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ALMOST θ-CONTRACTIONS AND RELATED FIXED POINT RESULTS IN PARTIAL METRIC SPACES WITH AN APPLICATION

Saadia Mahideb, Maroua Meneceur, Ahmed Ali, and Said Beloul

Abstract. The purpose of this paper is to introduce the notion of almost (α,θ)-contractions of Hardy-Rogers type in partial metric spaces and use it to present some fixed point results. Two examples are given to demonstrate the validity of our outcomes and as an application we give an existence theorems of solutions for a boundary value problem of fractional differential equations.

1. Introduction and preliminaries

The notion of partial metric spaces is one of various generalizations of ordinary metric spaces, its idea has been started by Matthews [**18**], after that several fixed point results were given in this way, see for example [**3, 6, 21, 23**].

Samet et al. [**20**] introduced a new concept called α-admissible mappings and they obtained some fixed point results using $\alpha-\psi$-contractive mappings, some results have were established by using such concept, for instance see [**5, 16, 17, 22**]. Recently, Jleli and Samet [**15**] introduced θ-contraction concept and proved the existence of fixed point. Remark that a contraction in the sense of Banach is a particular case of θ-contraction, while there are some θ-contractions that are not Banach contraction. After that, many authors studied different variations of θ-contraction, see for example [**1, 13, 14, 24**].

In this work, we combine the notion of α-admissible mappings with θ-contraction and Berinde type contraction concepts to introduce a new contractions type and related fixed point results in complete partial metric spaces. We also deduce the

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existence of fixed points in partially ordered metric spaces and in complete partial metric spaces endowed with a graph. Finally, we provide two examples and an application to the existence of the solutions for a boundary value problem of fractional differential equations to illustrate the importance of the obtained results.

Definition 1.1. [**10**] *Let (X, d) be a metric space. A map $T : X \to X$ is said to be a almost ((δ-L) weak) contraction if there exist $\delta \in [0, 1)$ and $L \geqslant 0$ such that*

$$d(Tx, Ty) \leqslant \delta d(x, y) + Ld(y, Tx),$$

for all $x, y \in X$.

Definition 1.2. [**4**] *A self mapping T on a metric space (X, d) is said satisfies the condition (B) if there exist $\delta \geqslant 0$ and $L \geqslant 0$ such that and for all $x, y \in X$ we have*

$$d(Tx, Ty) \leqslant \delta d(x, y) + L \min(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)).$$

Definition 1.3. [**12**] *A self mapping T on a metric space (X, d) is said a Ćiric type strong almost contraction if there exist $\delta \geqslant 0$ and $L \geqslant 0$ such that for all $x, y \in X$ we have*

$$d(Tx, Ty) \leqslant \delta M(x, y) + Ld(y, Tx),$$

where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}(d(x, Ty) + d(y, Tx))\}$.

Definition 1.4. [**18**] *Let $X \neq \phi$ and $p : X \times X \to [0, \infty)$ is said to be a partial metric on X if and only if it satisfies the following assumptions:*

(1) *$p(x, x) = p(y, x) = p(x, y)$ if and only if $x = y$.*
(2) *$p(x, x) \leqslant p(x, y)$.*
(3) *$p(x, y) = p(y, x)$.*
(4) *$p(x, z) \leqslant p(x, y) + p(y, z) - p(y, y)$.*

The space (X, p) is called a partial metric space.

Clearly that if $p(x, y) = 0$ then the two conditions (1) and (2) implies that $x = y$.
Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family open p-balls $B_p(x, \varepsilon) = \{y \in X, p(x, y) < p(x, x) + \varepsilon\}$, for all $x \in X$ and $\varepsilon > 0$.

Definition 1.5. [**18**]*Let (X, p) be a partial metric space.*

- *A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if $\lim_{n\to\infty} p(x_n, x_m$ exists and is finite.*
- *(X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p to a point $x \in X$ such that $\lim\limits_{n\to\infty} p(x, x_n) = p(x, x)$.*
  In this case, we say that the partial metric p is complete.

If p is a partial metric on X, then the functions $d_p, p^w : X \times X \to \mathbb{R}_+$ given by

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y),$$
$$p^w(x, y) = p(x, y) - \min\{p(x, x), p(y, y)\}$$

are ordinary metrics on X.

LEMMA 1.1. [**18**]*Let (X,p) be a partial metric space. Then we have:*

(1) *$\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,d_p),*
(2) *X is complete if and only if the metric space (X,d_p) is complete.*

DEFINITION 1.6. [**3**] *Let (X,p) be a partial metric space. A map $T: X \to X$ is called (δ,L)-weak contraction if there exist a $\delta \in [0,1)$ and $L \geqslant 0$ such that*

$$p(Tx,Ty) \leqslant \delta p(x,y) + Lp^w(y,Tx),$$

for all$x,y \in X$.

DEFINITION 1.7. [**3**] *Let (X,p) be a partial metric space. A map $T: X \to X$ is called (δ,L)-weak contraction if there exists a comparison function ϕ and $L \geqslant 0$ such that*

$$p(Tx,Ty) \leqslant \phi(p(x,y)) + Lp^w(y,Tx),$$

for all$x,y \in X$.

DEFINITION 1.8. [**15**] *Let Θ be the set of all functions $\theta : (0,+\infty) \to (1,+\infty)$ satisfying:*

(θ_1) : *θ is non decreasing,*
(θ_2) : *for each sequence $\{\varepsilon_n\}$ in $(0,+\infty)$, $\lim_{n\to\infty} \varepsilon_n = 1$ if and only if $\lim_{n\to\infty} \varepsilon_n = 0$,*
(θ_3) : *there exist $\rho \in (0,1)$ and $\varrho \in [0,\infty)$ such that $\lim_{t\to 0^+} \dfrac{\theta(t)-1}{t^\rho} = \varrho$.*

EXAMPLE 1.1. *The following functions are elements of Θ.*
1) $\theta_1(t) = e^t$. 2) $\theta_2(t) = e^{te^t}$. 3) $\theta_3(t) = e^{\sqrt{t}}$. 4) $\theta_4(t) = e^{\sqrt{t}e^t}$.

2. Main results

DEFINITION 2.1. *A self mapping T on a partial metric space (X,p) is an almost (α,θ)-contraction of Hardy-Rogers type, if there exists $k \in [0,1)$, $\theta \in \Theta$, and $\alpha : X \times X \to (0,+\infty)$ such that $p(Tx,Ty) > 0$ implies*

$$\alpha(x,y)\theta(p(Tx,Ty)) \leqslant \theta(M(x,y) + LN(x,y), \tag{2.1}$$

for all $x,y \in X$, where

$$M(x,y) = a_1p(x,y) + a_2p(x,Tx) + a_3p(y,Ty) + a_4p(x,Ty) + a_5p(y,Tx),$$

with $a_1 + a_2 + a_3 + 2a_4 + a_5 = 1, a_3 \neq 1, L \geqslant 0$ and
$N(x,y) = \min\{p^w(x,Ty)), p^w(y,Tx)\}$.

THEOREM 2.1. *Let (X,p) be a complete partial metric space and $T: X \to X$ be an almost (α,θ)-contraction of Hardy-Rogers type satisfies the following conditions:*

(i) *there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geqslant 1$,*
(ii) *T is α-admissible,*
(iii) *X is α-regular, that is, for every sequence $\{x_n\}$ in X such that $x_n \to x$ and $\alpha(x_n,x_{n+1}) \geqslant 1$, then $\alpha(x_n,x) \geqslant 1$.*

Then, T has a fixed point $x^* \in X$.

PROOF. From (i) there is x_0 such $\alpha(x_0, x_1) \geqslant 1$, where $x_1 = Tx_0$. If $x_0 = x_1$ then x_1 is a fixed point, suppose the contrary so $p(Tx_0, Tx_1) = p(x_1, Tx_1) > 0$ then by using (2.1) we get

$$\theta(p(Tx_0, Tx_1)) \leqslant [\theta(M(x_0, x_1))]^k + p^w(N(x_0, x_1)),$$

$$\begin{aligned} M(x_0, x_1) &= a_1 p(x_0, x_1) + a_2 p(x_0, Tx_0) + a_3 p(x_1, Tx_1) + a_4 p(x_0, Tx_1) + a_5 p(x_1, Tx_0) \\ &\leqslant (a_1 + a_2 + a_4) p(x_0, x_1) + (a_3 + a_4) p(x_1, x_2) + (a_5 - a_4) p(x_1, x_1) \\ &\leqslant (a_1 + a_2 + a_4) p(x_0, x_1) + (a_3 + a_4 + a_5) p(x_1, x_2) \end{aligned}$$

and

$$N(x_0, x_1) = \min\{p^w(x_0, x_1), p^w(x_0, x_2)\} = 0.$$

Then we have

$$\theta(p(x_1, x_2)) \leqslant [\theta(M(x_0, x_1))]^k < \theta(M(x_0, x_1)),$$

since θ is non decreasing function we get

$$p(x_1, x_2) < M(x_0, x_1) \leqslant (a_1 + a_2 + a_4) p(x_0, x_1) + (a_3 + a_4 + a_5) p(x_1, x_2),$$

which implies that

$$p(x_1, x_2) < \frac{(a_1 + a_2 + a_4)}{1 - a_3 - a_4 - a_5} p(x_0, x_1) = p(x_0, x_1).$$

Hence

$$\theta(p(x_1, x_2) = \theta(p(Tx_0, Tx_1) < \Big[\theta(p(x_0, x_1))\Big]^{k^n}.$$

Since T is α-admissible we have $\alpha(x_1, x_2) \geqslant 1$. If $x_1 \neq x_2$ we get $p(Tx_1, Tx_2) > 0$, then by using (2.1) we get

$$\theta(p(Tx_1, Tx_2)) \leqslant [\theta(M(x_1, x_2))]^k + p^w(N(x_1, x_2))$$

as in the first step, we obtain

$$\theta(p(x_2, x_3) = \theta(p(Tx_0, Tx_1) < \Big[\theta(p(x_1, x_2))\Big]^k < \Big[\theta(p(x_0, x_1))\Big]^{k^2}.$$

Continuing in this manner, we construct a sequence (x_n) defined as $x_{n+1} = Tx_n$ verifies $\alpha(x_n, x_{n+1}) \geqslant 1$ and $p(x_n, x_{n+1}) > 0$, so we have

$$1 < \theta(p(x_n, x_{n+1}) = \theta(p(Tx_{n-1}, Tx_n) < \Big[\theta(p(x_0, x_1))\Big]^{k^n}.$$

Letting $n \to \infty$, we get

$$\lim_{n\to\infty} \theta(p(x_n, x_{n+1}) = 1,$$

which implies that

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0.$$

We prove $\{x_n\}$ is a Cauchy sequence, from (θ_3) there exists $\rho \in (0, 1)$ and $\varrho \in [0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(p(x_n, x_{n+1})) - 1}{(p(x_n, x_{n+1}))^\rho} = \varrho.$$

If $\varrho < \infty$, let $2\varepsilon = \varrho$, so from the definition of limit there exists $n_0 \in \mathbb{N}$ such that $n_0 \geqslant n$ and for all $n \geqslant n_0$, we have

$$\varepsilon = \varrho - \varepsilon \leqslant \frac{\theta(p(x_n, x_{n+1}) - 1}{(p(x_n, x_{n+1}))^\rho} = \varrho,$$

which gives

$$p(x_n, x_{n+1})^\rho \leqslant \frac{\theta(p(x_n, x_{n+1})) - 1}{\varepsilon},$$

which implies

$$n\big(p(x_n, x_{n+1})\big)^\rho \leqslant \frac{n[(\theta(p(x_0, x_1)))^{k^n} - 1]}{\varepsilon}, \tag{2.2}$$

If $\varrho = \infty$, let A be an arbitrary positive real number, so from the definition of the limit there exists $n_1 \in \mathbb{N}$ such that for all $n \geqslant n_1$ we have

$$\frac{\theta(p(x_n, x_{n+1})) - 1}{(p(x_n, x_{n+1}))^\rho} > A,$$

which implies that

$$n(p(x_n, x_{n+1}))^\rho \leqslant \frac{n(\theta(p(x_0, x_1))^{k^n} - 1)}{A}. \tag{2.3}$$

Letting $n \to \infty$ in (2.2)(resp in (2.3)), we obtain

$$\lim_{n\to\infty} n(p(x_n, x_{n+1}))^\rho = 0.$$

From the definition of the limit, there exists $n_2 \geqslant \max\{n_0, n_1\}$ such that for all $n \geqslant n_2$, we have

$$p(x_n, x_{n+1}) \leqslant \frac{1}{n^{\frac{1}{\rho}}}.$$

Then the series $\displaystyle\sum_{n=1}^{\infty} p(x_n, x_{n+1})$ is convergent, so its rest tends to 0, which implies that for all $n \geqslant m \geqslant n_0$, we have

$$p(x_n, x_m) \leqslant \sum_{i=n}^{m-1} p(x_i, x_{i+1}) \leqslant \sum_{i=n}^{\infty} p(x_i, x_{i+1}) \to 0.$$

Then $\{x_n\}$ is a Cauchy sequence.
Since (X, p) is complete, so $\{x_n\}$ converges to some $x \in X$ and we have

$$\lim_{n\to\infty} p(x_{n+1}, x) = \lim_{n,m\to\infty} p(x_n, x_m) = p(x, x) = 0.$$

By (H_3), we have $\alpha(x_n, x) \geqslant 1$, then using (2.1) we get

$$1 < \theta(p(x_{n+1}, Tx)) \leqslant \Big[\theta(M(x_n, x))\Big]^k < \theta(p(x_n, x)),$$

which implies that

$$\lim_{n\to\infty} \theta(p(x_{n+1}, Tx)) = 1,$$

applying (θ_2) we obtain

$$\lim_{n\to\infty} p(x_{n+1}, Tx)) = p(x, Tx) = 0.$$

Then $p(x,x) = p(x,Tx) = 0$, so the first property on partial metric gives $x = Tx$. $\square$

If $\alpha(x,y) = 1$, for all $x, y \in X$ we get the following corollary.

COROLLARY 2.1. *Let (X, p) be a partial metric space and let $T : X \to X$ be a self mapping such that for all $x, y \in X$, we have*

$$\theta(p(Tx, Ty)) \leqslant \left[\theta(M(x,y)\right]^k + LN(x,y),$$

where $\theta \in \Theta$ and $k \in (0,1)$. Then T has a fixed point in X.

COROLLARY 2.2. *Let $(X, p, \preceq)$ be a complete partially ordered partial metric space and let $T : X \to X$ be an increasing self mapping such that for all $x, y \in X$ with $x \preceq y$, we have*

$$\theta(p(Tx, Ty)) \leqslant \left[\theta(M(x,y)\right]^k + L\min\{p(x,Ty), p(y,Tx)\},$$

where $\theta \in \Theta$ and $k \in (0,1)$.
If the following assertions hold:

(1) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$.*
(2) *For every nondecreasing sequence $\{x_n\}$ in X with $x_n \to x$, we have $x_n \preceq x$.*

Then T has a fixed point in X.

PROOF. Define a function $\alpha : X \times X \to \mathbb{R}_+$ by

$$\begin{cases} 1, & \text{if } x \preceq y, \\ 0, & \text{otherwise.} \end{cases}$$

Now, we check the conditions of Theorem 2.1, in fact, the existence of $x_0 \in X$ with $x_0 \preceq Tx_0$ implies that $\alpha(x_0, Tx_0) = 1$, also the monotonicity of T implies that it is admissible. Consequently, all the conditions of Theorem 2.1 are satisfied, then T has a fixed point. $\square$

Now, we introduce the notion of almost (α,θ)-Suzuki contraction of Hardy-Rogers type in a partial metric space.

DEFINITION 2.2. *Let (X, p) be a partial metric space and $T : X \to X$ be a self mapping. T is said to be an almost (α,θ)-Suzuki contraction of Hardy-Rogers type, if there exist $\theta \in \Theta$ and $L \geqslant 0$ such that*

$$\frac{1}{2} p(x, Tx) \leqslant p(Tx, Ty) \textit{implies}$$

$$\theta(p(Tx, Ty)) \leqslant \left[\theta(M(x,y)\right]^k + LN(x,y), \tag{2.4}$$

for all $x, y \in X$.

THEOREM 2.2. *Let $X, p)$ be a complete partial metric space and $T : X \to X$ be an almost (α, θ)-Suzuki contraction of Hardy-Rogers type such that:*

(i) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$,*
(ii) *T is α-admissible,*
(iii) *X is α-regular, that's, for every sequence $\{x_n\}$ such that $x_n \to x$ and $\alpha(x_n, x_{n+1}) \geqslant 1$, then $\alpha(x_n, x) \geqslant 1$.*

Then, T has a fixed point in X.

PROOF. From the hypothesis(1), there exits x_0 such that $\alpha(x_0, x_1) \geqslant 1$. If $x_0 = x_1$, then x_0 is a fixed point and the proof is finished. Suppose $x_0 \neq x_1$, then $p(x_0, x_1) > 0$ and

$$\frac{1}{2}p(x_0, Tx_0) = \frac{1}{2}p(x_0, x_1) < p(x_0, x_1,$$

then by using (2.4) we get

$$\theta(p(x_1, x_2)) = \theta(p(Tx_0, Tx_1)) \leqslant \Big[\theta(M(x_0, x_1))\Big]^k + LN(p(x_0, x_1)$$

as in the proof of Theorem 2.1, we get

$$p(x_1, x_2) < p(x_0, x_1).$$

Continuing in this manner we construct a sequence (x_n) such that

$$x_{n+1} = Tx_n \ \ ; \ and \ \ \ \alpha(x_n, x_{n+1}) \geqslant 1.$$

If there exists n_0 such that $x_{n_0}) = x_{n_0+1}$, then x_{n_0} is a fixed point. Suppose $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$, since T is α-admissible we have $\alpha(x_n, x_{n+1}) \geqslant 1$ and $\frac{1}{2}p(x_n, Tx_n) = \frac{1}{2}p(x_n, x_{n+1}) < p(x_n, x_{n+1})$, then by using (2.4) we get

$$\theta(p(x_n, x_{n+1})) = \theta(p(Tx_{n-1}, Tx_n)) \leqslant \Big[\theta(M(x_{n-1}, x_n))\Big]^k + LN(x_{n-1}, x_n).$$

The rest of the proof is like in the proof of Theorem 2.1. $\square$

COROLLARY 2.3. *Let (X, p) be a complete partial metric space endowed with a graph G, that is $G = (V(G), E(G))$, where $V(G)$ is its vertices and $E(G)$ its edges, moreover suppose the G ha no parallels edges and $T : X \to X$ be a self mapping. Assume that the following assertions hold:*

(*i*) *For each $x, y \in X$ such that $(x, y) \in E(G)$ we have $(Tx, Ty) \in E(G)$.*
(*ii*) *There exist $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$;*
(*iii*) *There exist $\theta \in \Theta$, $L \geqslant 0$ and $k \in [0, 1)$ such that $\frac{1}{2}p(x, Tx) \leqslant p(x, y)$ implies*

$$\theta(p(Tx, Ty)) \leqslant [\theta(M(x, y)]^k + LN(x, y),$$

for all $x, y \in X$ with $(x, y) \in E(G)$ and $p(Tx, Ty) > 0$. Then T has a fixed point.

PROOF. Define $\alpha : X \times X \to \mathbb{R}_+$ by

$$\alpha\colon X \times X \to [0, +\infty), \quad \alpha\,(x, y) = \begin{cases} 1, & \text{if } x \preceq y, \\ 0, & \text{otherwise.} \end{cases}$$

From (iii), we have $(x,y) \in E(G)$ so $\alpha(x,y) = 1 \geqslant 1$, which implies T is an almost (α,θ)-Suzuki contraction.
Also from (i) for $x \in X$ and $y \in Tx$ such that $(x,y) \in E(G)$, i.e., $\alpha(x,y) = 1 \geqslant 1$ we have $(Tx,Ty) \in E(G)$, then T is α-admissible. From (ii) there exists $x_0 \in X$ such that $(x_0,Tx_0) \in E(G)$, which implies $\alpha(x_0,Tx_0) \geqslant 1$.
Then all hypotheses of Theorem 2.2 are satisfied, then T has a fixed point. □

EXAMPLE 2.1. *Let $X = \{0,1,2,3\}$ and $p(x,y) = \max\{x,y\}$. Define $T\colon X \to X$ and $\alpha : X \times X \to [0,+\infty)$ by*

$$Tx = \begin{cases} 0, & x \in \{0,1,2\} \\ 1, & x = 3 \end{cases}$$

and

$$\alpha(x,y) = \begin{cases} 1, & x,y \in \{0,1,2\} \\ 0, & \textit{otherwise} \end{cases}$$

Taking $\theta(t) = e^t$, $a_1 = \frac{1}{5}$, $a_2 = a_3 = a_4 = 0$, $a_5 = \frac{4}{5}$, $L = 0$ and $k = \frac{1}{2}$.
Let x,y in X such that $\alpha(x,y) \geqslant 1$, so $x,y \in \{0,1,2\}$ and for this case we have:

$$\alpha(Tx,Ty) = \alpha(0,0) = 1 \geqslant 1.$$

Then T is α-admissible.
Since p is symmetric, so we have the following cases:

(1) *For $x = y = 0$ we have*

$$e^{p(T0,T0)} = 1 \leqslant e^{\frac{1}{2}p(0,0)} = 1$$

(2) *For $x \in \{0,1\}$ and $y = 1$ we have*

$$e^{p(T0,T1)} = e^{p(T1,T1)} = 1 \leqslant e^{\frac{1}{2}}$$

(3) *For $x \in \{0,1,2\}$ and $y = 2$ we have*

$$e^{p(T0,T2)} = e^{p(T1,T2)} = e^{p(T2,T2)} = 1 \leqslant e$$

If $\{x_n\}$ a sequence in X converges to x with $\alpha(x_n,x_{n+1}) \geqslant 1$, so $x_n \in \{0,1,2\}$, for all $n \in \mathbb{N}$, thus $x \in \{0;1,2\}$ which implies $\alpha(x_n,x) \geqslant 1$. Then all hypotheses of Theorem 2.1 hold, so T has a fixed point. Here T has a fixed point 0.

EXAMPLE 2.2. *Let $X = [0,\infty)$ and $p(x,y) = \max\{x,y\}$. Define $T\colon X \to X$ and $\alpha : X \times X \to [0,\infty)$ by*

$$Tx = \begin{cases} \frac{x}{8}, & x \in [0,1] \\ \frac{2x+1}{3}, & x > 1 \end{cases}$$

and

$$\alpha(x,y) = \begin{cases} 1, & x,y \in [0,1] \\ \frac{1}{5}, & \textit{otherwise} \end{cases}$$

Taking $\theta(t) = e^t$, $L = 0$, $a_1 = \frac{3}{4}, a_2 = \frac{1}{4}$, $a_3 = a_4 = a_5 = 0$ and $k = \frac{1}{4}$.
Let x,y in X such that $\alpha(x,y) \geqslant 1$, so $x,y \in [0,4]$ and for this case we have

$Tx, Ty \in [0, \frac{1}{8}]$, which implies that $\alpha(Tx, Ty) = 1 \geqslant 1$. Then T is α-admissible. For $x, y \in [0, 1]$ with $x \geqslant y$, we have

$$e^{p(Tx,Ty)} = e^{\frac{x}{8}} \leqslant e^{\frac{x)}{4}}.$$

Let $\{x_n\}$ be a sequence in X converges to x with $\alpha(x_n, x_{n+1}) \geqslant 1$, so $x_n \in [0, 1]$, for all $n \in \mathbb{N}$, then $x \in [0, 1]$ which implies $\alpha(x_n, x) \geqslant 1$. Then all hypotheses of Theorem 2.1 are satisfied, so T has a fixed point which is 0.

3. Application to fractional differential equations

Consider the following boundary value problem:

$$\left\{ \begin{array}{l} D_1^q x(t) = f(t, x(t)),\ t \in J = [1, e], \\ x(1) - x'(e) = 0, \\ x(e) = \int_1^e g(s, x(s))ds, \end{array} \right. \tag{3.1}$$

where D_1^q with $1 < q \leqslant 2$ is the Caputo-Hadamard fractional derivative, $\lambda > 0$ and $f : J \times \mathbb{R} \to \mathbb{R}$.

Let $X = C(J, \mathbb{R})$ be the space of all continuous functions on J, we consider on X the partial metric defined by:

$$p(x, y) = \|x - y\|_\infty + \|x\|_\infty + \|y\|_\infty, \text{for all} x, y \in X \quad \text{and} \quad t \in J\}.$$

Since $d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y) = \|x - y\|_\infty$ is an ordinary metric and $(X, \|.\|_\infty)$ is complete, then from Lemma 1.1 (X, p) is complete.

Lemma 3.1. *A function x is a solution of the problem* (3.1) *if and only if, x is a solution of the following integral equation:*

$$x(t) = \frac{1}{\Gamma(q)} \int_1^t G(t, s) f(s, x(s))ds + \frac{1 + \log t}{2} \int_1^e g(s, x(s))ds,$$

for all $t \in J$, where

$$G(t, s) = \frac{1}{s\Gamma(q)} \left\{ \begin{array}{ll} (\log \frac{t}{s})^{q-1} - \frac{(1+\log t)(1-\log s)^{q-1}}{2}, & 1 \leqslant s \leqslant t \leqslant e \\ -\frac{(1+\log t)(1-\log s)^{q-1}}{2} & 1 \leqslant t \leqslant s \leqslant e. \end{array} \right. \tag{3.2}$$

Proof. We have

$$I^q(D_1^q x(t)) = x(t) - c_0 - c_1 \log t = \frac{1}{\Gamma(q)} \int_0^t (\log \frac{t}{s})^{q-1} \frac{f(s, x(s))}{s} ds,$$

using the boundary values we get

$$x(1) - x(e) = c_0 - c_1 = 0$$

$$x(e) = 2c_1 + \frac{1}{\Gamma(q)} \int_0^1 (1 - s)^{q-1} f(s, x(s)ds = \int_0^1 g(s, x(s))ds,$$

which implies that

$$c_1 = -\frac{1}{2\Gamma(q)} \int_1^e ((1 - \log s)^{q-1} \frac{f(s, x(s))}{s} ds + \frac{1}{2} \int_1^e g(s, x(s))ds.$$

Then we have

$$x(t) = \frac{1}{\Gamma(q)}\int_1^t (\log\frac{t}{s})^{q-1}\frac{f(s,x(s))}{s}ds - \frac{1+\log t}{2}\Big[\frac{1}{\Gamma(q)}\int_1^e (1-\log s)^{q-1}\frac{f(s,x(s))}{s}ds - \int_1^e g(s,x(s))ds\Big],$$

which implies that

$$x(t) = \int_0^1 G(t,s)f(s,x(s))ds + \frac{1+\log t}{2}\int_1^e g(s,x(s))ds,$$

where

$$G(t,s) = \frac{1}{s\Gamma(q)}\begin{cases} (\log\frac{t}{s})^{q-1} - \frac{(1+\log t)(1-\log s)^{q-1}}{2}, & 1\leqslant s\leqslant t\leqslant e\\ -\frac{(1+\log t)(1-\log s)^{q-1}}{2} & 1\leqslant t\leqslant s\leqslant e. \end{cases}$$

□

$$\begin{aligned}\int_1^e G(t,s)ds &= \frac{1}{\Gamma(q)}[\int_1^e (\log\frac{t}{s})^{q-1} - \log t(1-\log s)^{q-1}ds - \int_t^e t(1-\log s)^{q-1}ds\\ &= \frac{1}{\Gamma(q)}[(\log t)^q + 1] \leqslant \frac{2}{\Gamma(q)}.\end{aligned}$$

Assume that the following assumptions hold:

(A_1) : f and g are continuous.

(A_2) : There exist two functions $\varphi_1, \varphi_2 : [1,e] \to \mathbb{R}_+$ such that for all $x_1, x_2 \in \mathbb{R}$, we have

$$|f(t,x_1(t)) - f(t,x_2(t))| \leqslant \varphi_1(t)_1|x_1 - x_2|)$$
$$\text{and}|f(t,x)| \leqslant \varphi_2(t)|x|.$$

(A_3) : There exist two functions $\psi_1, \psi_2 : [1,e] \to \mathbb{R}_+$ such that for all $x_1, x_2 \in \mathbb{R}$, we have

$$|g(t,x_1(t)) - g(t,x_2(t)| \leqslant \psi_1(t)|x_1 - x_2|)$$
$$\text{and} \quad |g(t,x)| \leqslant \psi_2(t)|x|.$$

where $\beta = G_0(\varphi_1^* + \varphi_2^*)\psi_1^* + \psi_2^* < \frac{1}{2}$, such that $G_0 = \sup\int_1^e G(t,s)ds$, $\varphi_i^* = \sup\limits_{1\leqslant t\leqslant e}|\varphi(t)|$ and $\psi_i^* = \sup\limits_{1\leqslant t\leqslant e}|\psi(t)|$.

Theorem 3.1. *Under the assumptions $(A_1) - (A_3)$, the problem (3.1) has a solution in X.*

Proof. For $x, y \in X$ and $t \in J$ we have

$$\begin{aligned}|Tx(t) - Ty(t)| &\leqslant \int_1^e G(t,s)|f(s,x(s)) - f(s,y(s))|ds\\ &+\lambda\log t\int_1^e |g(s,x(s)) - g(s,y(s)|ds\\ &\leqslant (G_0\varphi_1^* + \psi_1^*)|x-y|.\end{aligned}$$

This yields

$$\|Tx(t) - Ty(t)\|_\infty \leqslant G_0\varphi_1^* + \psi_1^*\|x-y\|_\infty.$$

On the other hand we have

$$\begin{aligned}|Tx(t)| &\leqslant \int_1^e G(t,s)|f(s,x(s))|ds + \int_1^e |g(s,x(s))|ds \\ &\leqslant (G_0\varphi_2^* + \psi_2^*)|x|,\end{aligned}$$

which implies that

$$\|Tx\|_\infty \leqslant (G_0\varphi_2^* + \psi_2^*)\|x\|_\infty.$$

Similarly, we find:

$$\|Ty(t)\|_\infty \leqslant (G_0\varphi_2^* + \psi_2^*)\|y\|_\infty.$$

Consequently we obtain

$$\begin{aligned}p(Tx,Ty) &\leqslant (G_0\varphi_1^* + \psi_1^*)\|x-y\|_\infty + (G_0\varphi_2^* + \psi_2^*)(\|x\|_\infty + \|y\|_\infty) \\ &\leqslant \Big(G_0\varphi_1^* + \psi_1^*) + G_0\varphi_2^* + \psi_2^*\Big)p(x,y) \leqslant \beta M(x,y).\end{aligned}$$

Hence we have

$$e^{\sqrt{p(Tx,Ty)}} \leqslant \big(e^{\sqrt{2\beta M(x,y)}}\big)^{\frac{\sqrt{2}}{2}}.$$

Then all the hypotheses of Corollary 2.1 are satisfied with $\theta = e^t$, $a_1 = 2\beta$, $k = \frac{\sqrt{2}}{2}$, $L = 0$ and $\theta(t) = e^{\sqrt{t}}$. So T has a fixed point which is a solution of the problem (3.1). □

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Saadia Mahideb, Department of Mathematics, Higher Normal School of Constantine, Algeria.
Email address: saadiamahideb@gmail.com

Maroua Meneceur, Laboratory of Operators Theory and PDE "LABTHOP",, Department of Mathematics, Exact Sciences Faculty,, University of El Oued, P.O.Box789, El Oued39000, Algeria
Email address: chaimanour9@gmail.com

Ahmed Ali, Laboratory of Analysis and Control of Differential Equations "ACED",, Faculty of Mathematics and Computer Sciences and Material Sciences, University of 8 may 1945, Guelma, Algeria.
Email address: alipromath@gmail.com

Said Beloul, Laboratory of Operators Theory and PDE "LABTHOP",, Department of Mathematics, Exact Sciences Faculty,, University of El Oued, P.O.Box789, El Oued39000, Algeria.
Email address: beloulsaid@gmail.com