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CONCIRCULAR CURVATURE TENSOR ON THE ALMOST $C(\alpha)$-MANIFOLD

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Abstract. In this article, the concircular curvature tensor on an almost $C(\alpha)$-manifold is discussed. Some special curvature conditions provided by the concircular curvature tensor on the Riemann, Ricci, projective, concircular curvature tensors have been investigated.

1. Introduction

A transformation of an $n-$dimensional Riemannian manifold, which transforms every geodesic circle of Riemannian manifold into a geodesic circle, is called a concircular transformation ([**14**],[**6**]). A concircular transformation is always a conformal transformation [**14**]. In general, a geodesic circle does not transform into a geodesic circle by the conformal transformation. The transformation which preserves geodesic circles was first introduced by Yano [**14**]. The conformal transformation transforms a geodesic circle into a geodesic circle with the help of a special partial differential equation. Such a transformation is known as the concircular transformation and the geometry which deals with such transformation is called the concircular geometry [**13**]. The concircular curvature tensor is very important in the differential geometry of some F-structures, such as complex, almost complex, Kaehler, almost Kaehler, contact, almost contact structures ([**2**]-[**16**]). Concircular curvature tensor is the next most important $(1,3)-$type curvature tensor from Riemannian point wiev.

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The concircular curvature tensor is one of the tensors that characterizes the important properties of the manifold on which it is defined. Again, many geometers investigated the properties of manifolds on different curvature tensors ([**5**]-[**9**]).

In this article, some special curvature conditions are investigated for the concircular curvature tensor on a $(2n+1)$-dimensional almost $C(\alpha)-$manifold. The special relationship between the concircular curvature tensor and Riemann, Ricci, the projective curvature tensors and the effect of the concircular curvature tensor on itself are discussed. Under these special curvature conditions, some important properties of the almost $C(\alpha)-$manifold are obtained.

2. Preliminaries

Let's take an $(2n+1)-$dimensional differentiable M manifold. If the R Riemann curvature tensor of the M almost contact metric manifold satisfies the condition

$$\begin{aligned} R(\beta_1,\beta_2,\beta_3,\beta_4) = \; & R(\beta_1,\beta_2,\phi\beta_3,\phi\beta_4) + \alpha\{-g(\beta_1,\beta_3)g(\beta_2,\beta_4) \\ & +g(\beta_1,\beta_4)g(\beta_2,\beta_3) + g(\beta_1,\phi\beta_3)g(\beta_2,\phi\beta_4) \\ & -g(\beta_1,\phi\beta_4)g(\beta_2,\phi\beta_3)\} \end{aligned}$$

for all $\beta_1,\beta_2,\beta_3,\beta_4 \in \chi(M)$, for $\alpha \in \mathbb{R}$, then M is called the almost $C(\alpha)$-manifold where ϕ is $(1,1)-$type tensor field. Also, the Riemann curvature tensor of a almost $C(\alpha)$-manifold with $c-$constant sectional curvature is given by

$$\begin{aligned} R(\beta_1,\beta_2)\beta_3 \; & = \left(\frac{c+3\alpha}{4}\right)\{g(\beta_2,\beta_3)\beta_1 - g(\beta_1,\beta_3)\beta_2\} \\ & + \left(\frac{c-\alpha}{4}\right)\{g(\beta_1,\phi\beta_3)\phi\beta_2 - g(\beta_2,\phi\beta_3)\phi\beta_1 \\ & +2g(\beta_1,\phi\beta_2)\phi\beta_3 + \eta(\beta_2)\eta(\beta_3)\beta_1 \\ & +g(\beta_1,\beta_3)\eta(\beta_2)\xi - g(\beta_2,\beta_3)\eta(\beta_1)\xi\}. \end{aligned} \tag{1}$$

If putting $\beta_1=\xi, \beta_2=\xi$ and $\beta_3=\xi$ in (1), the following relations are obtained.

$$R(\xi,\beta_2)\beta_3 = \alpha[g(\beta_2,\beta_3)\xi - \eta(\beta_3)\beta_2], \tag{2}$$

$$R(\beta_1,\xi)\beta_3 = \alpha[-g(\beta_1,\beta_3)\xi + \eta(\beta_3)\beta_1], \tag{3}$$

$$R(\beta_1,\beta_2)\xi = \alpha[\eta(\beta_2)\beta_1 - \eta(\beta_1)\beta_2]. \tag{4}$$

Also, if we take the inner product of both sides of equation (1) with the vector $\xi \in \chi(M)$, we have

$$\eta(R(\beta_1,\beta_2)\beta_3) = \alpha[g(\beta_2,\beta_3)\eta(\beta_1) - g(\beta_1,\beta_3)\eta(\beta_2)]. \tag{5}$$

For M is a $(2n+1)$-dimensional Riemann manifold, the projective curvature tensor P defined as

$$P(\beta_1,\beta_2)\beta_3 = R(\beta_1,\beta_2)\beta_3 - \frac{1}{2n}\left[S(\beta_2,\beta_3)\beta_1 - S(\beta_1,\beta_3)\beta_2\right], \tag{6}$$

for each $\beta_1,\beta_2,\beta_3 \in \chi(M)$,[**3**]. If $\beta_1=\xi,\beta_2=\xi$ and $\beta_3=\xi$ are selected respectively in (6), the following relations are obtained.

$$P(\xi,\beta_2)\beta_3 = \alpha g(\beta_2,\beta_3)\xi - \frac{1}{2n}S(\beta_2,\beta_3)\xi, \tag{7}$$

$$P(\beta_1,\xi)\beta_3 = -\alpha g(\beta_1,\beta_3)\xi + \frac{1}{2n}S(\beta_1,\beta_3)\xi, \tag{8}$$

$$P(\beta_1,\beta_2)\xi = 0. \tag{9}$$

Also, if we take the inner product of both sides of (6) with the vector $\xi \in \chi(M)$, we have

(10)

$$\eta(P(\beta_1,\beta_2)\beta_3) = \eta(\beta_1)\left[\alpha g(\beta_2,\beta_3) - \frac{1}{2n}S(\beta_2,\beta_3)\right] - \eta(\beta_2)\left[\alpha g(\beta_1,\beta_3) - \frac{1}{2n}S(\beta_1,\beta_3)\right].$$

As M is a $(2n+1)$-dimensional Riemann manifold, the tensor $\tilde{Z}$ defined as

$$\tilde{Z}(\beta_1,\beta_2)\beta_3 = R(\beta_1,\beta_2)\beta_3 - \frac{r}{2n(2n+1)}\left[g(\beta_2,\beta_3)\beta_1 - g(\beta_1,\beta_3)\beta_2\right] \tag{11}$$

for each $\beta_1,\beta_2,\beta_3 \in \chi(M)$, is called the concircular curvature tensor [**7**]. If $\beta_1=\xi,\beta_2=\xi$ and putting $\beta_3=\xi$ in (11), the following relations are obtained.

$$\tilde{Z}(\xi,\beta_2)\beta_3 = \left[\alpha - \frac{r}{2n(2n+1)}\right]\left[g(\beta_2,\beta_3)\xi - \eta(\beta_3)\beta_2\right], \tag{12}$$

$$\tilde{Z}(\beta_1,\xi)\beta_3 = \left[\alpha - \frac{r}{2n(2n+1)}\right]\left[-g(\beta_1,\beta_3)\xi + \eta(\beta_3)\beta_1\right], \tag{13}$$

$$\tilde{Z}(\beta_1,\beta_2)\xi = \left[\alpha - \frac{r}{2n(2n+1)}\right]\left[\eta(\beta_2)\beta_1 - \eta(\beta_1)\beta_2\right]. \tag{14}$$

Also, if we take the inner product of both sides of equation (11) with the vector $\xi \in \chi(M)$, we have

$$\eta\left(\tilde{Z}(\beta_1,\beta_2)\beta_3\right) = \left[\alpha - \frac{r}{2n(2n+1)}\right]\left[g(\beta_2,\beta_3)\eta(\beta_1) - g(\beta_1,\beta_3)\eta(\beta_2)\right]. \tag{15}$$

For a $(2n+1)$ −dimensional M almost $C(\alpha)$ −manifold, the following equations hold.

$$S(\beta_1,\beta_2) = \left[\frac{\alpha(3n-1)+c(n+1)}{2}\right]g(\beta_1,\beta_2) + \frac{(\alpha-c)(n+1)}{2}\eta(\beta_1)\eta(\beta_2) \tag{16}$$

$$S(\beta_1,\xi) = 2n\alpha\eta(\beta_1) \tag{17}$$

$$Q\beta_1 = \left[\frac{\alpha(3n-1)+c(n+1)}{2}\right]\beta_1 + \frac{(\alpha-c)(n+1)}{2}\eta(\beta_1)\xi \tag{18}$$

$$Q\xi = 2n\alpha\xi \tag{19}$$

for each $\beta_1, \beta_2, \in \chi(M)$, where Q and S are the Ricci operator and Ricci tensor of manifold M, respectively.

3. Concircular curvature tensor on the almost $C(\alpha)$-manifold

Let M be a $(2n+1)-$dimensional almost $C(\alpha)-$manifold. Let us first examine a special curvature condition established between the concircular curvature tensor and the Riemann curvature tensor. Let us state and prove the following theorem.

THEOREM 3.1. *If a $(2n+1)-$ dimensional almost $C(\alpha)-$manifold satisfies the curvature condition*

$$\tilde{Z}(\beta_1, \beta_2).R = \lambda_1 Q(S, R),$$

then the almost $C(\alpha)-$manifold is either co-Kaehler manifold or $\lambda_1 = 0$.

PROOF. Let's assume that a $(2n+1)-$dimensional almost $C(\alpha)-$manifold satisfies the curvature condition

$$\left(\tilde{Z}(\beta_1, \beta_2).R\right)(\beta_5, \beta_4, \beta_3) = \lambda_1 Q(S, R)(\beta_5, \beta_4, \beta_3; \beta_1, \beta_2),$$

for each $\beta_1, \beta_2, \beta_5, \beta_4, \beta_3 \in \chi(M)$. In this situation, we can write

$$\begin{aligned}
&\tilde{Z}(\beta_1, \beta_2) R(\beta_5, \beta_4) \beta_3 - R\left(\tilde{Z}(\beta_1, \beta_2) \beta_5, \beta_4\right) \beta_3 - R\left(\beta_5, \tilde{Z}(\beta_1, \beta_2) \beta_4\right) \beta_3 \\
&-R(\beta_5, \beta_4) \tilde{Z}(\beta_1, \beta_2) \beta_3 = -\lambda_1 \{S(\beta_2, \beta_5) R(\beta_1, \beta_4) \beta_3 - S(\beta_1, \beta_5) R(\beta_2, \beta_4) \beta_3 \\
&+S(\beta_2, \beta_4) R(\beta_5, \beta_1) \beta_3 - S(\beta_1, \beta_4) R(\beta_5, \beta_2) \beta_3 + S(\beta_2, \beta_3) R(\beta_5, \beta_4) \beta_1 \\
&-S(\beta_1, \beta_3) R(\beta_5, \beta_4) \beta_2\}.
\end{aligned} \tag{20}$$

If we choose $\beta_1 = \xi$ in (20) and make use of $(2), (3), (4), (12)$, we obtain

$$\begin{aligned}
&\left(\alpha - \tfrac{r}{2n(2n+1)}\right) \{g(\beta_2, R(\beta_5, \beta_4) \beta_3) \xi - \eta(R(\beta_5, \beta_4) \beta_3) \beta_2 \\
&-g(\beta_2, \beta_5) R(\xi, \beta_4) \beta_3 + \eta(\beta_5) R(\beta_2, \beta_4) \beta_3 - g(\beta_2, \beta_4) R(\beta_5, \xi) \beta_3 \\
&+\eta(\beta_4) R(\beta_5, \beta_2) \beta_3 - g(\beta_2, \beta_3) R(\beta_5, \beta_4) \xi + \eta(\beta_3) R(\beta_5, \beta_4) \beta_2\} \\
&= -\lambda_1 \{\alpha S(\beta_2, \beta_5) g(\beta_4, \beta_3) \xi - \alpha S(\beta_2, \beta_5) \eta(\beta_3) \beta_4 - 2n\alpha\eta(\beta_5) R(\beta_2, \beta_4) \beta_3 \\
&-\alpha S(\beta_2, \beta_4) g(\beta_5, \beta_3) \xi + \alpha S(\beta_2, \beta_4) \eta(\beta_3) \beta_5 - 2n\alpha\eta(\beta_4) R(\beta_5, \beta_2) \beta_3 \\
&+\alpha S(\beta_2, \beta_3) \eta(\beta_4) \beta_5 - \alpha S(\beta_2, \beta_3) \eta(\beta_5) \beta_4 - 2n\alpha\eta(\beta_3) R(\beta_5, \beta_4) \beta_2\}.
\end{aligned} \tag{21}$$

If we choose $\beta_5 = \xi$ in (21) and from (2) , we get

(22)
$$\left(\alpha - \frac{r}{2n(2n+1)}\right)\left[R\left(\beta_2,\beta_4\right)\beta_3 - \alpha g\left(\beta_4,\beta_3\right)\beta_2 + \alpha g\left(\beta_2,\beta_3\right)\beta_4\right]$$
$$= -\lambda_1\left\{2n\alpha^2 g\left(\beta_4,\beta_3\right)\eta\left(\beta_2\right)\xi - 2n\alpha R\left(\beta_2,\beta_4\right)\beta_3 - 2n\alpha^2 g\left(\beta_2,\beta_3\right)\eta\left(\beta_4\right)\xi -\right.$$
$$+2n\alpha^2\eta\left(\beta_4\right)\eta\left(\beta_3\right)\beta_2 + \alpha S\left(\beta_2,\beta_3\right)\eta\left(\beta_4\right)\xi - \alpha S\left(\beta_2,\beta_3\right)\beta_4$$
$$\left.-2n\alpha^2 g\left(\beta_4,\beta_2\right)\eta\left(\beta_3\right)\xi\right\}.$$

If we take the inner product of both sides of equation (22) by $\xi \in \chi(M)$ and we choose $\beta_3 = \xi$ in (22), we get

$$-2n\alpha^2\lambda_1\left[-g\left(\beta_2,\beta_4\right) + \eta\left(\beta_2\right)\eta\left(\beta_4\right)\right]. \tag{23}$$

It is clear from equation (23) that

$$\alpha = 0 \text{ or } \lambda_1 = 0$$

This completes the proof. □

Corollary 3.1. *Let M be $(2n+1)$ dimensional almost $C(\alpha)$ −manifold provided*

$$\tilde{Z}\left(\beta_1,\beta_2\right).R = \lambda_1 Q\left(S,R\right).$$

Then M is a real space form with $c = \alpha$ if and only if $\lambda_1 = 0$.

Theorem 3.2. *If a $(2n+1)-$ dimensional almost $C(\alpha)-$manifold satisfies the curvature condition*

$$\tilde{Z}\left(\beta_1,\beta_2\right).R = \lambda_2 Q\left(g,R\right),$$

then the almost $C(\alpha)-$manifold is either co-Kaehler manifold or $\lambda_2 = 0$.

Corollary 3.2. *Let M be $(2n+1)$ dimensional almost $C(\alpha)$ −manifold provided*

$$\tilde{Z}\left(\beta_1,\beta_2\right).R = \lambda_2 Q\left(g,R\right).$$

Then M is a real space form with $c = \alpha$ if and only if $\lambda_2 = 0$.

Let us secondly examine a special curvature condition established between the concircular curvature tensor and the projective curvature tensor. Let us state and prove the following theorem.

Theorem 3.3. *If a $(2n+1)-$ dimensional almost $C(\alpha)-$manifold satisfies the curvature condition*

$$\tilde{Z}\left(\beta_3,\beta_1\right).P = \lambda_3 Q\left(S,P\right),$$

then the almost $C(\alpha)-$manifold is an Einstein manifold provided $r \neq 2n(2n+1)\alpha$.

Proof. Let's assume that a $(2n+1)$ −dimensional almost $C(\alpha)-$manifold satisfies the curvature condition

$$\left(\tilde{Z}\left(\beta_3,\beta_1\right).P\right)\left(\beta_2,\beta_5,\beta_4\right) = \lambda_3 Q\left(S,P\right)\left(\beta_2,\beta_5,\beta_4;\beta_3,\beta_1\right),$$

for each $\beta_1, \beta_2, \beta_5, \beta_4, \beta_3 \in \chi(M)$. In this situation, we can write

(24)

$$\tilde{Z}(\beta_3, \beta_1) P(\beta_2, \beta_5) \beta_4 - P\left(\tilde{Z}(\beta_3, \beta_1) \beta_2, \beta_5\right) \beta_4 - P\left(\beta_2, \tilde{Z}(\beta_3, \beta_1) \beta_5\right) \beta_4$$

$$-P(\beta_2, \beta_5) \tilde{Z}(\beta_3, \beta_1) \beta_4 = -\lambda_3 \{S(\beta_1, \beta_2) P(\beta_3, \beta_5) \beta_4 - S(\beta_3, \beta_2) P(\beta_1, \beta_5) \beta_4$$

$$+S(\beta_1, \beta_5) P(\beta_2, \beta_3) \beta_4 - S(\beta_3, \beta_5) P(\beta_2, \beta_1) \beta_4 + S(\beta_1, \beta_4) P(\beta_2, \beta_5) \beta_3$$

$$-S(\beta_3, \beta_4) P(\beta_2, \beta_5) \beta_1\}.$$

If we choose $\beta_3 = \xi$ in (24) and make use of $(7), (8), (9), (12)$ we obtain

(25)

$$\left(\alpha - \frac{r}{2n(2n+1)}\right) [g(\beta_1, P(\beta_2, \beta_5) \beta_4) \xi - \eta(P(\beta_2, \beta_5) \beta_4) \beta_1$$

$$-g(\beta_1, \beta_2) P(\xi, \beta_5) \beta_4 + \eta(\beta_2) P(\beta_1, \beta_5) \beta_4 - g(\beta_1, \beta_5) P(\beta_2, \xi) \beta_4$$

$$+\eta(\beta_5) P(\beta_1, \beta_2) \beta_4 - g(\beta_1, \beta_4) P(\beta_2, \beta_5) \xi + \eta(\beta_4) P(\beta_2, \beta_5) \beta_1]$$

$$= -\lambda_3 \left\{\alpha S(\beta_1, \beta_2) g(\beta_5, \beta_4) \xi - \frac{1}{2n} S(\beta_1, \beta_2) S(\beta_5, \beta_4) \xi\right.$$

$$-2n\alpha\eta(\beta_2) P(\beta_1, \beta_5) \beta_4 - \alpha S(\beta_1, \beta_5) g(\beta_2, \beta_4) \xi + \frac{1}{2n} S(\beta_1, \beta_5) S(\beta_2, \beta_4) \xi$$

$$-2n\alpha\eta(\beta_5) P(\beta_2, \beta_1) \beta_4 - 2n\alpha\eta(\beta_4) P(\beta_2, \beta_5) \beta_1\}.$$

If we choose $\beta_2 = \xi$ in (25) and make use of (7), we have

(26)

$$\left(\alpha - \frac{r}{2n(2n+1)}\right) [\alpha g(\beta_1, \beta_5) \eta(\beta_4) \xi - \alpha g(\beta_5, \beta_4) \beta_1 - \alpha g(\beta_1, \beta_4) \eta(\beta_5) \xi$$

$$+\frac{1}{2n} [S(\beta_5, \beta_4) \beta_1 + S(\beta_1, \beta_4) \eta(\beta_5) \xi - S(\beta_5, \beta_1) \eta(\beta_4) \xi] + P(\beta_5, \beta_1) \beta_4]$$

$$= -\lambda_3 \left\{2n\alpha^2 g(\beta_5, \beta_4) \eta(\beta_1) \xi - \alpha S(\beta_5, \beta_4) \eta(\beta_1) \xi - 2n\alpha P(\beta_1, \beta_5) \beta_4\right.$$

$$-2n\alpha^2 g(\beta_1, \beta_4) \eta(\beta_5) \xi + \alpha S(\beta_1, \beta_4) \eta(\beta_5) \xi - 2n\alpha^2 g(\beta_5, \beta_1) \eta(\beta_4) \xi$$

$$+\alpha S(\beta_5, \beta_1) \eta(\beta_4) \xi\}.$$

If we take the inner product of both sides of equation (26) by $\xi \in \chi(M)$, we get

(27)
$$\left(\alpha - \frac{r}{2n(2n+1)}\right)\left[\alpha g(\beta_1,\beta_5)\eta(\beta_4) - \alpha g(\beta_5,\beta_4)\eta(\beta_1) - \alpha g(\beta_1,\beta_4)\eta(\beta_5)\right.$$
$$\left. + \tfrac{1}{2n}\left[S(\beta_5,\beta_4)\eta(\beta_1) + S(\beta_1,\beta_4)\eta(\beta_5) - S(\beta_5,\beta_1)\eta(\beta_4)\right] + \eta(P(\beta_5,\beta_1)\beta_4)\right]$$
$$= -\lambda_3\left\{2n\alpha^2 g(\beta_5,\beta_4)\eta(\beta_1) - \alpha S(\beta_5,\beta_4)\eta(\beta_1) - 2n\alpha\eta(P(\beta_1,\beta_5)\beta_4)\right.$$
$$-2n\alpha^2 g(\beta_1,\beta_4)\eta(\beta_5) + \alpha S(\beta_1,\beta_4)\eta(\beta_5) - 2n\alpha^2 g(\beta_5,\beta_1)\eta(\beta_4)$$
$$\left. +\alpha S(\beta_5,\beta_1)\eta(\beta_4)\right\}.$$

If we choose $\beta_4 = \xi$ in (27), we have

$$\left(\alpha - \frac{r}{2n(2n+1)}\right)\left[\alpha g(\beta_5,\beta_1) - \frac{1}{2n}S(\beta_5,\beta_1)\right] = 2n\alpha^2\lambda_3 g(\beta_1,\beta_5). \tag{28}$$

If necessary arrangements are made, we obtain

$$S(\beta_5,\beta_1) = \left[\frac{4n^2\alpha^2(2n+1)(1-2n\lambda_3) - 2n\alpha r}{2n\alpha(2n+1) - r}\right] g(\beta_5,\beta_1).$$

This completes the proof. □

Corollary 3.3. *If a $(2n+1)-$ dimensional almost $C(\alpha)-$manifold satisfies the curvature condition*

$$\tilde{Z}(\beta_3,\beta_1).P = \lambda_4 Q(g,P),$$

then the almost $C(\alpha)-$manifold is an Einstein manifold provided $r \neq 2n(2n+1)\alpha$.

Let us as the third examine a special curvature condition established on the concircular curvature tensor itself. Let us state and prove the following theorem.

Theorem 3.4. *If a $(2n+1)-$ dimensional almost $C(\alpha)-$manifold satisfies the curvature condition*

$$\tilde{Z}(\beta_3,\beta_1).\tilde{Z} = \lambda_5 Q\left(S,\tilde{Z}\right),$$

then the almost $C(\alpha)-$manifold is either co-Kaehler manifold or $\lambda_5 = 0$.

Proof. Let's assume that a $(2n+1)-$dimensional almost $C(\alpha)-$manifold satisfies the curvature condition

$$\left(\tilde{Z}(\beta_3,\beta_1).\tilde{Z}\right)(\beta_2,\beta_5,\beta_4) = \lambda_5 Q\left(S,\tilde{Z}\right)(\beta_2,\beta_5,\beta_4;\beta_3,\beta_1),$$

for each $\beta_1, \beta_2, \beta_5, \beta_4, \beta_3 \in \chi(M)$. In this situation, we can write

(29)
$$\tilde{Z}(\beta_3,\beta_1)\tilde{Z}(\beta_2,\beta_5)\beta_4 - \tilde{Z}\left(\tilde{Z}(\beta_3,\beta_1)\beta_2,\beta_5\right)\beta_4 - \tilde{Z}\left(\beta_2,\tilde{Z}(\beta_3,\beta_1)\beta_5\right)\beta_4$$
$$-\tilde{Z}(\beta_2,\beta_5)\tilde{Z}(\beta_3,\beta_1)\beta_4 = -\lambda_5\left\{S(\beta_1,\beta_2)\tilde{Z}(\beta_3,\beta_5)\beta_4 - S(\beta_3,\beta_2)\tilde{Z}(\beta_1,\beta_5)\beta_4\right.$$
$$+S(\beta_1,\beta_5)\tilde{Z}(\beta_2,\beta_3)\beta_4 - S(\beta_3,\beta_5)\tilde{Z}(\beta_2,\beta_1)\beta_4 + S(\beta_1,\beta_4)\tilde{Z}(\beta_2,\beta_5)\beta_3$$
$$\left.-S(\beta_3,\beta_4)\tilde{Z}(\beta_2,\beta_5)\beta_1\right\}.$$

If we choose $\beta_3 = \xi$ in (29) and make use of (12), (13), (14), we obtain

$$\tilde{Z}(\xi,\beta_1)R(\beta_2,\beta_5)\beta_4 - \tilde{Z}(R(\xi,\beta_1)\beta_2,\beta_5)\beta_4 - \tilde{Z}(\beta_2,R(\xi,\beta_1)\beta_5)\beta_4$$
$$-\tilde{Z}(\beta_2,\beta_5)R(\xi,\beta_1)\beta_4 + \left(\tfrac{r}{2n(2n+1)}\right)\left[g(\beta_2,\beta_4)\tilde{Z}(\xi,\beta_1)\beta_5\right.$$
$$-g(\beta_5,\beta_4)\tilde{Z}(\xi,\beta_1)\beta_2 + g(\beta_1,\beta_2)\tilde{Z}(\xi,\beta_5)\beta_4 - \eta(\beta_2)\tilde{Z}(\beta_1,\beta_5)\beta_4$$
$$+g(\beta_1,\beta_5)\tilde{Z}(\beta_2,\xi)\beta_4 - \eta(\beta_5)\tilde{Z}(\beta_2,\beta_1)\beta_4 + g(\beta_1,\beta_4)\tilde{Z}(\beta_2,\beta_5)\xi$$
$$\left.-\eta(\beta_4)\tilde{Z}(\beta_2,\beta_5)\beta_1\right] = -\lambda_5\left\{\left(\alpha - \tfrac{r}{2n(2n+1)}\right)[S(\beta_1,\beta_2)g(\beta_5,\beta_4)\xi\right. \tag{30}$$
$$-S(\beta_1,\beta_2)\eta(\beta_4)\beta_5 - S(\beta_1,\beta_5)g(\beta_2,\beta_4)\xi + S(\beta_1,\beta_5)\eta(\beta_4)\beta_2$$
$$+S(\beta_1,\beta_4)\eta(\beta_5)\beta_2 - S(\beta_1,\beta_4)\eta(\beta_2)\beta_5] - 2n\alpha\eta(\beta_2)\tilde{Z}(\beta_1,\beta_5)\beta_4$$
$$\left.-2n\alpha\eta(\beta_5)\tilde{Z}(\beta_2,\beta_1)\beta_4 - 2n\alpha\eta(\beta_4)\tilde{Z}(\beta_2,\beta_5)\beta_1\right\}$$

If we choose $\beta_2 = \xi$ in (30) and make use of (12), (13), we have

$$\left(\alpha - \tfrac{r}{2n(2n+1)}\right)[R(\beta_1,\beta_5)\beta_4 - \alpha g(\beta_5,\beta_4)\beta_1 + \alpha g(\beta_1,\beta_4)\beta_5]$$
$$= -\lambda_5\left\{\left(\alpha - \tfrac{r}{2n(2n+1)}\right)[2n\alpha g(\beta_5,\beta_4)\eta(\beta_1) - 2n\alpha g(\beta_1,\beta_4)\eta(\beta_5)\xi\right.$$
$$+2n\alpha\eta(\beta_5)\eta(\beta_4)\beta_1 + S(\beta_1,\beta_4)\eta(\beta_5)\xi - S(\beta_1,\beta_4)\beta_5 \tag{31}$$
$$-2n\alpha g(\beta_1,\beta_5)\eta(\beta_4)\xi] - 2n\alpha R(\beta_1,\beta_5)\beta_4 + \tfrac{\alpha r}{2n+1}g(\beta_5,\beta_4)\beta_1$$
$$\left.-\tfrac{\alpha r}{2n+1}g(\beta_1,\beta_4)\beta_5\right\}.$$

If we take inner product both sides of (31) by $\xi \in \chi(M)$ and then choose $\beta_4 = \xi$, we obtain

$$2n\alpha\lambda_5 \left(\alpha - \frac{r}{2n(2n+1)}\right)[g(\beta_5, \beta_1) - \eta(\beta_5)\eta(\beta_1)]$$

This completes the proof. □

COROLLARY 3.4. *Let M be $(2n+1)$ dimensional almost $C(\alpha)-$manifold provided*

$$\tilde{Z}(\beta_3, \beta_1).\tilde{Z} = \lambda_5 Q\left(S, \tilde{Z}\right).$$

Then M is a real space form with $c = \alpha$ if and only if $\lambda_5 = 0$.

THEOREM 3.5. *If a $(2n+1)-$ dimensional almost $C(\alpha)-$manifold satisfies the curvature condition*

$$\tilde{Z}(\beta_3, \beta_1).\tilde{Z} = \lambda_6 Q\left(g, \tilde{Z}\right),$$

then the almost $C(\alpha)-$manifold is either co-Kaehler manifold or $\lambda_6 = 0$.

COROLLARY 3.5. *Let M be $(2n+1)$ dimensional almost $C(\alpha)-$manifold provided*

$$\tilde{Z}(\beta_3, \beta_1).\tilde{Z} = \lambda_6 Q\left(g, \tilde{Z}\right).$$

Then M is a real space form with $c = \alpha$ if and only if $\lambda_6 = 0$.

Let us as the finally examine a special curvature condition established between the concircular curvature tensor and the Ricci curvature tensor. Let us state and prove the following theorem.

THEOREM 3.6. *If a $(2n+1)-$ dimensional almost $C(\alpha)-$manifold satisfies the curvature condition*

$$\tilde{Z}(\beta_1, \beta_2).S = \lambda_7 Q(g, S),$$

then the almost $C(\alpha)-$manifold is an Einstein manifold.

PROOF. Let's assume that a $(2n+1)-$dimensional almost $C(\alpha)-$manifold satisfies the curvature condition

$$\left(\tilde{Z}(\beta_1, \beta_2).S\right)(\beta_5, \beta_4) = \lambda_7 Q(g, S)(\beta_5, \beta_4; \beta_1, \beta_2),$$

for each $\beta_1, \beta_2, \beta_5, \beta_4 \in \chi(M)$. This mean

$$\begin{aligned} &-S\left(\tilde{Z}(\beta_1, \beta_2)\beta_5, \beta_4\right) - S\left(\beta_5, \tilde{Z}(\beta_1, \beta_2)\beta_4\right) = -\lambda_7 \{g(\beta_2, \beta_5)S(\beta_1, \beta_4) \\ &-g(\beta_1, \beta_5)S(\beta_2, \beta_4) + g(\beta_2, \beta_4)S(\beta_5, \beta_1) - g(\beta_1, \beta_4)S(\beta_5, \beta_2)\}. \end{aligned} \tag{32}$$

If we choose $\beta_1 = \xi$ in (32) and make use of (12), we have
(33)

$$2n\alpha^2 g(\beta_2,\beta_5)\eta(\beta_4) - \alpha\eta(\beta_5) S(\beta_2,\beta_4) - \left(\frac{r}{2n(2n+1)}\right)[2n\alpha g(\beta_2,\beta_5)\eta(\beta_4)$$

$$-\eta(\beta_5) S(\beta_2,\beta_4)] + 2n\alpha^2 g(\beta_2,\beta_4)\eta(\beta_5) - \alpha\eta(\beta_4) S(\beta_2,\beta_5)$$

$$-\left(\frac{r}{2n(2n+1)}\right)[2n\alpha g(\beta_2,\beta_4)\eta(\beta_5) - \eta(\beta_4) S(\beta_2,\beta_5)]$$

$$= -\lambda_7\{2n\alpha g(\beta_2,\beta_5)\eta(\beta_4) - \eta(\beta_5) S(\beta_2,\beta_4)$$

$$+2n\alpha g(\beta_2,\beta_4)\eta(\beta_5) - \eta(\beta_4) S(\beta_5,\beta_2)\}.$$

If we choose $\beta_5 = \xi$ in (33), we obtain

$$S(\beta_2,\beta_4) = 2n\alpha g(\beta_2,\beta_4).$$

This completes the proof. □

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