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ON REGULAR CLOSED SETS IN IDEAL NANO TOPOLOGICAL SPACES

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ABSTRACT. The concepts of regular-nI-closed sets in perfect nano topological spaces are introduced, and their characteristics are examined, in this study.

1. Introduction

An ideal I [**13**] on a space (X, τ) is a non-empty collection of subsets of X which satisfies the following conditions.

(1) $A \in I$ and $B \subset A$ imply $B \in I$ and
(2) $A \in I$ and $B \in I$ imply $A \cup B \in I$.

Given a space (X, τ) with an ideal I on X if $\wp(X)$ is the set of all subsets of X, a set operator $(.)^\star : \wp(X) \to \wp(X)$, called a local function of A with respect to τ and I is defined as follows: for $A \subset X$, $A^\star(I, \tau) = \{x \in X : U \cap A \notin I$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau : x \in U\}$ [**2**]. The closure operator defined by $cl^\star(A) = A \cup A^\star(I, \tau)$ [**12**] is a Kuratowski closure operator which generates a topology $\tau^\star(I, \tau)$ called the $\star$-topology which is finer then τ. We will simply write $A^\star$ for $A^\star(I, \tau)$ and $\tau^\star$ for $\tau^\star(I, \tau)$. If I is an ideal on X, then (X, τ, I) is called an ideal topological space or an ideal space.

In this paper, we introduce the notions of regular-nI-closed sets in ideal nano topological spaces and investigate their properties.

2. Preliminaries

DEFINITION 2.1. [**8**] *Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation.*

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Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space. Let $X \subseteq U$.

(1) *The lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and it is denoted by $L_R(X)$. That is, $L_R(X) = \bigcup_{x \in U}\{R(x) : R(x) \subseteq X\}$, where R(x) denotes the equivalence class determined by x.*
(2) *The upper approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by $U_R(X)$. That is, $U_R(X) = \bigcup_{x \in U}\{R(x) : R(x) \cap X \neq \phi\}$.*
(3) *The boundary region of X with respect to R is the set of all objects, which can be classified neither as X nor as not - X with respect to R and it is denoted by $B_R(X)$. That is, $B_R(X) = U_R(X) - L_R(X)$.*

Definition 2.2. [**3**] *Let U be the universe, R be an equivalence relation on U and $\tau_R(X) = \{U, \phi, L_R(X), U_R(X), B_R(X)\}$ where $X \subseteq U$. Then R(X) satisfies the following axioms:*

(1) *U and $\phi \in \tau_R(X)$,*
(2) *The union of the elements of any sub collection of $\tau_R(X)$ is in $\tau_R(X)$,*
(3) *The intersection of the elements of any finite subcollection of $\tau_R(X)$ is in $\tau_R(X)$.*

Thus $\tau_R(X)$ is a topology on U called the nano topology with respect to X and $(U, \tau_R(X))$ is called the nano topological space. The elements of $\tau_R(X)$ are called nano-open sets (briefly n-open sets). The complement of a n-open set is called n-closed.

In the rest of the paper, we denote a nano topological space by $(U, \mathcal{N})$, where $\mathcal{N} = \tau_R(X)$. The nano-interior and nano-closure of a subset O of U are denoted by $I_n(O)$ and $C_n(O)$, respectively.

A nano topological space $(U, \mathcal{N})$ with an ideal I on U is called [**5**] an ideal nano topological space and is denoted by $(U, \mathcal{N}, I)$. $G_n(x) = \{G_n \mid x \in G_n, G_n \in \mathcal{N}\}$, denotes [**5**] the family of nano open sets containing x.

In future an ideal nano topological spaces $(U, \mathcal{N}, I)$ is referred as a space.

Definition 2.3. [**5**] *Let $(U, \mathcal{N}, I)$ be a space with an ideal I on U. Let $(.)_n^\star$ be a set operator from $\wp(U)$ to $\wp(U)$ ($\wp(U)$ is the set of all subsets of U).*

For a subset $O \subseteq U$, $O_n^\star(I, \mathcal{N}) = \{x \in U : G_n \cap O \notin I$, for every $G_n \in G_n(x)\}$ is called the nano local function (briefly, n-local function) of A with respect to I and $\mathcal{N}$. We will simply write $O_n^\star$ for $O_n^\star(I, \mathcal{N})$.

Theorem 2.1. [**5**] *Let $(U, \mathcal{N}, I)$ be a space and O and B be subsets of U. Then*

(1) $O \subseteq B \Rightarrow O_n^\star \subseteq B_n^\star$,
(2) $O_n^\star = C_n(O_n^\star) \subseteq C_n(O)$ ($O_n^\star$ *is a n-closed subset of* $C_n(O)$),
(3) $(O_n^\star)_n^\star \subseteq O_n^\star$,
(4) $(O \cup B)_n^\star = O_n^\star \cup B_n^\star$,
(5) $V \in \mathcal{N} \Rightarrow V \cap O_n^\star = V \cap (V \cap O)_n^\star \subseteq (V \cap O)_n^\star$,
(6) $J \in I \Rightarrow (O \cup J)_n^\star = O_n^\star = (O - J)_n^\star$.

THEOREM 2.2. [**5**] *Let* $(U, \mathcal{N}, I)$ *be a space with an ideal* I *and* $O \subseteq O_n^{\star}$, *then* $O_n^{\star} = C_n(O_n^{\star}) = C_n(O)$.

DEFINITION 2.4. [**7**] *A subset* A *of a space* $(U, \mathcal{N}, I)$ *is n⋆-dense in itself (resp. n⋆-perfect and n⋆-closed) if* $O \subseteq O_n^{\star}$ *(resp.* $O = O_n^{\star}$, $O_n^{\star} \subseteq O$*).*

The complement of a n⋆-closed set is said to be n⋆-open.

DEFINITION 2.5. [**4**] *A subset* O *of* U *in a nano topological space* $(U, \mathcal{N})$ *is called nano-codense (briefly n-codense) if* $U - O$ *is n-dense.*

DEFINITION 2.6. [**5**] *Let* $(U, \mathcal{N}, I)$ *be a space. The set operator* $C_n^{\star}$ *called a nano ⋆-closure is defined by* $C_n^{\star}(O) = O \cup O_n^{\star}$ *for* $O \subseteq U$.

It can be easily observed that $C_n^{\star}(O) \subseteq C_n(O)$.

THEOREM 2.3. [**6**] *In a space* $(U, \mathcal{N}, I)$, *if* O *and* B *are subsets of* U, *then the following results are true for the set operator n-cl**.

(1) $O \subseteq C_n^{\star}(O)$,
(2) $C_n^{\star}(\phi) = \phi$ *and* $C_n^{\star}(U) = U$,
(3) $If O \subset B$, *then* $C_n^{\star}(O) \subseteq C_n^{\star}(B)$,
(4) $C_n^{\star}(O) \cup C_n^{\star}(B) = C_n^{\star}(O \cup B)$.
(5) $C_n^{\star}(C_n^{\star}(O)) = C_n^{\star}(O)$.

DEFINITION 2.7. *A subset* O *of a space* $(U, \mathcal{N})$ *is said to be*

(1) *nano nowhere dense (resp. n-nowhere dense)* [**3**] *if* $I_n(C_n(O) = \phi$.
(2) *nano regular closed (resp. nr-closed)* [**3**] *if* $O = I_n(C_n(O))$.
(3) *nano* $\mathcal{A}$*-set (resp. n*$\mathcal{A}$*-set)* [**1**] *if* $O = S \cap K$ *where* S *is n-open and* K *is nr-closed.*

DEFINITION 2.8. *A subset* O *of a space* $(U, \mathcal{N}, I)$ *is said to be*

(1) *nano α-I-open (resp. α-nI-open)* [**10**] *if* $O \subseteq I_n(C_n^{\star}(I_n(O)))$.
(2) *nano semi-I-open (resp. semi-nI-open) if* $O \subseteq C_n^{\star}(I_n(O))$.
(3) *nano pre-I-open (resp. pre-nI-open)* [**10**] *if* $O \subseteq I_n(C_n^{\star}(O))$.
(4) *nano I-locally closed (resp. nI-locally closed)* [**9**] *if* $O = S \cap K$ *where* S *is n-open and* K *is n⋆-perfect.*
(5) *nano* t_α*-I-set (resp.* t_α*-nI-set)* [**11**] *if* $I_n(O) = I_n(C_n^{\star}(I_n(O)))$.
(6) *nano* $\mathcal{R}_\alpha$*-I-set (resp.* $\mathcal{R}_\alpha$*-nI-set)* [**11**] *if* $O = S \cap K$, *where* S *is n-open and* K *is* t_α*-nI-set.*

THEOREM 2.4. [**10**] *In an ideal nano space* $(U, \mathcal{N}, I)$,

(1) *every n-open set is α-nI-open.*
(2) *every α-nI-open set is pre-nI-open.*

3. Regular closed sets in ideal nano space

DEFINITION 3.1. *A subset* O *of an ideal nano space* $(U, \mathcal{N}, I)$, *is called a nano regular-I-closed (resp. regular-nI-closed) if* $O = (I_n(O))_n^{\star}$.

EXMAMPLE 3.1. *Let* $U = \{o_1, o_2, o_3, o_4\}$ *with* $U/R = \{\{o_2\}, \{o_4\}, \{o_1, o_3\}\}$ *and* $X = \{o_3, o_4\}$. *Then the nano topology* $\mathcal{N} = \{\phi, \{o_4\}, \{o_1, o_3\}, \{o_1, o_3, o_4\}, U\}$ *and* $I = \{\phi, \{o_3\}\}$. *Clearly the set* $\{\phi, \{o_2, o_4\}, \{o_1, o_2, o_3\}, U\}$ *is regular-nI-closed.*

PROPOSITION 3.1. *For a subset O of an ideal nano space $(U, \mathcal{N}, I)$, the following properties hold:*

(1) *If O is regular-nI-closed, then the set O is t_α-nI-open and semi-nI-open.*
(2) *If O is regular-nI-closed, then the set O is $n\star$-perfect.*

PROOF.

(1) Let O be a regular-nI-closed set. Then, we have $C_n^\star(I_n(O)) = I_n(O) \cup (I_n(O))_n^\star = I_n(O) \cup O = O$. Thus, $I_n(C_n^\star(I_n(O))) = I_n(O)$ and $O \subseteq C_n^\star(I_n(O))$. Therefore, O is t_α-nI-open and semi-nI-open.
(2) Let O be a regular-nI-closed set. Then we have $O = (I_n(O))_n^\star$. Since $I_n(O) \subseteq O$, $(I_n(O))_n^\star \subseteq O_n^\star$ by Theorem 2.1. Then we have $O = (I_n(O))_n^\star \subseteq O_n^\star$.
   On the other hand, by Theorem 2.1 it follows from $O = (I_n(O))_n^\star$ that $O_n^\star = ((I_n(O))_n^\star)_n^\star \subseteq (I_n(O))_n^\star = O$. Therefore, we obtain $O = O_n^\star$. This shows that O is $n\star$-perfect.

□

REMARK 3.1. The converses of Proposition 3.1 need not be true.

EXMAMPLE 3.2. *In Example 3.1,*

(1) *the set $\{o_4\}$ is t_α-nI-open and semi-nI-open but not regular-nI-closed.*
(2) *the set $\{o_2\}$ is $n\star$-perfect but not regular-nI-closed.*

PROPOSITION 3.2. *In an ideal nano topological spaces, if O is regular-nI-closed, then the set O is $n\star$-closed and $n\star$-dense.*

PROOF.
The proof is obvious from Proposition 3.1. □

EXMAMPLE 3.3. *In Example 3.1, the set $\{o_2\}$ is $n\star$-closed and $n\star$-dense but not regular-nI-closed.*

PROPOSITION 3.3. *In an ideal nano space $(U, \mathcal{N}, I)$, if O is regular-nI-closed set, then the set O is nr-closed.*

PROOF.
Let O be any regular-nI-closed set. Then we have $(I_n(O))_n^\star = O$. Thus, we obtain that $C_n(O) = C_n((I_n(O))_n^\star) = (I_n(O))_n^\star = O$ by Theorem 2.1. Additionally, by Theorem 2.1, we have $(I_n(O))_n^\star \subseteq C_n(I_n(O))$ and hence $O = (I_n(O))_n^\star \subseteq C_n(I_n(O)) \subseteq C_n(O) = O$. Then we have $O = C_n(I_n(O))$ and hence O is a nr-closed set. □

REMARK 3.2. The converse of Proposition 3.3 need not be true.

EXMAMPLE 3.4. *In Example 3.1, the set $\{o_4\}$ is nr-closed but not regular-nI-closed.*

PROPOSITION 3.4. *Let $(U, \mathcal{N}, I)$ be an ideal nano topological space and $I = \phi$ (or $\mathcal{K}$, where $\mathcal{K}$ is the ideal of all n-nowhere dense sets). Then a subset O of U is a regular-nI-closed set $\iff$ O is nr-closed.*

PROOF.
By Proposition 3.3, every regular-nI-closed set is nr-closed. If $I = \phi$ (resp. $\mathcal{K}$), then it is well-known $O_n^\star = C_n(O)$ (resp. $O_n^\star = C_n(I_n(C_n(O))))$. Therefore, we obtain $(I_n(O))_n^\star = C_n(I_n(O))$ (resp. $(I_n(O))_n^\star = C_n(I_n(C_n(I_n(O)))) = C_n(I_n(O))$. Thus, regular-$nI$-closed set and nr-closed are equivalent. □

REMARK 3.3. In an ideal nano space $(U, \mathcal{N}, I)$,

(1) If O is n-open set, then the set O is α-nI-open.
(2) every regular-nI-closed set is α-nI-open sets.

EXMAMPLE 3.5. *In Example 3.1,*

(1) *the* $\{o_1, o_2, o_3\}$ *is* α-nI*-open but not* n*-open.*
(2) *the* $\{o_4\}$ *is* α-nI*-open sets but not regular-*nI*-closed.*

4. On nano $\mathcal{R}_I$-set in ideal nano spaces

DEFINITION 4.1. *A subset* O *of a space* $(U, \mathcal{N})$*, is called a nano* $\mathcal{R}_I$*-set (or) nano* $\mathcal{O}_I$*-set (resp.* $n\mathcal{R}_I$*-set)* [**9**] *if* $O = S \cap K$ *where* S *is* n*-open and* K *is* $(I_n(K))_n^\star$.

EXMAMPLE 4.1. *In Example 3.1, the set* $\{\phi, \{o_4\}, \{o_1, o_3\}, \{o_2, o_4\}, \{o_1, o_2, o_3\}, \{o_1, o_3, o_4\}, U\}$ *is* $n\mathcal{R}_I$*-set.*

REMARK 4.1. In an ideal nano spaces, If O is a regular-nI-closed set, then O is $n\mathcal{R}_I$-set.

EXMAMPLE 4.2. *In Example 3.1, the set* $\{o_1, o_3\}$ *is* $n\mathcal{R}_I$*-set but not regular-*nI*-closed.*

PROPOSITION 4.1. *Let* $(U, \mathcal{N}, I)$ *be an ideal nano topological space and* O *is a subset of* U*. Then the following conditions hold:*

(1) *If* O *is* $n\mathcal{R}_I$*-set, then* O *is a* $\mathcal{R}_\alpha$-nI*-set and* nI*-locally closed set.*
(2) *If* O *is an* $n\mathcal{R}_I$*-set, then* O *is a* $n\mathcal{A}$*-set.*

PROOF.
This is an immediate consequence of Propositions 3.1 and 3.3. □

REMARK 4.2. The converses of Proposition 4.1 need not be true as the following Examples.

EXMAMPLE 4.3. *In Example 3.1,*

(1) *the set* $\{o_2\}$ *is a* $\mathcal{R}_\alpha$-nI*-set and* nI*-locally closed set but not* $n\mathcal{R}_I$*-set.*
(2) *the set* $\{o_4\}$ *is a* $n\mathcal{A}$*-setbut not* $n\mathcal{R}_I$*-set.*

PROPOSITION 4.2. *For a subset* O *of a ideal nano space* $(U, \mathcal{N}, I)$*, the following properties are equivalent:*

(1) O *is* n*-open set.*
(2) O *is* α-nI*-open set and* $n\mathcal{R}_I$*-set.*
(3) O *is a pre-*nI*-open set and* $n\mathcal{R}_I$*.*

Proof.

(1) $\Longrightarrow$ (2): Let O is a n-open set. Hence O is α-nI-open set by Theorem 2.4.

On the other hand, $O = O \cap U$ where O in n-open and U is a regular-nI-closed set. Hence O is $n\mathcal{R}_I$set.

(2) $\Longrightarrow$ (3): This is obvious since every α-nI-open set is pre-nIopen by Theorem 2.4.

(3) $\Longrightarrow$ (1): Let O be pre-nI-open and $n\mathcal{R}_I$-set. Then $O = S \cap K$, where S is n-open and K is $n\mathcal{R}_I$-set. Since O is pre-nI-open , we have $O = S \cap K \subseteq I_n(C_n^\star(S \cap K)) \subseteq I_n(C_n^\star(S) \cap C_n^\star(K))$. By Proposition 3.2, K is $n\star$-closed and $C_n^\star(K) = K$. Therefore, we have $I_n(C_n^\star(S) \cap C_n^\star(K)) = I_n(C_n^\star(S) \cap K) = I_n(C_n^\star(S)) \cap I_n(K)$ and $S \cap K \subseteq S \cap I_n(C_n^\star(S)) \cap I_n(K) = I_n(S \cap C_n^\star(S) \cap K) = I_n(S \cap K)$.

Consequently, we obtain $S \cap K \cap I_n(S \cap K)$ and $O = S \cap K$ is n-open.

□

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