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AN EXTENSION OF GOTTSCHALK-HEDLUND THEOREM

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ABSTRACT. We extend the famous Gottschalk-Hedlund theorem [**9**] which guarantees continuous solutions to a certain cohomological equation over classical minimal dynamical systems to the cohomological equation for vector-valued cocycles over a minimal group action.

1. Introduction

Many problems in dynamics can be reduced to solving a certain functional equation in appropriate topology, the so-called *cohomological equation*. For the classical dynamical system given by iterating (forward and backward) some invertible map T on a set M the (untwisted) cohomological equation is

$$h \circ T - h = g \tag{1.1}$$

for a given $g\colon M \to \mathbb{R}$ and $h\colon M \to \mathbb{R}$ is unknown. However, the iteration of the map T can be seen as an action of the additive group of integers $(\mathbb{Z}, +)$ on M. Therefore, we look at the cohomological equation in a more general context, namely for group actions.

1.1. Cohomology in dynamics. Let $\rho\colon G \times M \to M$ be a group action of a group $(G, *)$ on a set M and let $(\Gamma, +)$ be an abelian group.

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Definition 1.1. *A 1-cocycle over ρ twisted with a representation $\alpha\colon G \to GL(n,\mathbb{R})$ and with values in Γ is a map*

$$\beta\colon G \times M \to \Gamma,$$

such that

$$\beta(gh,x) = \alpha(h)\beta(g,\rho(h,x)) + \beta(h,x),$$

for all $g,h \in G$ and all $x \in M$.

If α is the identity representation then β is called *untwisted* 1-cocycle. In this paper, we deal only with untwisted 1-cocycles. $\mathcal{Z}(\rho,\Gamma)$ denotes the group of all 1-cocycles over ρ with values in Γ under the pointwise addition. One can put some regularity on the action ρ as well as on the 1-cocycle β. For example, if ρ and β are continuous or differentiable r times, $r \in \mathbb{N} \cup \{\infty,\omega\}$ we denote $\mathcal{Z}(\rho,\Gamma)$ by $\mathcal{Z}^0(\rho,\Gamma)$ and by $\mathcal{Z}^r(\rho,\Gamma)$. In particular, in this paper, we are mostly interested in continuous and analytic categories, i.e. $\mathcal{C}^0$ and $\mathcal{C}^\omega$. Very often Γ is the additive group of real numbers $(\mathbb{R},+)$.

A very special and very important class of smooth 1-cocycles is that of 1-coboundaries.

Definition 1.2. *A 1-coboundary over ρ with values in Γ is a 1-cocycle β given by a function $f\colon M \to \Gamma$*

$$\beta(g,x) = f(\rho(g,x)) - \alpha(g)f(x).$$

$\mathcal{B}(\rho,\Gamma)$ denotes the subgroup of all 1-coboundaries over ρ with values in Γ. We will often drop the prefixes and just write cocycles and coboundaries instead of 1-cocycles and smooth 1-coboundaries. There is a natural notion of equivalence between two cocycles, they are equivalent if they differ by a coboundary.

Definition 1.3. *Two cocycles $\beta,\gamma\colon G \times M \to \Gamma$ are cohomologous if there exists a map $f\colon M \to \Gamma$ such that*

$$\beta(g,x) = f(\rho(g,x)) + \gamma(g,x) - \alpha(g)f(x),$$

for all $x \in M$ and all $g \in G$.

The map f is called the transfer map. Constant cocycles are the ones that do not depend on M. In these terms, we can say that a cocycle is a coboundary if and only if it is cohomologous to the zero cocycle.

Of course, all these notions come from cohomology theory. The set of (continuous) functions on M $\mathrm{C}^0(M,\mathbb{R})$ and the set of continuous vector fields on M $\mathrm{Vect}^0(M)$ under the action ρ become naturally $G-$modules, if M has appropriate manifold structure. Hence, we can define the Chevalley-Eilenberg complex $H^*(G,V)$, over the action ρ with values in V, where V is either $\mathrm{C}^0(M,\mathbb{R})$ or $\mathrm{Vect}^0(M)$. Moreover, in this case, $H^1(G,V)$, which is of principal importance to us, turns out to be naturally isomorphic to the quotient vector space of all (continuous) cocycles over ρ by the subspace of all (continuous) coboundaries over ρ.

Definition 1.4. *The first cohomology group over the action ρ with values in Γ is the quotient group*

$$\mathcal{H}^1(\rho, \Gamma) = \frac{\mathcal{Z}^1(\rho, \Gamma)}{\mathcal{B}^1(\rho, \Gamma)}.$$

As usual, $\mathcal{H}^1(\rho, \Gamma)$ measures the extent to which cocycles fail to be coboundaries. If the case group action ρ has some regularity, we give $\mathcal{H}^1(\rho, \Gamma)$ the quotient topology.

1.2. Applications of cohomology to dynamical systems. As we already mentioned above, many questions in dynamical systems such as rigidity properties of smooth actions, the existence of invariants, questions about extensions, and many more can be reduced to the cohomological considerations about the action itself. We mention only a few.

Cocycles over a group action allow us to build more complex group actions. One of the results in this direction is obtained by Furstenberg [**8**]. We extend the group action in the base to an extended group action on a fibered space. These are usually called skew-products and they can be defined in the following way

$$\tilde{\rho}\colon G \times M \times \Gamma \to M \times \Gamma$$
$$\tilde{\rho}(g, (x, \gamma)) = (\rho(g, x), \gamma + \beta(g, x)),$$

for all $g \in G$, $x \in M$ and $\gamma \in \Gamma$. This is indeed a group action if β is a cocycle over ρ with values in Γ. Moreover, note that if β is a coboundary generated by a map f, then $\tilde{\rho}$ is conjugated to the dynamical system $I(x, \gamma) = (x, \gamma + f(x))$ on $M \times \Gamma$. Therefore, the first cohomology classifies all possible extensions of the base dynamics.

Other applications of cohomology in dynamical systems include problems such as the existence of invariant volume forms and the stability of hyperbolic systems (see [**13**]).

In all the applications in dynamical systems, the main problem is to show whether a given real-valued cocycle is or is not a coboundary, whether it is or it is not cohomologous to another given real-valued cocycle and whether or not the transfer map has some (higher) regularity or not. That is the main reason why it is so important to investigate the solutions $h\colon M \to \mathbb{R}$ of the cohomological equation

$$h(\rho(g, x)) - h(x) = \beta(g, x), \tag{1.2}$$

for all $g \in G$ and all $x \in M$, where β is a given real-valued cocycle over the group action ρ. Usually, the representation α is the linear part of the action ρ, and this is the case in this thesis as well. In that case, there is a huge difference between hyperbolic and non-hyperbolic α (in particular, the untwisted case). For example, in the continuous category, if α is hyperbolic, then there is a unique continuous solution to the twisted cohomological equation, while in the non-hyperbolic case that is very rarely the case, and even if the (formal) solution exists, the regularity of the solution is a big issue (see [**18**], [**19**], [**17**] and [**16**]).

1.3. Obstructions to solving a cohomological equation. Usually, it is very difficult to determine whether the cohomological equation has a solution, and even when it does, whether the solution is in some particular regularity class or not. The reason for this is the set of *obstructions*. Even in the case of C^r actions where it is possible to solve the cohomological equation, the solution itself is usually not as regular as the action itself. In fact, it can be very discontinuous while the action itself is analytic. This phenomenon is called *loss of regularity* or *loss of derivatives*. Therefore, if possible, it would be a very important task to determine the set of all obstructions for a particular regularity class. For example, a very obvious set of obstructions for solving the cohomological equation (1.2) is the set of all Borel probability measures that are invariant under the action, if any. Since $\mathcal{H}^1(\rho,\Gamma)$ measures the extent to which cocycles fail to be coboundaries, it is natural that the obstructions to the cohomological equation parameterize the first cohomology.

2. Gottschlak-Hedlund theorem

It is very rare that one is able to completely characterize the set of obstructions. Classical examples where this is possible are Gottschalk and Hedlund for continuous systems and Livšic's theorem for hyperbolic systems. The first one deals with a classical continuous dynamical system on a compact metric space that is minimal, i.e. every orbit is dense in M. They proved the following theorem.

THEOREM 2.1. [**9**] *Let M be a compact metric space, and $T\colon M\to M$ a continuous dynamical system which is minimal, and let $g\colon M\to\mathbb{R}$ be a given continuous function. The cohomological equation*

$$h\circ T-h=g$$

has a continuous solution $h\colon M\to\mathbb{R}$ if and only if the sequence of functions

$$\left\{\sum_{i=0}^{n} g\circ T^i(x_0))\right\}_{n\geqslant 1}$$

is uniformly bounded in $\mathrm{C}^0(M,\mathbb{R})$ for some $x_0\in M$.

REMARK 2.1. Note that boundedness of $\left\{\sum_{i=0}^{n} g\circ T^i(x_0))\right\}_{n\geqslant 1}$ gives its boundedness at any other point $y\in M$. Otherwise, if $\left|\sum_{i=0}^{n} g\circ T^i(y))\right|>2K$, where $K=\sup_{n\geqslant 1}\left|\sum_{i=0}^{n} g\circ T^i(x_0)\right|$. By continuity, the same inequality holds for some $z\in M$ which is sufficiently close to y, and then, by minimality, for some iterate $T^{n_0}(x_0)$ of x_0. But then

$$2K<\left|\sum_{i=n_0}^{n_0+N} g\circ T^i(x_0)\right|\leqslant\left|\sum_{i=0}^{n_0+N} g\circ T^i(x_0)\right|+\left|\sum_{i=0}^{n_0-1} g\circ T^i(x_0)\right|$$

which contradicts the choice of K. One easily sees now that we can take

$$h(x) = \sup_{n \geqslant 1} -\sum_{i=0}^{n} g \circ T^i(x)$$

as a solution to the cohomological equation. The only thing left to prove Theorem 2.1 is the continuity of h. We will not do that here, since we prove an extension of Theorem 2.1. An interested reader can find proof of Theorem 2.1 in [**13**].

The second classical example where it is possible to characterize the complete set of obstructions is due Livšic [**20**] who showed that the only obstructions to the continuous solutions of the cohomological equation in the case of a C^2 $\mathbb{R}$ action, i.e. a C^2 flow, are given as the set of all probability measures supported on the periodic orbits. This is commonly known as the *periodic data* condition.

More recent advances in the direction of completely characterizing the set of obstructions are obtained by Forni [**10**], for the area-preserving flows on higher genus surfaces, by Forni and Flaminio [**11**] for actions on $SL(2,\mathbb{R})/\Gamma$, then by Marmi, Moussa, and Yoccoz [**22**] for the interval exchange maps and by Flaminio and Forni [**12**] for flows on nilmanifolds.

We extend Theorem 2.1 to the case of cocycles over minimal continuous group actions, with values in $\mathbb{R}^k$, $k \geqslant 1$.

THEOREM 2.2. *Let M be a compact metric space and $\rho\colon G \times M \to M$ a continuous group action. A cocycle $\beta \in \mathcal{Z}^0(\rho, \mathbb{R}^k)$ over ρ is a coboundary from $\mathcal{B}^0(\rho, \mathbb{R}^k)$, i.e. there is a continuous function $h\colon M \to \mathbb{R}^k$ such that, if and only if there is a constant $C > 0$ and a point $x_0 \in M$ such that*

$$||\beta(g, x_0)|| \leqslant C,$$

for every $g \in G$.

REMARK 2.2. Arguing in a similar way as in Remark 2.1, the previous theorem can be formulated as follows. A continuous cocycle β is a coboundary if and only if it is uniformly bounded in $\mathcal{Z}^0(\rho, \mathbb{R}^k)$.

PROOF. If $\beta \in \mathcal{Z}^0(\rho, \mathbb{R}^k)$ is a coboundary, then let $h\colon M \to \mathbb{R}^k$ be a continuous function such that

$$h(\rho(g, x)) - h(x) = \beta(g, x),$$

for all $g \in G$ and all $x \in M$. Obviously, β is bounded in $\mathcal{Z}^0(\rho, \mathbb{R}^k)$ since now $||\beta(g, x)|| \leqslant 2||h||_{\mathrm{C}^0}$ and M is compact. Here $||\cdot||$ is any norm on $\mathbb{R}^k$ Hence, one can take $C = 2||h||_{\mathrm{C}^0}$ and x_0 to be any point from M.

Let now $\beta \in \mathcal{Z}^0(\rho, \mathbb{R}^k)$ be such that there is a constant $C > 0$ and a point x_0 such that $||\beta(g, x_0)|| \leqslant C$, for every $g \in G$. We need to construct a continuous function $h\colon M \to \mathbb{R}^k$ such that $h(\rho(g, x)) - h(x) = \beta(g, x)$, for all $g \in G$ and all $x \in M$. We define a skew-shift action over ρ

$$\tilde{\rho}\colon G \times M \times \mathbb{R}^k \to M \times \mathbb{R}^k$$

in the following way

$$\tilde{\rho}(g,(x,y)) = (\rho(g,x), y + \beta(g,x)).$$

That $\tilde{\rho}$ is also a G-action follows easily from the fact that β is a cocycle, and hence it satisfies the equality from Definition 1.1. By assumption, the orbit of $(x_0, 0) \in M \times \mathbb{R}^k$ under the skew-shift action $\tilde{\rho}$ is bounded, i.e. the set

$$\mathcal{O}_{\tilde{\rho}}(x_0,0) = \{\tilde{\rho}(g,(x_0,0)) : g \in G\} = \{(\rho(x_0), \beta(g,x_0)) : g \in G\}$$

is bounded. Then $\overline{\mathcal{O}_{\tilde{\rho}}(x_0,0)}$ is a compact set invariant under the action of $\tilde{\rho}$. By Zorn's lemma, there is a minimal compact set $K \subset \overline{\mathcal{O}_{\tilde{\rho}}(x_0,0)}$ that is invariant under $\tilde{\rho}$. Minimality of the action ρ on the base M guarantees that the set

$$K_x = K \cap (\{x\} \times \mathbb{R}^k)$$

is not empty, for any $x \in M$. Now, if K_x is a singleton for every $x \in M$, one can easily conclude that is enough to define $h(x) = \pi_2(K_x)$, where $\pi_2 \colon M \times \mathbb{R}^k \to \mathbb{R}^k$ is a projection on the second component. Let us show that this is indeed the case, i.e. that $\{K_x : x \in M\}$ is a graph of a continuous function $h \colon M \to \mathbb{R}^k$. Now, we suppose that there are at least two points in some K_x, i.e. $(x, y_1), (x, y_2) \in K_x$. We also observe that $\tilde{\rho}$ commutes with vertical translations $T_z(x,y) = (x, y+z)$, for any $z \in \mathbb{R}^k$, since for any $g \in G$, any $x \in M$ and any $y, z \in \mathbb{R}^k$

$$\tilde{\rho}(g, T_z(x,y)) = (\rho(g,x), y + z + \beta(g,x)) = T_z((\rho(g,x), y + \beta(g,x))).$$

From here, we get that the set $T_{y_0}(K)$ is also minimal, compact and invariant for $\tilde{\rho}$, where $y_0 = (|y_1^1 - y_2^1|, \ldots, |y_1^k - y_2^k|)$. Since both minimal and non-empty, we conclude that

$$K = T_{y_0}(K).$$

Iterating the horizontal translation T_{y_0} on the set K, we conclude that the set K cannot be bounded which is a contradiction. Therefore, $\cup_{x \in M} K_x$ is a graph of a continuous function $H \colon M \to \mathbb{R}^k$ such that $h(\rho(g,x)) - h(x) = \beta(g,x)$ holds for all $g \in G$ and all $x \in M$. Hence, β is a continuous coboundary over ρ. □

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