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THE QUATERNION-TYPE CYCLIC-PELL SEQUENCES IN FINITE GROUPS

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Abstract. In this study, we give three different quaternion-type cyclic-Pell sequences and present some properties, such as, the Cassini formula, generating function. Then, we study quaternion-type cyclic-Pell sequences modulo m. Also we present the relationships between the lengths of periods of the quaternion-type cyclic-Pell sequences of the first, second and third kind modulo m and the generating matrices of these sequences. Finally, we introduce the quaternion-type cyclic-Pell sequences in finite groups. We calculate the lengths of periods for these sequences of the generalized quaternion groups and obtain the 1st quaternion-type cyclic-Pell orbit of the quaternion group Q_8 as applications of the results.

1. Introduction

In [**10**], by Sir William Rowan Hamilton defined the quaternions. Quaternions consist of a noncommutative, associative algebra over $\mathbb{R}$

$$\mathbb{H} = \{a_1 + a_2 i + a_3 j + a_4 k \mid a_1, a_2, a_3, a_4 \in \mathbb{R}\}$$

where $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$ are familiar with Hamilton's rules (see [**10, 18**]).

It is well known that the Pell sequence $\{P_n\}$ is defined by the following homogeneous linear recurrence relation:

$$P_n = 2P_{n-1} + P_{n-2}$$

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for $n \geqslant 2$, where $P_0 = 0$ and $P_1 = 1$. In [**16**], it can be obtained miscellaneous properties involving Pell numbers. The initial work began with Fibonacci sequences in algebraic structures that Wall [**20**] investigated in cyclic groups. Number theoretic properties such as these get from homogeneous linear reccurence relations relevant to this subject have been researched recently by many authors; see for example, [**1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 17, 19, 21**]. The authors investigated the properties of the Pell and Pell-Lucas quaternions in [**3**]. Then, Deveci and Shannon [**5**] extended the theory to the quaternions of the Pell sequence.

If a sequence consists only of repetitions of a fixed subsequence after a certain point, it is periodic. The period of the sequence is the number of elements in the shortest repetition subsequence. For instance, the sequence $x, y, z, w, y, z, w, y, \dots$ is periodic after the first element x and has period 3. As a special case, a sequence is simply periodic with period u if the initial u elements in the sequence form a repeating subsequence. For example, the sequence $x, y, z, w, x, y, z, w, \dots$ is simply periodic with period 4.

In Section 2, we define three different quaternion-type cyclic-Pell sequences and then present some properties, such as, the Cassini formulas, generating function. Also, we get the relationship between the Pell sequence and these quaternions. In Section 3, we study quaternion-type cyclic-Pell sequences modulo m and then, we give the relationships between the lengths of periods of the quaternion-type cyclic-Pell sequences of the first, second and third kind modulo m and the generating matrices of these sequences. In Section 4, we introduce the quaternion-type cyclic-Pell sequences in groups. After, we calculate the quaternion Pell lengths of generalized quaternion groups. Finally, we give specific example for the first type sequence of quaternion group Q_8.

2. The quaternion-type cyclic-Pell sequences

In this section, we will introduce three different quaternion-type cyclic-Pell sequences for $n \geqslant 2$ any positive integer numbers. Then, we will present miscellaneous properties of these sequences.

Definition 2.1. *Define the quaternion-type cyclic-Pell sequences of the first, second and third kind, respectively:*

$$x_n^1 = \begin{cases} 2kx_{n-1}^1 + jx_{n-2}^1 & \text{if } n \equiv 0\ (3), \\ 2jx_{n-1}^1 + ix_{n-2}^1 & \text{if } n \equiv 1\ (3), \\ 2ix_{n-1}^1 + kx_{n-2}^1 & \text{if } n \equiv 2\ (3), \end{cases}$$
$$x_n^2 = \begin{cases} 2ix_{n-1}^2 + kx_{n-2}^2 & \text{if } n \equiv 0\ (3), \\ 2kx_{n-1}^2 + jx_{n-2}^2 & \text{if } n \equiv 1\ (3), \\ 2jx_{n-1}^2 + ix_{n-2}^2 & \text{if } n \equiv 2\ (3), \end{cases}$$
$$x_n^3 = \begin{cases} 2jx_{n-1}^3 + ix_{n-2}^3 & \text{if } n \equiv 0\ (3), \\ 2ix_{n-1}^3 + kx_{n-2}^3 & \text{if } n \equiv 1\ (3), \\ 2kx_{n-1}^3 + jx_{n-2}^3 & \text{if } n \equiv 2\ (3), \end{cases}$$

the initial conditions for all type are $x_0^\tau = 0$ *and* $x_1^\tau = 1$ $(1 \leqslant \tau \leqslant 3)$.

Let the entries of the matrices A and B be the element of the quaternion-type cyclic-Pell sequences,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},$$

then the following properties are hold:

(i). $A \times B = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$.

(ii). $\det A = a_{11}a_{22} - a_{12}a_{21}$.

(iii). $\det(A \cdot B) = \det A \cdot \det B$.

(iv). $A^n = A^{n-1} \times A \quad (n \in \mathbb{Z}^+)$.

Since the multiplication of quaternions is not commutative, the above properties are given considering multiplicative order. Therefore, it is easy to see that

$$\det A \cdot \det B \neq \det B \cdot \det A$$

and

$$A^{n-1} \times A \neq A \times A^{n-1}.$$

In order to easy in our operations, we define $\epsilon(\eta)$ as follows:

$$\epsilon(\eta) = \begin{cases} j & \text{if } \eta \equiv 0 \ (3), \\ k & \text{if } \eta \equiv 1 \ (3), \\ i & \text{if } \eta \equiv 2 \ (3), \end{cases} \tag{2.1}$$

where $\eta \in \mathbb{Z}^+$. We will give relation these sequences to the well-known classic Pell sequence

$$x_n^\tau = \begin{cases} -(-1)^{\frac{n}{3}} P_n \epsilon(\tau + 2) & \text{if } n \equiv 0 \ (3), \\ (-1)^{\frac{n-1}{3}} P_n & \text{if } n \equiv 1 \ (3), \\ (-1)^{\frac{n-2}{3}} P_n \epsilon(\tau + 1) & \text{if } n \equiv 2 \ (3), \end{cases}$$

where $\tau = 1, 2, 3$ and $\epsilon(\tau)$ is as defined in the Equation (2.1). We can write for the quaternion-type cyclic-Pell sequences

$$G_\tau = \begin{bmatrix} -12 & -5\epsilon(\tau + 2) \\ 5\epsilon(\tau + 2) & -2 \end{bmatrix} \text{ for } \tau = 1, 2, 3. \tag{2.2}$$

By iterative operations on n, we find

$$(G_\tau)^n = \begin{bmatrix} x_{3n+1}^\tau & -x_{3n}^\tau \\ x_{3n}^\tau & x_{3n-1}^\tau \epsilon(\tau + 1) \end{bmatrix} \text{ for } \tau = 1, 2, 3, \tag{2.3}$$

where $n \geqslant 1$.

Now we obtain the Cassini formula for the quaternion-type cyclic-Pell sequences. By using the determinant function and the Equations (2.2), (2.3), we have

$$x_{3n+1}^\tau x_{3n-1}^\tau \epsilon(\tau + 1) + (x_{3n}^\tau)^2 = (-1)^n \text{ for } \tau = 1, 2, 3. \tag{2.4}$$

Lemma 2.1. *We give the recurrence relation for the quaternion-type cyclic-Pell sequences as follows:*

$$x_n^\tau = -14x_{n-3}^\tau + x_{n-6}^\tau,$$

where $\tau = 1, 2, 3$.

Proof. The proof will only be done for the case $\tau = 1$, the others are done similarly. By Definition 2.1, we get

$$\begin{cases} x^1_{3n} = 2kx^1_{3n-1} + jx^1_{3n-2} \ , \\ x^1_{3n+1} = 2jx^1_{3n} + ix^1_{3n-1} \ , \\ x^1_{3n+2} = 2ix^1_{3n+1} + kx^1_{3n} \ . \end{cases}$$

Thus, we have

$$\begin{aligned} x^1_{3n+2} =& 2ix^1_{3n+1} + kx^1_{3n} \\ =& 5kx^1_{3n} - 2x^1_{3n-1} \\ =& -2x^1_{3n-1} + 5k\left(2kx^1_{3n-1} + jx^1_{3n-2}\right) \\ =& -12x^1_{3n-1} + k5jx^1_{3n-2}. \end{aligned}$$

And then, since $5jx^1_{3n-2} = k\left(2x^1_{3n-1} - x^1_{3n-4}\right)$, we obtain

$$x^1_{3n+2} = -14x^1_{3n-1} + x^1_{3n-4}. \tag{2.5}$$

Similarly, we can write

$$\begin{aligned} x^1_{3n+1} =& 2jx^1_{3n} + ix^1_{3n-1} \\ =& 5ix^1_{3n-1} - 2x^1_{3n-2} \\ =& -2x^1_{3n-2} + 5i\left(2ix^1_{3n-2} + kx^1_{3n-3}\right) \\ =& -12x^1_{3n-2} + i5kx^1_{3n-3}. \end{aligned}$$

And then, since $5kx^1_{3n-3} = i\left(2x^1_{3n-2} - x^1_{3n-5}\right)$, we acquire

$$x^1_{3n+1} = -14x^1_{3n-2} + x^1_{3n-5}. \tag{2.6}$$

Similarly, we have

$$\begin{aligned} x^1_{3n} =& 2kx^1_{3n-1} + jx^1_{3n-2} \\ =& 5jx^1_{3n-2} - 2x^1_{3n-3} \\ =& -2x^1_{3n-3} + 5j\left(2jx^1_{3n-3} + ix^1_{3n-4}\right) \\ =& -12x^1_{3n-3} + j5ix^1_{3n-4}. \end{aligned}$$

And then, since $5ix^1_{3n-4} = j\left(2x^1_{3n-3} - x^1_{3n-6}\right)$, we get

$$x^1_{3n} = -14x^1_{3n-3} + x^1_{3n-6}. \tag{2.7}$$

From the Equations (2.5), (2.6) and (2.7), we obtain $x^1_n = -14x^1_{n-3} + x^1_{n-6}$, as required. □

In the following Theorem, we develop the generating function for the quaternion-type cyclic-Pell sequences.

Theorem 2.1. *The generating function of the* $\{x_n^\tau\}$ *is*

$$\sum_{n=0}^{\infty} x_n^\tau t^n = \frac{t + 2\epsilon(\tau+1)t^2 + 5\epsilon(\tau+2)t^3 + 2t^4 - \epsilon(\tau+1)t^5}{1 + 14t^3 - t^6},$$

where $\tau = 1, 2, 3$.

Proof. Assume that $f(t)$ is the generating function of the $\{x_n^\tau\}$ for $\tau = 1, 2, 3$. Then we have

$$f(t) = \sum_{n=0}^{\infty} x_n^\tau t^n$$

From Lemma 2.1, we obtain

$$\begin{aligned} f(t) =& x_0^\tau + x_1^\tau t + x_2^\tau t^2 + x_3^\tau t^3 + x_4^\tau t^4 + x_5^\tau t^5 + \sum_{n=6}^{\infty} \left(-14x_{n-3}^\tau + x_{n-6}^\tau\right) t^n \\ =& x_1^\tau t + x_2^\tau t^2 + x_3^\tau t^3 + x_4^\tau t^4 + x_5^\tau t^5 - 14\left(f(t) - x_0^\tau - x_1^\tau t - x_2^\tau t^2\right) t^3 + f(t)t^6 . \end{aligned}$$

Now rearrangement the equation implies that

$$f(t) = \frac{x_1^\tau t + x_2^\tau t^2 + x_3^\tau t^3 + \left(x_4^\tau + 14x_1^\tau\right) t^4 + \left(x_5^\tau + 14x_2^\tau\right) t^5}{1 + 14t^3 - t^6},$$

which equal to the $\sum_{n=0}^{\infty} x_n^\tau t^n$ in Theorem. □

3. The quaternion-type cyclic-Pell sequence modulo m

In this section, we study quaternion-type cyclic-Pell sequences modulo m. Then, we give the relationships between the lengths of periods of the quaternion-type cyclic-Pell sequences of the first, second and third kind modulo m and the generating matrices of these sequences.

Let p_n denote the nth member of the Pell sequences $p_0 = a$, $p_1 = b$, $p_{n+1} = 2p_n + p_{n-1}$ $(n \geqslant 1)$.

Theorem 3.1. *(* [**4**]*)* $p_n \,(\mathrm{mod}\, m)$ *forms a simply periodic sequence. That is, the sequence is periodic and repeats by returning to its starting values.*

The length of the period of the ordinary Pell sequence $\{P_n\}$ modulo m was denoted by $k(m)$.

If we reduce the quaternion-type cyclic-Pell sequences of the first, second and third kind modulo m, taking least nonnegative residues, then we get the following recurrence sequences:

$$\{x_n^\tau(m)\} = \{x_1^\tau(m),\ x_2^\tau(m),\ \ldots,\ x_u^\tau(m),\ \ldots\}$$

for every integer $1 \leqslant \tau \leqslant 3$, where $x_u^\tau(m)$ is used to mean the uth element of the τth quaternion-type cyclic-Pell sequence when read modulo m. We note here that the recurrence relations in the sequences $\{x_n^\tau(m)\}$ and $\{x_n^\tau\}$ are the same.

Theorem 3.2. *The sequences* $\{x_n^\tau(m)\}$ *are periodic and the lengths of their periods are divisible by* 3.

Proof. Let us consider the quaternion-type cyclic-Pell sequence of the first kind $\left\{x_n^1\right\}$ as an example. Consider the set

$$Q = \{(q_1, q_2) \mid q_u\text{'s are quaternions } a_u + b_u i + c_u j + d_u k \text{ where } a_u,\ b_u,\ c_u \text{ and } d_u \text{ are integers such that } 0 \leqslant a_u, b_u, c_u, d_u \leqslant m-1 \text{ and } u \in \{1,2\}\}.$$

Suppose that the cardinality of the set Q is denoted by the notation $|Q|$. Since the set Q is finite, there are $|Q|$ distinct 2-tuples of the quaternion-type cyclic-Pell sequences of the first kind $\left\{x_n^1\right\}$ modulo m. Thus, it is clear that at least one of these 2-tuples appears twice in the sequence $\left\{x_n^1(m)\right\}$. Let $x_\alpha^1(m) \equiv x_\beta^1(m)$ and $x_{\alpha+1}^1(m) \equiv x_{\beta+1}^1(m)$. If $\beta - \alpha \equiv 0 \,(\mathrm{mod}\ 3)$, then we get $x_{\alpha+2}^1(m) \equiv x_{\beta+2}^1(m)$, $x_{\alpha+3}^1(m) \equiv x_{\beta+3}^1(m)$, ... So, it is easy to see that the subsequence following this 2 -tuple repeats; that is, $\left\{x_n^1(m)\right\}$ is a periodic sequence and the length of its period must be divisible by 3.

The proofs for the sequences $\left\{x_n^2\right\}$ and $\left\{x_n^3\right\}$ are similar to the above and are omitted. □

We next denote the lengths of periods of the sequences $\{x_n^\tau(m)\}$ by $l_{x_n^\tau}(m)$.

Consider the matrices

$$A_1 = \begin{bmatrix} 2i & k \\ 1 & 0 \end{bmatrix},\quad A_2 = \begin{bmatrix} 2k & j \\ 1 & 0 \end{bmatrix} \text{ and } A_3 = \begin{bmatrix} 2j & i \\ 1 & 0 \end{bmatrix}.$$

Suppose that $G_1 = A_3A_2A_1$, $G_2 = A_2A_1A_3$ and $G_3 = A_1A_3A_2$. Using the above, we define the following matrices:

$$(M_1)^n = \begin{cases} (G_1)^{\frac{n}{3}} & \text{if } n \equiv 0\ (3), \\ A_1 (G_1)^{\frac{n-1}{3}} & \text{if } n \equiv 1\ (3), \\ A_2A_1 (G_1)^{\frac{n-2}{3}} & \text{if } n \equiv 2\ (3), \end{cases} \qquad (M_2)^n = \begin{cases} (G_2)^{\frac{n}{3}} & \text{if } n \equiv 0\ (3), \\ A_3 (G_2)^{\frac{n-1}{3}} & \text{if } n \equiv 1\ (3), \\ A_1A_3 (G_2)^{\frac{n-2}{3}} & \text{if } n \equiv 2\ (3), \end{cases}$$

$$(M_3)^n = \begin{cases} (G_3)^{\frac{n}{3}} & \text{if } n \equiv 0\ (3), \\ A_2 (G_3)^{\frac{n-1}{3}} & \text{if } n \equiv 1\ (3), \\ A_3A_2 (G_3)^{\frac{n-2}{3}} & \text{if } n \equiv 2\ (3). \end{cases}$$

Then we get

$$(M_\tau)^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_{n+1}^\tau \\ x_n^\tau \end{pmatrix},$$

where τ is an integer such that $1 \leqslant \tau \leqslant 3$. Therefore, we immediately deduce that $l_{x_n^\tau}(m)$ is the smallest positive integer α such that $(M_\tau)^\alpha \equiv I(\mathrm{mod}\, m)$ for every integer $1 \leqslant \tau \leqslant 3$.

4. The quaternion-type cyclic-Pell sequence in groups

In this section, we will define three different quaternion-type cyclic-Pell sequences in finite groups. Subsequently, we will examine the quaternion-type cyclic-Pell orbits of the first, second and third kind of the generalized quaternion group. Finally, we will give specific example for the 1st type sequences of quaternion group Q_8.

The G is a 2-generator group and

$$X = \{(x_1, x_2) \in G \times G \mid \langle \{x_1, x_2\} \rangle = G\}.$$

We indicate (x_1, x_2) a generating pair for G.

Definition 4.1. *Let G be a 2-generator group. For the generating pair (x, y), we define the quaternion-type cyclic-Pell orbits of the first, second and third kind of G as follows, respectively:*

$$a_n^1 = \begin{cases} (a_{n-2}^1)^j (a_{n-1}^1)^{2k} & \text{if } n \equiv 0\ (3), \\ (a_{n-2}^1)^i (a_{n-1}^1)^{2j} & \text{if } n \equiv 1\ (3), \\ (a_{n-2}^1)^k (a_{n-1}^1)^{2i} & \text{if } n \equiv 2\ (3), \end{cases}$$

$$a_n^2 = \begin{cases} (a_{n-2}^2)^k (a_{n-1}^2)^{2i} & \text{if } n \equiv 0\ (3), \\ (a_{n-2}^2)^j (a_{n-1}^2)^{2k} & \text{if } n \equiv 1\ (3), \\ (a_{n-2}^2)^i (a_{n-1}^2)^{2j} & \text{if } n \equiv 2\ (3), \end{cases}$$

$$a_n^3 = \begin{cases} (a_{n-2}^3)^i (a_{n-1}^3)^{2j} & \text{if } n \equiv 0\ (3), \\ (a_{n-2}^3)^k (a_{n-1}^3)^{2i} & \text{if } n \equiv 1\ (3), \\ (a_{n-2}^3)^j (a_{n-1}^3)^{2k} & \text{if } n \equiv 2\ (3), \end{cases}$$

for $n \geqslant 2$, with initial conditions $a_0^\tau = x$ and $a_1^\tau = y$ $(1 \leqslant \tau \leqslant 3)$, where the following conditions hold for every $x, y \in G$:

- (i). *Let $q = a + bi + cj + dk$ such that a, b, c and d are integers and let e be the identity of G, then*
  - $* \; x^q = x^{a(mod|x|)+b(mod|x|)i+c(mod|x|)j+d(mod|x|)k}$ $= x^{a(mod|x|)} x^{b(mod|x|)i} x^{c(mod|x|)j} x^{d(mod|x|)k}$.
  - $* \; (x^u)^a = (x^a)^u$, *where $u \in \{i, j, k\}$ and a is an integer.*
  - $* \; e^q = e$ *and* $x^{0+0i+0j+0k} = e$.
- (ii). *Let $q_1 = a_1 + b_1 i + c_1 j + d_1 k$ and $q_2 = a_2 + b_2 i + c_2 j + d_2 k$ such that $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2$ are integers, then $(x^{q_1} x^{q_2})^{-1} = x^{-q_2} x^{-q_1}$.*
- (iii). *If $xy \neq yx$, then $x^u y^u \neq y^u x^u$ for $u \in \{i, j, k\}$.*
- (iv). $(xy)^u = y^u x^u$ *for* $u \in \{i, j, k\}$.
- (v). $(x^{u_1} y^{u_2})^{u_3} = x^{u_3 u_1} y^{u_3 u_2}$, $(xy^{u_1})^{u_2} = x^{u_2} y^{u_2 u_1}$ *and* $(x^{u_1} y)^{u_2} = x^{u_2 u_1} y^{u_2}$ *for $u_1, u_2, u_3 \in \{i, j, k\}$ and so* $(x^{u_1} y^{u_1})^{u_1} = x^{-1} y^{-1}$.
- (vi). *For $u_1, u_2 \in \{i, j, k\}$ such that $u_1 \neq u_2$, $x^{u_1} y^{u_2} = y^{u_2} x^{u_1}$, $xy^{u_1} = y^{u_1} x$, $x^{u_1} y = y x^{u_1}$ and so $(xy^{u_1})^{u_1} = x^{u_1} y^{-1}$ and $(x^{u_1} y)^{u_1} = x^{-1} y^{u_1}$.*

Let the notation $P_{(x,y)}^{q,\tau}(G)$ denote the τth quaternion-type cyclic-Pell orbit of the group G for the generating pair (x, y). From the definition of the orbit $P_{(x,y)}^{q,\tau}(G)$ it is clear that the length of the period of this sequence in a finite group depend on the chosen generating pair and the order in which the assignments of x, y are made.

Theorem 4.1. *Let G be a 2-generator group. If G is finite, then the quaternion-type cyclic-Pell orbits of the first, second and third kind of G are periodic and the lengths of their periods are divisible by 3.*

Proof. Let us consider the 2nd quaternion-type cyclic-Pell orbit of the group G. We take the set

$$S=\Big\{(s_1)^{a_1(\mathrm{mod}|s_1|)+b_1(\mathrm{mod}|s_1|)i+c_1(\mathrm{mod}|s_1|)j+d_1(\mathrm{mod}|s_1|)k},\\ (s_2)^{a_2(\mathrm{mod}|s_2|)+b_2(\mathrm{mod}|s_2|)i+c_2(\mathrm{mod}|s_2|)j+d_2(\mathrm{mod}|s_2|)k}:\\ s_1,s_2\in G \text{ and } a_1,a_2,b_1,b_2,c_1,c_2,d_1,d_2\in\mathbb{Z}\Big\}.$$

Since the group G is finite, S is a finite set. Hence there exists $v>u$ such that $a_u^2=a_v^2$ and $a_{u+1}^2=a_{v+1}^2$ for any $u\geqslant 0$. If $v-u\equiv 0\,(\mathrm{mod}\ 3)$, then we get $a_{u+2}^2=a_{v+2}^2$, $a_{u+3}^2=a_{v+3}^2$, …. Because of the repeating, for all generating pairs, the sequence $P_{(x,y)}^{q,2}(G)$ is periodic and the length of its period must be divisible by 3.

The proofs for the orbits $P_{(x,y)}^{q,1}(G)$ and $P_{(x,y)}^{q,3}(G)$ are similar to the above and are omitted. □

We next denote the lengths of the periods of the orbits $P_{(x,y)}^{q,\tau}(G)$ by $LP_{(x,y)}^{q,\tau}(G)$.

We shall now address the lengths of the periods of the orbits $P_{(x,y)}^{q,1}(Q_{2^{m+1}})$, $P_{(x,y)}^{q,2}(Q_{2^{m+1}})$ and $P_{(x,y)}^{q,3}(Q_{2^{m+1}})$. It is well-known that the generalized quaternion group $Q_{2^{m+1}}$ of order 2^m is defined by the presentation

$$Q_{2^{m+1}}=\langle x,y\mid x^{2^m}=y^4=1,\ \ x^{2^{m-1}}=y^2,\ \ y^{-1}xy=x^{-1}\rangle.$$

Theorem 4.2. *For $m\geqslant 2$,*

$$LP_{(x,y)}^{q,1}(Q_{2^{m+1}})=LP_{(x,y)}^{q,2}(Q_{2^{m+1}})=LP_{(x,y)}^{q,3}(Q_{2^{m+1}})=3.2^m.$$

Proof. By direct calculation, we obtain the orbits $P_{(x,y)}^{q,1}(Q_{2^{m+1}})$, $P_{(x,y)}^{q,2}(Q_{2^{m+1}})$ and $P_{(x,y)}^{q,3}(Q_{2^{m+1}})$ as follows, respectively. Firstly, the orbit $P_{(x,y)}^{q,1}(Q_{2^{m+1}})$ is

$$\begin{array}{l} a_0^1=x,\ \ a_1^1=y,\ a_2^1=y^{2i}x^k,\quad a_3^1=y^jx^{-2},\quad a_4^1=x^{-5j},\quad a_5^1=y^{-i}x^{-12k},\cdots,\\ a_{12}^1=x^{5741},a_{13}^1=yx^{13860j},a_{14}^1=y^{2i}x^{33461k},\ a_{15}^1=y^jx^{-80782},\ a_{16}^1=x^{-195025j},\cdots,\\ a_{24}^1=x^{225058681},a_{25}^1=yx^{543339720j},a_{26}^1=y^{2i}x^{1311738121k},a_{27}^1=y^jx^{-3166815962},\cdots,\\ \quad\cdots\\ a_{12n}^1=x^{P_{12n-1}},\ \ a_{12n+1}^1=yx^{P_{12n}j},\ a_{12n+2}^1=y^{2i}x^{P_{12n+1}k},\ a_{12n+3}^1=y^jx^{-P_{12n+2}},\\ a_{12n+4}^1=x^{-P_{12n+3}j},a_{12n+5}^1=y^{-i}x^{-P_{12n+4}k},a_{12n+6}^1=y^{2j}x^{P_{12n+5}},a_{12n+7}^1=yx^{P_{12n+6}j},\\ a_{12n+8}^1=x^{P_{12n+7}k},a_{12n+9}^1=y^jx^{-P_{12n+8}},a_{12n+10}^1=y^2x^{-P_{12n+9}j},a_{12n+11}^1=y^{-i}x^{-P_{12n+10}k}. \end{array}$$

Secondly, we take into account the orbit $P^{q,2}_{(x,y)}(Q_{2^{m+1}})$. We have the sequence

$$a^2_0 = x, a^2_1 = y, a^2_2 = y^{2j}x^i, a^2_3 = y^k x^{-2}, a^2_4 = x^{-5k}, a^2_5 = y^{-j}x^{-12i}, \cdots,$$
$$a^2_{12} = x^{5741}, a^2_{13} = yx^{13860k}, a^2_{14} = y^{2j}x^{33461i}, a^2_{15} = y^k x^{-80782}, a^2_{16} = x^{-195025k}, \cdots,$$
$$a^2_{24} = x^{225058681}, a^2_{25} = yx^{543339720k}, a^2_{26} = y^{2j}x^{1311738121i}, a^2_{27} = y^k x^{-3166815962}, \cdots,$$
$$\ldots$$
$$a^2_{12n} = x^{P_{12n-1}}, a^2_{12n+1} = yx^{P_{12n}k}, a^2_{12n+2} = y^{2j}x^{P_{12n+1}i}, a^2_{12n+3} = y^k x^{-P_{12n+2}},$$
$$a^2_{12n+4} = x^{-P_{12n+3}k}, a^2_{12n+5} = y^{-j}x^{-P_{12n+4}i}, a^2_{12n+6} = y^{2k}x^{P_{12n+5}}, a^2_{12n+7} = yx^{P_{12n+6}k},$$
$$a^2_{12n+8} = x^{P_{12n+7}i}, a^2_{12n+9} = y^k x^{-P_{12n+8}}, a^2_{12n+10} = y^2 x^{-P_{12n+9}k}, a^2_{12n+11} = y^{-j}x^{-P_{12n+10}i}.$$

Finally, we consider the 3rd quaternion-type cyclic-Pell orbit of the generalized quaternion group $Q_{2^{m+1}}$ with respect to the generating pair (x, y), $P^{q,3}_{(x,y)}(Q_{2^{m+1}})$. Using a similar argument to the above, we obtain the following sequence:

$$a^3_0 = x, \quad a^3_1 = y, \quad a^3_2 = y^{2k}x^j, \quad a^3_3 = y^i x^{-2}, \quad a^3_4 = x^{-5i}, \quad a^3_5 = y^{-k}x^{-12j}, \cdots,$$
$$a^3_{12} = x^{5741},\ a^3_{13} = yx^{13860i},\ a^3_{14} = y^{2k}x^{33461j},\ a^3_{15} = y^i x^{-80782},\ a^3_{16} = x^{-195025i}, \cdots,$$
$$a^3_{24} = x^{225058681},\ a^3_{25} = yx^{543339720i},\ a^3_{26} = y^{2k}x^{1311738121j},\ a^3_{27} = y^i x^{-3166815962}, \cdots,$$
$$\ldots$$
$$a^3_{12n} = x^{P_{12n-1}}, \quad a^3_{12n+1} = yx^{P_{12n}i},\ a^3_{12n+2} = y^{2k}x^{P_{12n+1}j},\ a^3_{12n+3} = y^i x^{-P_{12n+2}},$$
$$a^3_{12n+4} = x^{-P_{12n+3}i},\ a^3_{12n+5} = y^{-k}x^{-P_{12n+4}j},\ a^3_{12n+6} = y^{2i}x^{P_{12n+5}},\ a^3_{12n+7} = yx^{P_{12n+6}i},$$
$$a^3_{12n+8} = x^{P_{12n+7}j}, a^3_{12n+9} = y^i x^{-P_{12n+8}}, a^3_{12n+10} = y^2 x^{-P_{12n+9}i}, a^3_{12n+11} = y^{-k}x^{-P_{12n+10}j}.$$

where P_n is the nth term of the ordinary Pell sequence $\{P_n\}$. It is known that $k(2^m) = 2^m$; see [**4**] for proof. So we get that the lengths of the periods of the sequences $P^{q,1}_{(x,y)}(Q_{2^{m+1}})$, $P^{q,2}_{(x,y)}(Q_{2^{m+1}})$ and $P^{q,3}_{(x,y)}(Q_{2^{m+1}})$ are $\operatorname{lcm}[12, k(2^m)] = \operatorname{lcm}[12, 2^m] = 3.2^m$. □

Now, for the generating pair (x, y), we give the 1st quaternion-type cyclic-Pell orbits of the quaternion group $Q_8 = \langle x, y \mid x^4 = 1,\ x^2 = y^2,\ y^{-1}xy = x^{-1}\rangle$ which is a non-abelian group of order eight.

EXAMPLE 4.1. *The sequence* $P^{q,1}_{(x,y)}(Q_8)$ *is*

$$x,\ y,\ y^{2i}x^k,\ y^{j-2},\ x^{-j},\ y^{-i},\ y^{2j}x,\ yx^{2j},\ x^k,\ y^j,\ y^2x^{-j},$$
$$y^{-i-2k},\ x,\ y,\ y^{2i}x^k,\ y^{j-2},\ x^{-j},\ y^{-i},\ y^{2j}x,\ yx^{2j}, \ldots,$$

which implies that $LP^{q,1}_{(x,y)}(Q_8) = 12$.

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