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A NOTE ON CHROMATIC D-POLYNOMIALS OF GRAPHS

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ABSTRACT. Graph colouring problem is to assign colours to certain elements of a graph subject to certain constraints. A proper vertex colouring of a graph is an assignment of colours to the vertices of the graph so that no two adjacent vertices have the same colour. In this note, we determine the two types of chromatic D-polynomials of a star graph, the line graph, and line cut graph of a star graph.

1. Introduction

The concept of *distance* in graph theory is associated with the number of edges in the shortest path between two vertices of a graph. A graph is said to be *connected* if there is a path between any pair of its vertices. If the two vertices concerned belong to different connected components of a graph, then the distance is usually defined to be $infinite$. For undefined graph theoretic terminologies and notions, we refer to [**1**].

Graph colouring problem is to assign colours to certain elements of a graph subject to certain constraints. By a graph colouring we mean vertex colouring. A proper vertex colouring of a graph is defined in [**4**] as follows.

A proper vertex colouring of a graph G is an assignment $\phi : V(G) \rightarrow \zeta$ of the vertices of G, where $\zeta = \{c_1, c_2, c_3, \ldots, c_l\}$ is a set of colours to the vertices of G so that no two adjacent vertices have the same colour. The *chromatic number* $\chi(G)$ of G is the smallest number of colours needed to colour the vertices of G so that no two adjacent vertices share the same colour. The set of all vertices of G which

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have the colour c_i is called the *colour class* of that colour c_i in G. Other types of colourings on graphs also exist, most notably edge colourings that may be subject to various constraints. A vertex colouring consisting of the colours with minimum subscripts may be called a *minimum parameter colouring* [**2**].

Kok et al. [**3**] introduced the concepts of χ^-- *colouring* and χ^+- *colouring* of a graph G as follows:

The vertices of G are coloured in such a way that the color c_1 is assigned to maximum possible number of vertices, then the color c_2 is assigned to maximum possible number of remaining uncoloured vertices and proceed in this way until all vertices are coloured. This type of colouring is called χ^-- *colouring* of G. In a similar manner, if c_l is assigned to maximum possible number of vertices, then the color c_{l-1} is assigned to maximum possible number of remaining uncoloured vertices and proceed in this way until all vertices are coloured. This type of colouring is called χ^+- *colouring* of G.

Motivated by the two types of colouring mentioned above, Sudev et al. [**4**] first introduced a new polynomial called the *chromatic D-polynomial* with respect to a proper vertex colouring of G; and then introduced two types of chromatic D-polynomials corresponding to χ^-- colouring and χ^+- colouring of G.

Definition 1.1. *A function $\varphi : V(G) \to \{1, 2, 3, \dots, l\}$ such that $\varphi(v_i) = s$ if and only if $\phi(v_i) = c_s, c_s \in \zeta$, where $l = \chi(G)$.*

Definition 1.2. *Let G be a connected graph with chromatic number $\chi(G)$. Then for real variables x and y, the chromatic D-polynomial $D_\chi(G; x, y)$ of G is defined as*

$$D_\chi(G; x, y) = \sum_{v_i, v_j \in V(G)} d(v_i, v_j) x^{\varphi(v_i)} y^{\varphi(v_j)}, i < j.$$

Taking note of Definition 1.2, the two types of chromatic D-polynomials corresponding to χ^-- colouring and χ^+- colouring of G are defined as follows:

Definition 1.3. *Let G be a connected graph; and χ^- and χ^+ be the minimal and maximal parameter colouring of G, respectively. Then for real variables x and y,*

(i) the χ^-- chromatic D-polynomial $D_{\chi^-}(G; x, y)$ of G is

$$D_{\chi^-}(G; x, y) = \sum_{v_i, v_j \in V(G)} d(v_i, v_j) x^{\varphi_{\chi^-}(v_i)} y^{\varphi_{\chi^-}(v_j)}$$

(ii) the χ^+- chromatic D-polynomial $D_{\chi^+}(G; x, y)$ of G is

$$D_{\chi^+}(G; x, y) = \sum_{v_i, v_j \in V(G)} d(v_i, v_j) x^{\varphi_{\chi^+}(v_i)} y^{\varphi_{\chi^+}(v_j)}$$

The chromatic D-polynomials of paths, cycles, complete graph, complete bipartite graph, wheel graph, and double wheel graphs are computed in [**4**]. Recently, Sudev et al. [**5**] computed the chromatic D-polynomials of Mycielskian of paths and cycles.

Motivated by the studies mentioned above, in this note, we determine the two types of chromatic D-polynomials of a star graph; the line graph and line cut graph of a star graph.

2. Chromatic D-polynomials of star graphs

The star graph $K_{1,n}$ is a tree on $n+1$ vertices. In a star graph one vertex has degree n and the remaining n vertices have degree 1.

In the next theorem, we obtain the result for chromatic D-polynomial of star graph using χ^-- colouring and χ^+- colouring.

THEOREM 2.1. *Let $G = K_{1,n}, n \geqslant 3$, be a star graph. Then* $D_{\chi^-}(G; x, y) = nxy^2 + \binom{n}{2}xy$.

PROOF. Let $G = K_{1,n}, n \geqslant 3$, be a star graph. Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of G, where v_1 is the central vertex; and $v_2, v_3, \ldots, v_n$ are the pendant vertices of G. It is obvious that the star graph has chromatic number 2. Also, since the star graph has diameter 2, the distance $d(v_i, v_j)$ is either 1 or 2.

Now, according to χ^-- colouring, the vertices $v_2, v_3, \ldots, v_n$ are coloured with the colour c_1; and the vertex v_1 is coloured with the colour c_2. The possible colour pairs and their numbers in G in terms of the distances between them are listed in Table 1.

Distance $d(v_i, v_j)$	Colour pairs	Number of pairs
1	(c_1, c_2)	n
2	(c_1, c_1)	$\binom{n}{2}$

TABLE 1. Table (CDP) for star graph with χ^-- colouring.

From Table 1, we have $D_{\chi^-}(G; x, y) = nxy^2 + \binom{n}{2}xy$. □

THEOREM 2.2. *Let $G = K_{1,n}, n \geqslant 3$, be a star graph. Then* $D_{\chi^+}(G; x, y) = nx^2y + \binom{n}{2}x^2y^2$.

PROOF. In the χ^-- colouring of a star graph, if we interchange the colours c_1 and c_2, we get χ^+- colouring. The possible colour pairs and their numbers in G in terms of the distances between them are listed in Table 2.

Distance $d(v_i, v_j)$	Colour pairs	Number of pairs
1	(c_2, c_1)	n
2	(c_2, c_2)	$\binom{n}{2}$

TABLE 2. Table (CDP) for star graph with χ^+- colouring.

From Table 2, we have $D_{\chi^+}(G; x, y) = nx^2y + \binom{n}{2}x^2y^2$. □

3. Chromatic D-polynomials of the line graph and line cut graph of a star graph

There are many graph operators (or graph valued functions) with which one can construct a new graph from a given graph, such as the line graphs, the line cut graphs, the total graphs, and their generalizations.

DEFINITION 3.1. *The line graph of a graph G, written $L(G)$, is the graph whose vertices are the edges of G, with two vertices of $L(G)$ adjacent whenever the corresponding edges of G have a vertex in common.*

DEFINITION 3.2. *The line cut graph of G, written $L_c(G)$, is the graph whose vertices are the edges and cut-vertices of G, with two vertices of $L_c(G)$ adjacent whenever the corresponding edges of G have a vertex in common; or one corresponds to an edge e_i of G and the other corresponds to a cut-vertex c_j of G such that e_i is incident with c_j.*

Clearly, $L(G) \subseteq L_c(G)$, where $\subseteq$ is the subgraph notation.

The following Theorem in [**4**] determines the chromatic D-polynomial of a complete graph $K_n, n \geqslant 3$.

THEOREM 3.1. *Let K_n be a complete graph on $n \geqslant 3$ vertices. Then,*

$$D_{\chi^-}(K_n; x, y) = \sum_{i=1}^{n-1} x^i y^{i+1} + x^n y$$

$$D_{\chi^+}(K_n; x, y) = \sum_{i=1}^{n-1} x^{i+1} y^i + x y^n$$

We use Theorem 3.1 to obtain the result for chromatic D-polynomial of line graph and line cut graph of a star graph using χ^-- colouring and χ^+- colouring.

Since the line graph of a star graph is a complete graph of order n, the chromatic D-polynomial of line graph of a star graph follows Theorem 3.1. Further, since the line cut graph of a star graph is also a complete graph of order $n+1$, the chromatic D-polynomial of line cut graph of a star graph is given by

$$D_{\chi^-}(K_{n+1}; x, y) = \sum_{i=1}^{n} x^i y^{i+1} + x^n y$$

$$D_{\chi^+}(K_{n+1}; x, y) = \sum_{i=1}^{n} x^{i+1} y^i + x y^n.$$

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