

$$\left| \int_1^y f(t) d_p t \right| \leqslant \int_1^y g(t) d_p t,$$

(2)

$$\left|\int_1^x f(t)d_pt\right| \leqslant -\int_1^x g(t)d_pt,$$

(3)

$$\left|\int_x^y f(t)d_pt\right| \leqslant \int_x^y g(t)d_pt.$$

Therefore, assuming for all $t \in [a,b]_p$, $0 \leqslant g(t)$, *then we obtain*

$$\int_1^b g(t)d_pt \geqslant 0 \text{ and} \int_a^b g(t)d_pt \geqslant 0.$$

PROOF. (1) Since $y > 1$, then $y^{p^{j+1}} < y^{p^j}$, $j \in N_0$, $y \in [a,b]_p$,

$$\left|\int_1^y f(t)d_pt\right| = \left|\sum_{j=0}^{\infty}(y^{p^j} - y^{p^{j+1}})f(y^{p^j})\right| \leqslant \sum_{j=0}^{\infty}(y^{p^j} - y^{p^{j+1}})\left|f(y^{p^j})\right|$$

$$\leqslant \sum_{j=0}^{\infty}(y^{p^j} - y^{p^{j+1}})g(y^{p^j}) = \int_1^y g(t)d_pt.$$

(2)

$$\left|\int_x^1 f(t)d_pt\right| = \left|\sum_{j=0}^{\infty}(x^{p^{j+1}} - x^{p^j})f(x^{p^j})\right| \leqslant \sum_{j=0}^{\infty}(x^{p^j} - x^{p^{j+1}})\left|f(x^{p^j})\right|$$

$$\leqslant -\sum_{j=0}^{\infty}(x^{p^j} - x^{p^{j+1}})g(x^{p^j}) = -\int_1^x g(t)d_pt.$$

(3) The proof is similar previous ones.

□

3. Main results

In this section, we introduce a new type variational calculus which is called p-variational calculus. For this purpose, we consider the following variational problem as expected:

(P)

$$\mathcal{L}[y] = \int_a^b L\big(t, y(t^p), D_p[y](t)\big)d_pt \longrightarrow \textit{extremize},$$

$$y \in \mathcal{Y}\big([a,b]_p, \mathbb{R}\big),\ y(a) = \alpha,\ y(b) = \beta,$$

where by extremize we mean minimize or maximize and $y \in \mathcal{Y}$, where

$$\mathcal{Y} := \{y : J \to \mathbb{R} \mid y \text{ and } D_p[y] \text{ are bounded on } [a,b]_p \text{ and continuous at 0 and 1}\}$$

equipped with the norm

$$||y|| = \sup_{t\in[a,b]_p} |y(t)| + \sup_{t\in[a,b]_p} |D_p[y](t)|.$$

DEFINITION 3.1. Assuming that $y \in \mathcal{Y}([a,b]_p, \mathbb{R})$ is merely required to the boundary condition $y(a) = \alpha,\ y(b) = \beta$. In this case, we call the function y an admissible function for Problem (P).

DEFINITION 3.2. Assuming that there exists $\delta > 0$ such that

$$\mathcal{L}[y_*] \leqslant \mathcal{L}[y] \quad (\text{resp. } \mathcal{L}[y_*] \geqslant \mathcal{L}[y])$$

for all admissible y with $||y_* - y|| < \delta$. In this case, we call the admissible function y_* a local minimizer (resp. local maximizer) for Problem (P).

DEFINITION 3.3. Assuming that $\eta(a) = 0 = \eta(b)$. In this case, we call the function η in $\mathcal{Y}([a,b]_p, \mathbb{R})$ an admissible variation for Problem (P).

3.1. Basic Lemmas. In order to get our main findings, we need the following lemma which is the p-variational form of the fundamental lemma of classical variational calculus [**5**].

LEMMA 3.1 (**Fundamental Lemma of p-variational Calculus**). *Given any function f suppose that belongs to $\mathcal{Y}([a,b]_p, \mathbb{R})$. Then, for all functions $h \in \mathcal{Y}([a,b]_p, \mathbb{R})$ which satisfies the boundary conditions $h(a) = h(b) = 0$, we have*

$$\int_a^b f(t)h(t^p)d_pt = 0$$

if and only if $f(t) = 0$ for all $t \in [a,b]_p$.

PROOF. The implication "$\Leftarrow$" is pretty obvious from definition of p-integral. It remains to prove the implication "$\Rightarrow$". To obtain a contradiction, assume that there exists a $q \in [a,b]_p$ ensures that $f(q) \neq 0$.

(1) Setting $q \neq 0$, we obtain $q = a^{p^{-j}}$ or $q = b^{p^{-j}}$ for some $j \in \mathbb{N}_0$.

 (a) Let $a \neq 0,\ b \neq 0$. There is no loss of generality in assuming that $q = b^{p^j}$. Set

$$h(t) = \begin{cases} f(a^{p^{-j-1}}), & \text{if } t = a^{p^{-j}}, \\ 0, & \text{otherwise.} \end{cases}$$

Then, we get

$$\begin{aligned}\int_a^b f(t)h(t^p)d_pt &= \int_0^b f(t)h(t^p)d_pt - \int_0^a f(t)h(t^p)d_pt \\ &= \int_0^1 f(t)h(t^p)d_pt + \int_1^b f(t)h(t^p)d_pt - \int_0^a f(t)h(t^p)d_pt \\ &= \sum_{j=-\infty}^{\infty} (p^{p^{j+1}} - p^{p^j})f(p^{p^j})h((p^{p^j})^p) \\ &\quad + \sum_{j=0}^{\infty}(b^{p^{-j}} - b^{p^{-j-1}})f(b^{p^{-j-1}})h((b^p)^{p^{-j-1}}) \\ &\quad - \sum_{j=0}^{\infty}(a^{p^{-j}} - a^{p^{-j-1}})f(a^{p^{-j-1}})h((a^p)^{p^{-j-1}}) \\ &= -\left(a^{p^{-j}} - a^{p^{-j-1}}\right)\left[f(a^{p^{-j-1}})\right]^2 \neq 0.\end{aligned}$$

This contradicts our assumption.

(b) Setting $a = 0$ and $b \neq 0$, we obtain $q = b^{p^{-j}}$ for some $j \in \mathbb{N}_0$. Let

$$h(t) = \begin{cases} f(b^{p^{-j-1}}), & t = b^{p^{-j}}, \\ 0, & \text{otherwise}, \end{cases}$$

and as in the proof (a), which is impossible.

(c) Setting $b = 0$ and $a \neq 0$, we obtain a result same as the previous case.

(2) Set $q = 0$. There is no loss of generality in assuming that $0 < f(q)$. Since

$$\lim_{j\to\infty} a^{p^{-j-1}} = \lim_{j\to\infty} b^{p^{-j-1}} = 1,$$

and f is continuous at 1, one has

$$\lim_{j\to\infty} f(a^{p^{-j-1}}) = \lim_{j\to\infty} f(b^{p^{-j-1}}) = f(1).$$

Thus, we obtain an order $j_0 \in \mathbb{N}$ satisfies that for all $j > j_0$

$$f(a^{p^{-j-1}}) > 1 \text{ and } f(b^{p^{-j-1}}) > 1.$$

(a) Setting $a, b \neq 0$, we define a function h for some $k > j_0$

$$h(t) = \begin{cases} f(b^{p^{-j-1}}), & \text{if } t = b^{p^{-j}}, \\ f(a^{p^{-j-1}}), & \text{if } t = a^{p^{-j}}, \\ 0, & \text{otherwise}. \end{cases}$$

Hence, we have

$$\int_a^b f(t)h(t^p)d_pt = (b^{p^{-j}} - b^{p^{-j-1}})f(b^{p^{-j-1}}) - (a^{p^{-j}} - a^{p^{-j-1}})f(a^{p^{-j-1}}) \neq 0.$$

(b) Setting $a = 0$, define

$$h(t) = \begin{cases} f(b^{p^{-j-1}}), & \text{if } t = b^{p^{-j}}, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, we have

$$\int_0^b f(t)h(t^p)d_pt = (b^{p^{-j}} - b^{p^{-j-1}}) \left[f(b^{p^{-j-1}})\right]^2 \neq 0.$$

(c) If $b = 0$, this follows by the same method as in case ((ii)-b).

□

Definition 3.4. For $s \in J$ and $g : J_p \times (-\theta, \theta) \to \mathbb{R}$, assuming that for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$0 < |\theta - \theta_0| < \delta \Longrightarrow \left| \frac{g(t,\theta) - g(t,\theta_0)}{\theta - \theta_0} - \partial_2 g(t,\theta_0) \right| < \epsilon$$

for all $t \in [s]_p$, where $\partial_2 g = \dfrac{\partial g}{\partial \theta}$. In this case, we call $g(t,\cdot)$ differentiable at θ_0 uniformly in $[s]_p$.

Lemma 3.2 (Cf. [**4**]). *Assume that* $s \in J$ *and* $g : J_p \times (-\theta, \theta) \to \mathbb{R}$ *is differentiable at* θ_0 *uniformly in* $[s]_p$*. Under the existence of* $\int_0^s g(t,\theta_0)d_pt$*, we obtain that* $G(\theta) := \int_0^s g(t,\theta)d_pt$ *for* θ *near* θ_0 *is differentiable at* θ_0 *and*

$$G'(\theta_0) = \int_0^s \partial_2 g(t,\theta_0)d_pt.$$

Proof.

i.) Let $s < 1$ be. Since $g(t,\cdot)$ is differentiable at θ_0 uniformly in $[s]_p$, then for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $t \in [s]_p$ and for $0 < |\theta - \theta_0| < \delta$, the following inequalities hold:

$$\left| \frac{g(t,\theta) - g(t,\theta_0)}{\theta - \theta_0} - \partial_2 g(t,\theta_0) \right| < \frac{\epsilon}{2s},$$

$$\left|\frac{G(\theta)-G(\theta_0)}{\theta-\theta_0}-G'(\theta_0)\right| \leqslant \int_0^s \left|\frac{g(t,\theta)-g(t,\theta_0)}{\theta-\theta_0}-\partial_2 g(t,\theta_0)\right| d_p t$$
$$< \int_0^s \frac{\epsilon}{2s} d_p t = \frac{\epsilon}{2} < \epsilon.$$

ii.) Let $s>1$ be. Since $g(t,\cdot)$ is differentiable at θ_0 uniformly in $[s]_p$, then for every $\epsilon>0$ there exists $\delta>0$ such that for all $t\in[s]_p$ and for $0<|\theta-\theta_0|<\delta$, the following inequalities hold:

$$\left|\frac{g(t,\theta)-g(t,\theta_0)}{\theta-\theta_0}-\partial_2 g(t,\theta_0)\right| < \frac{\epsilon}{2(s-1)},$$

$$\left|\frac{G(\theta)-G(\theta_0)}{\theta-\theta_0}-G'(\theta_0)\right| \leqslant \int_0^s \left|\frac{g(t,\theta)-g(t,\theta_0)}{\theta-\theta_0}-\partial_2 g(t,\theta_0)\right| d_p t$$
$$< \int_0^s \frac{\epsilon}{2(s-1)} d_p t = \frac{\epsilon}{2} < \epsilon.$$

Hence, $G(\cdot)$ is differentiable at θ_0 and $G'(\theta_0)=\int_0^s \partial_2 g(t,\theta_0) d_p t$. □

3.2. p-variational Problem. Given an admissible variation η, an admissible function y, we set the real function ϕ by

$$\phi(\varepsilon)=\phi(\varepsilon,y,\eta):=\mathcal{L}[y+\varepsilon\eta].$$

The first variation of the functional $\mathcal{L}$ of the problem (P) is defined by

$$\delta\mathcal{L}[y,\eta]:=\phi'(0).$$

We remark that

$$\mathcal{L}[y+\varepsilon\eta] = \int_a^b L\Big(t, y(t^p)+\varepsilon\eta(t^p), D_p[y](t)+\varepsilon D_p[\eta](t)\Big) d_p t$$
$$=\mathcal{L}_b[y+\varepsilon\eta]-\mathcal{L}_a[y+\varepsilon\eta],$$

where

$$\mathcal{L}_\zeta[y+\varepsilon\eta] = \int_0^\zeta L\Big(t, y(t^p)+\varepsilon\eta(t^p), D_p[y](t)+\varepsilon D_p[\eta]\Big)(t) d_p t$$

with $\zeta\in\{a,b\}$.

Thus, we have

$$\delta\mathcal{L}[y,\eta]=\delta\mathcal{L}_b[y,\eta]-\delta\mathcal{L}_a[y,\eta].$$

One can say that following lemma is a straight result of Lemma 3.2.

Lemma 3.3. *Given an admissible variation η, an admissible function y, define*

$$g(t,\varepsilon) := L\Big(t, y(t^p) + \varepsilon\eta(t^p), D_p[y](t) + \varepsilon D_p[\eta](t)\Big).$$

Suppose that

(1) $g(t,\cdot)$ *is differentiable at* 0 *uniformly in* $[a,b]_p$;

(2) $\mathcal{L}_a[y+\varepsilon\eta] = \int_0^a g(t,\varepsilon)d_pt$ *and* $\mathcal{L}_b[y+\varepsilon\eta] = \int_0^b g(t,\varepsilon)d_pt$ *exist for* $\varepsilon \approx 0$;

(3) $\int_0^a \partial_2 g(t,0)d_pt$ *and* $\int_0^b \partial_2 g(t,0)d_pt$ *exists.*

Then, we have

$$\begin{aligned}\phi'(0) =& \delta\mathcal{L}[y,\eta]\\ =& \int_a^b \Big(\partial_2 L\big(t, y(t^p), D_p[y](t)\big)\eta(t^q) + \partial_3 L\big(t, y(t^p), D_p[y](t)\big)D_p\eta(t)\Big)d_pt.\end{aligned}$$

3.3. Optimality conditions. In this section, we derive a necessary condition, called the p-Euler-Lagrange equation, and a sufficient condition to our Problem (P), respectively.

Theorem 3.1 (The p-Euler-Lagrange equation). *Assuming that Lagrangian L satisfy hypotheses (H1)-(H3) and conditions (i)-(iii) of Lemma 3.3. If $y_* \in \mathcal{Y}$ is a local extremizer for problem (P), then for all $t \in [a,b]_p$, y_* satisfies the p-Euler-Lagrange equation*

$$\partial_2 L\big(t, y(t^p), D_p[y](t)\big) = D_p\big[\partial_3 L(\cdot, y(\cdot^p), D_p[y](\cdot))\big](t). \tag{3.1}$$

Proof. Consider that y_* is a local minimizer (resp. maximizer) for problem (P) and η an admissible variation, we set $\phi : \mathbb{R} \to \mathbb{R}$ by

$$\phi(\varepsilon) := \mathcal{L}[y_* + \varepsilon\eta].$$

A necessary condition for y_* to be an extremizer is given by $\phi'(0) = 0$. By Lemma 3.3, it can be concluded that

$$\int_a^b \Big(\partial_2 L\big(t, y(t^p), D_p[y](t)\big)\eta(t^q) + \partial_3 L\big(t, y(t^p), D_p[y](t)\big)D_p\eta(t)\Big)d_pt = 0.$$

By integration by parts (2.5), it follows that

$$\begin{aligned}&\int_a^b \partial_3 L\big(t, y(t^p), D_p[y](t)\big)D_p\eta(t)d_pt\\ &= \partial_3 L\big(t, y(t^p), D_p[y](t)\big)\eta(t)\big|_a^b - \int_a^b D_p\partial_3 L\big(\cdot, y(\cdot^p), D_p[y](\cdot)\big)(t)\eta(t^p)d_pt.\end{aligned}$$

Hence $\eta(a) = \eta(b) = 0$, then we have

$$\int_a^b \Big(\partial_2 L\big(t, y(t^p), D_p[y](t)\big)\eta(t^p) - D_p\partial_3 L\big(\cdot, y(\cdot^p), D_p[y](\cdot)\big)(t)\eta(t^p)\Big)d_pt = 0.$$

Finally, by Lemma 3.1, it yields for all $t \in [a,b]_p$

$$\partial_2 L\Big(t, y(t^p), D_p[y](t)\Big) = D_p\partial_3 L\Big(\cdot, y(\cdot^p), D_p[y](\cdot)\Big)(t).$$

□

We conclude this section with a sufficient optimality condition for the Problem (P). Firstly, we need following definition.

Definition 3.5. Assuming that for a given function L has following properties: *i.)* $\partial_i L$, $i = 2, 3$, exist *ii.)* is continuous, *iii.)* satisfies the conditions

$$L(t, u + u_1, v + v_1) - L(t, u, v) \;\geqslant \text{(resp. } \leqslant\text{)}\; \partial_2 L(t, u, v)u_1 + \partial_3 L(t, u, v)v_1$$

for all $(t, u, v), (t, u + u_1, v + v_1) \in J \times \mathbb{R}^2$. In this case, we call $L(t, u, v)$ jointly convex (resp. concave) in (u, v).

Theorem 3.2. *For $a < b$ and $a, b \in [c]_p$ for some $c \in J$, suppose that L is a jointly convex(resp. concave) function in (u, v). If y_* satisfies the p-Euler-Lagrange equation* (3.1)*, then y_* is global minimizer (resp. maximizer) to the problem (P).*

Proof. Under the assumptions of Theorem 3.2 with "jointly convex in (u, v)", for any admissible variation η, one can obtain

$$\begin{aligned}
&\mathcal{L}[y_* + \eta] - \mathcal{L}[y_*] \\
&\quad= \int_a^b \left[L\Big(t, y_*(t^p) + \eta(t^p), D_p[y_*](t) + D_p[\eta](t)\Big) - L\Big(t, y_*(t^p), D_p[y_*](t)\Big)\right]d_pt \\
&\quad\geqslant \int_a^b \left[\partial_2 L\Big(t, y_*(t^p), D_p[y_*](t)\Big)\eta(t^p) + \partial_3 L\Big(t, y_*(t^p), D_p[y_*](t)\Big)D_p[\eta](t)\right]d_pt.
\end{aligned}$$

Using integration by part, formula (2.5), one obtains

$$\begin{aligned}\mathcal{L}[y_*+\eta]-\mathcal{L}[y_*] &\geqslant \int_a^b \Big[\partial_2 L\big(t, y_*(t^p), D_p[y_*](t)\big)\eta(t^p)\Big] d_pt \\ &\quad + \int_a^b \Big[\partial_3 L\big(t, y_*(t^p), D_p[y_*](t)\big) D_p[\eta](t)\Big] d_pt \\ &\geqslant \int_a^b \Big[\partial_2 L\big(t, y_*(t^p), D_p[y_*](t)\big)\eta(t^p)\Big] d_pt \\ &\quad + \partial_3 L\big(t, y_*(t^p), D_p[y_*](t)\big)\eta(t)\big|_a^b \\ &\quad - \int_a^b D_p\partial_3 L\big(\cdot, y(\cdot^p), D_p[y](\cdot)\big)(t)\eta(t^p) d_pt.\end{aligned}$$

Since y_* satisfies Theorem 3.1 and also η is an admissible variation, it is concluded that

$$\mathcal{L}[y_*+\eta]-\mathcal{L}[y_*] \geqslant 0.$$

Note that we have actually proved that y_* is a minimizer of Problem (P). The same conclusion can be drawn for the concave case. □

4. An example

We require that p is a fixed number different from 1. For $a < b$ in $[a, b]_p$, regard the following problem

$$\begin{cases}\mathcal{L}[y] = \int\limits_a^b \left(t + \frac{1}{2}\left(D_p[y](t)\right)^2\right) d_pt \to \text{ minimize} \\ y \in \mathcal{Y}([a, b]_p, \mathbb{R}) \\ y(a) = a, \\ y(b) = b.\end{cases}$$

Assume that y_* is a local minimizer of the problem. In this case y_* satisfies the p-Euler-Lagrange equation

$$D_p\left[D_p[y](p\cdot)\right](t) = 0 \text{ for all } t \in [a, b]_p.$$

It can be easily seen that the function $y_*(t) = t$ is a candidate solution of this problem. Hence that the Lagrangian function is jointly convex in (u, v), then by using Theorem 3.2, it follows immediately that the function y_* is a minimizer of the problem.

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