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NEW DISCONTINUITY AND FIXED DISC RESULTS VIA MEIR-KEELER AND CARISTI TECHNIQUES ON METRIC SPACES

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Abstract. Metric fixed-point theory has been extensively studied with effective approaches. There are some open problems about fixed-point theory. One of them is the Rhoades' discontinuity problem and another is the fixed-circle (or fixed-figure) problem. In this paper, we focus on these two open problems on metric spaces. To give some solutions, we combine Caristi and Meir-Keeler techniques. So, we present new answers to the stated problems.

1. Introduction

Fixed-point theory says that a self-mapping $f : X \to X$ has at least one fixed point. This theory has been studied on different areas such as mathematics, engineering, applied sciences etc. Metric fixed-point theory was started with Banach contraction principle and this theory has been generalized with some techniques.

	Some Generalization Techniques	
$\swarrow$	$\downarrow$	$\searrow$
To generalize the used contractive condition	To generalize the used metric space	A geometric approach: Fixed-circle problem

Also, in the literature, there are some open problems related to the fixed point of a self-mapping $f : X \to X$. For example,

(1) What are the contractive conditions which are strong enough to generate a fixed point but which do not force the map to be continuous at fixed point [**17**]?

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(2) What are the geometric properties of fixed point set in which case a self-mapping has more than one fixed point [9]?

Under the first open question, some solutions were given using different approaches. For example, in [4], Bisht and Rakočević obtained a solution using the number

$$m_6(x,y) = \max\left\{ \begin{array}{c} d(x,y), ad(x,fx) + (1-a)d(y,fy), \\ (1-a)d(x,fx) + ad(y,fy), \frac{b[d(x,fy)+d(y,fx)]}{2} \end{array} \right\},$$

where $0 < a < 1$ and $0 \leqslant b < 1$ as follows:

THEOREM 1.1. [4] *Let (X,d) be a complete metric space. Let f be a self-mapping on X such that for any $x, y \in X$*

(i) for a given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $\varepsilon < m_6(x,y) < \varepsilon + \delta$ implies $d(fx, fy) \leqslant \varepsilon$,

(ii) $d(fx, fy) < m_6(x,y)$ whenever $m_6(x,y) > 0$.

Then f has a unique fixed point, say z, and $f^n x \to z$ for each $x \in X$. Moreover, f is continuous at z if and only if $\lim\limits_{x\to z} m_6(x,z) = 0$.

A corrected version of Theorem 1.1 is the following [12]:

THEOREM 1.2. [12] *Let (X,d) be a complete metric space. Let f be a self-mapping on X such that for any $x, y \in X$*

(i) for a given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $\varepsilon < m_6(x,y) < \varepsilon + \delta$ implies $d(fx, fy) \leqslant \varepsilon$,

(ii) $d(fx, fy) \leqslant \phi(m_6(x,y))$, $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is such that $\phi(\mathsf{t}) < \mathsf{t}$ for each $\mathsf{t} > 0$.

Then f has a unique fixed point, say z, and $f^n x \to z$ for each $x \in X$. If $0 < a < 1$ then f is continuous at z and if $a = 0$ then f is continuous at z if and only if $\lim\limits_{x\to z} m_6(x,z) = 0$.

Many authors have studied this open problem via various aspects (for some examples, see [1], [2], [3], [5], [13], [14], [15], [16] and the references therein).

Under the second open question, the first solution was obtained in [9]. After this, new solutions were obtained using numerous techniques on both metric and generalized metric spaces (for example, see [6], [7], [10], [18], [19], [20] and the references therein).

In the light of these two open problems, we investigate some solutions on metric spaces. For this purpose, we inspire the used techniques in [20]. We prove a discontinuity fixed-point theorem with a corollary and obtain some fixed-circle (resp. fixed-disc) results on metric spaces. Also, we give two fixed-disc examples supporting our obtained results. Finally, we present an application to the Rectified Linear Unit Activation Functions (ReLU).

2. Main results

In this section,we give new solutions to the Rhoades' open problem and the fixed-circle problem on metric spaces using the Meir-Keeler type and Caristi type techniques and the number $m(x,y)$ defined as

$$m(x,y)=\max\left\{\begin{array}{c} d(x,y), ad(x,fx)+(1-a)d(y,fy), \\ (1-a)d(x,fx)+ad(y,fy), \frac{d(x,fy)+d(y,fx)}{2} \end{array}\right\}, 0\leqslant a<1.$$

2.1. New discontinuity results. We begin the following theorem.

THEOREM 2.1. *Let (X,d) be a complete metric space and f be a self-mapping on X. If the following condition holds for all $x,y\in X$*

(M) *Given $\varepsilon>0$ there exists a $\delta>0$ such that $d(x,fx)>0$ implies*

$$\varepsilon\leqslant[\varphi(x)-\varphi(fx)]m(x,y)<\varepsilon+\delta\Longrightarrow d(fx,fy)<\varepsilon,$$

then given $x\in X$, the sequence of iterates $\{f^nx\}$ is a Cauchy sequence and $\lim_{n\to\infty}f^nx=z$ for some $z\in X$.

PROOF. Using the condition (M), we obtain that if $d(x,fx)>0$ then

$$d(fx,fy)<[\varphi(x)-\varphi(fx)]m(x,y). \tag{2.1}$$

Let $x_0\in X$ and let us define a sequence $\{x_n\}$ in X by $x_n=fx_{n-1}$, that is , $x_n=f^nx_0$. If $x_n=x_{n+1}$ for some n then

$$x_n=x_{n+1}=x_{n+2}=...,$$

that is, $\{x_n\}=\{f^nx\}$ is a Cauchy sequence and x_n is a fixed point of f. Thus, without loss of generality, suppose that $x_n\neq x_{n+1}$ for each n and $c_n=d(x_{n-1},x_n)$. Using the inequality (2.1), we get

$$\begin{aligned}
c_{n+1} &= d(x_n,x_{n+1})=d(fx_{n-1},fx_n)\\
&< [\varphi(x_{n-1})-\varphi(x_n)]m(x_{n-1},x_n)\\
&= [\varphi(x_{n-1})-\varphi(x_n)]\max\left\{\begin{array}{c} d(x_{n-1},x_n), \\ ad(x_n,fx_n)+(1-a)d(x_{n-1},fx_{n-1}), \\ (1-a)d(x_n,fx_n)+ad(x_{n-1},fx_{n-1}), \\ \frac{d(x_{n-1},fx_n)+d(x_n,fx_{n-1})}{2} \end{array}\right\}\\
&= [\varphi(x_{n-1})-\varphi(x_n)]\max\left\{\begin{array}{c} d(x_{n-1},x_n), \\ ad(x_n,x_{n+1})+(1-a)d(x_{n-1},x_n), \\ (1-a)d(x_n,x_{n+1})+ad(x_{n-1},x_n), \\ \frac{d(x_{n-1},x_{n+1})}{2} \end{array}\right\}\\
&= [\varphi(x_{n-1})-\varphi(x_n)]d(x_{n-1},x_n). \qquad (2.2)
\end{aligned}$$

Using the inequality (2.2), we have

$$c_{n+1}=d(x_n,x_{n+1})<[\varphi(x_{n-1})-\varphi(x_n)]c_n$$

and so

$$0<\frac{c_{n+1}}{c_n}<\varphi(x_{n-1})-\varphi(x_n),$$

for each $n \in \mathbb{N}$. Therefore, the sequence $\{\varphi(x_n)\}$ is nonincreasing and positive whence it converges to some $f \geqslant 0$. For each $n \in \mathbb{N}$, we obtain

$$\begin{aligned}\sum_{m=1}^{n} \frac{c_{m+1}}{c_m} &< \sum_{m=1}^{n} [\varphi(x_{m-1}) - \varphi(x_m)] = \varphi(x_0) - \varphi(x_n) \\ &\to \varphi(x_0) - f < \infty \text{ as } n \to \infty\end{aligned}$$

and

$$\sum_{m=1}^{n} \frac{c_{m+1}}{c_m} < \infty \Longrightarrow \lim_{n\to\infty} \frac{c_{n+1}}{c_n} = 0.$$

Hence for $\alpha \in (0,1)$,there exists $n_0 \in \mathbb{N}$ such that

$$\frac{c_{n+1}}{c_n} \leqslant \alpha \text{ for all } n \geqslant n_0$$

and we have

$$d(x_n, x_{n+1}) \leqslant \alpha d(x_{n-1}, x_n) \text{ for all } n \geqslant n_0. \tag{2.3}$$

Now we show that $\{x_n\}$ is a Cauchy sequence and $\{x_n\}$ converges to some $a \in X$. From the inequality (2.3), we say that the sequence $\{d(x_n, x_{n+1})\}$ is bounded below an nonincreasing. Therefore, it converges to some $x \geqslant 0$. Since $\alpha < 1$,we easy prove $x = 0$.

For each $n_1, n_2 \in \mathbb{N}(n_1 > n_2)$, we get

$$d(x_{n_1}, x_{n_2}) \leqslant \sum_{m=n_2}^{n_1-1} d(x_m, x_{m+1}) \leqslant \frac{\alpha^{n_2}}{1-\alpha} d(x_0, x_1),$$

that is,

$$\lim_{n\to\infty} \sup \{d(x_{n_1}, x_{n_2}) : n_1 > n_2\} = 0.$$

Consequently, the sequence $\{x_n\}$ is Cauchy and there exists $a \in X$ such that $\{x_n\} \to a$ since (X, d) is complete metric space. □

A self-mapping f of a metric space X is called k-continuous, $k = 1, 2, 3, ...,$ if $f^k x_n \longrightarrow fa$ whenever $\{x_n\}$ is a sequence in X such that $f^{k-1} x_n \longrightarrow a$ (see [**13**] for more details). We note that the notion of k-continuity is stronger than orbital continuity.

THEOREM 2.2. *Let f satisfies the condition (M). If f is k-continuous, then f has a fixed point z. Also, f is continuous at z if and only if*

$$\lim_{x\to z} [\varphi(x) - \varphi(fx)] m(x, z) = 0.$$

PROOF. Let $x_0 \in X$ and let us define a sequence $\{x_n\}$ in X by $x_n = f x_{n-1}$, that is, $x_n = f^n x_0$. Using Theorem 2.1, we say that $\{x_n\}$ is a Cauchy sequence. Hence there exists a point $a \in X$ such that $\{x_n\} \to a$ since (X, d) is a complete metric space. Also we have $f^p x_n \longrightarrow a$ for each $p \geqslant 1$.

Let f be a k-continuity of f implies that $f^k x_n \longrightarrow fa$ since $f^{k-1}x_n \longrightarrow a$ and so we get $fa = a$ as $f^k x_n \longrightarrow a$. Therefore, a is a fixed point of f. It is also easy to prove that f is continuous at a if and only if

$$\lim_{x\to a}[\varphi(x) - \varphi(fx)]m(x,a) = 0.$$

□

Corollary 2.1. [**20**] *If the following condition holds for all $x, y \in X$*
(i) *Given $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x, fx) > 0$ implies*

$$\varepsilon \leqslant [\varphi(x) - \varphi(fx)]d(x,y) < \varepsilon + \delta \Longrightarrow d(fx, fy) < \varepsilon,$$

then given $x \in X$,the sequence of iterates $\{f^n x\}$is a Cauchy sequence and $\lim_{n\to\infty} f^n x = z$ *for some $z \in X$. If is k-continuous then f has a fixed point z.*

2.2. New fixed-disc results. In this section, let the number r be defined as

$$r = \inf\{d(x, fx) : x \neq fx, x \in X\} \tag{2.4}$$

and the function $\varphi : X \longrightarrow [0,\infty)$ defined as

$$\varphi(x) = d(x, fx), \tag{2.5}$$

for all $x \in X$.

At first, we recall the notions of a fixed circle and a fixed disc.

Let (X, d) be a metric space, $C_{x_0,r} = \{x \in X : d(x, x_0) = r\}$ a circle, $D_{x_0,r} = \{x \in X : d(x, x_0) \leqslant r\}$ a disc and $f : X \longrightarrow X$ a self-mapping.

(i) If $fx = x$ for every $x \in C_{x_0,r}$ then $C_{x_0,r}$ is called as the fixed circle of f [**9**].

(ii) If $fx = x$ for every $x \in D_{x_0,r}$ then $D_{x_0,r}$ is called as the fixed disc of f (see [**10**] and the references therein).

In the following theorem, we inspire from the Meir-Keeler type and Caristi type fixed-point theorems to obtain a new fixed-circle theorem.

Theorem 2.3. *Let (X, d) be a metric space, $f : X \longrightarrow X$ be a self-mapping, r be defined as in* (2.4) *and φ be defined as in* (2.5). *If there exists $x_0 \in X$ such that*
1. *$d(x_0, fx) \leqslant r$ and $0 \leqslant \varphi(x) \leqslant 1$ for all $x \in C_{x_0,r}$,*
2. *For all $x \in X$,*

$$\varphi(x) > 0 \Longrightarrow \varphi(x) < [\varphi(x) - \varphi(x_0)]m(x, x_0),$$

then $fx_0 = x_0$ and the circle $C_{x_0,r}$ is a fixed circle of f.

Proof. Let $r = 0$. Then we have $C_{x_0,r} = \{x_0\}$. On the contrary, we assume that $\varphi(x_0) > 0$. Using the condition (2), we get

$$\varphi(x_0) = d(x_0, fx_0) < [\varphi(x_0) - \varphi(x_0)]m(x_0, x_0) = 0,$$

a contradiction. Thus, it should be $\varphi(x_0) = 0$, that is,

$$fx_0 = x_0. \tag{2.6}$$

Let $r>0$ and $x\in C_{x_0,r}$. Now we show that f fixes the circle $C_{x_0,r}$. To do this,we suppose that $\varphi(x)>0$. Using the conditions (1), (2) and the equality (2.6) , we get

$$\begin{aligned}\varphi(x) &= d(x,fx)<[\varphi(x)-\varphi(x_0)]m(x,x_0)\\ &= d(x,fx)\max\left\{\begin{array}{c} d(x,x_0),ad(x,fx)+(1-a)d(x_0,fx_0),\\ (1-a)d(x,fx)+ad(x_0,fx_0),\frac{d(x,fx_0)+d(x_0,fx)}{2}\end{array}\right\}\\ &= d(x,fx)\max\left\{r,ad(x,fx),(1-a)d(x,fx),\frac{r+d(x_0,fx)}{2}\right\}\\ &\leqslant d(x,fx)\max\left\{r,ad(x,fx),(1-a)d(x,fx)\right\}. \qquad (2.7)\end{aligned}$$

Case 1: Let $\max\{r,ad(x,fx),(1-a)d(x,fx)\}=r$. Using the inequality (2.7) and the condition (1), we get

$$d(x,fx)<d(x,fx)r\leqslant d(x,fx)d(x,fx)=[d(x,fx)]^2,$$

a contradiction. Hence it should be $fx=x$.

Case 2: Let $\max\{r,ad(x,fx),(1-a)d(x,fx)\}=ad(x,fx)$. Using the inequality (2.7) and the condition (1), we get

$$d(x,fx)<d(x,fx)ad(x,fx)=a[d(x,fx)]^2,$$

a contradiction. Hence it should be $fx=x$.

Case 3: Let $\max\{r,ad(x,fx),(1-a)d(x,fx)\}=(1-a)d(x,fx)$. Using the inequality (2.7) and the condition (1), we get

$$d(x,fx)<d(x,fx)(1-a)d(x,fx)=(1-a)[d(x,fx)]^2,$$

a contradiction. Hence it should be $fx=x$.

Consequently, f fixes the circle $C_{x_0,r}$. □

As a natural consequence of Theorem 2.3, we give the following fixed-disc result.

COROLLARY 2.2. *Let (X,d) be a metric space, $f:X\longrightarrow X$ be a self-mapping, r be defined as in(2.4) and φ be defined as in (2.5). If there exists $x_0\in X$ such that*

1. *$d(x_0,fx)\leqslant r$ and $0\leqslant\varphi(x)\leqslant 1$ for all $x\in D_{x_0,r}$,*
2. *For all $x\in X$,*

$$\varphi(x)>0\Longrightarrow\varphi(x)<[\varphi(x)-\varphi(x_0)]m(x,x_0),$$

then $fx_0=x_0$ and the disc $D_{x_0,r}$ is a fixed disc of f.

EXAMPLE 2.1. *Let $X=\mathbb{R}$ be a usual metric space with the usual metric*

$$d(x,y)=|x-y|,$$

for all $x,y\in\mathbb{R}$. Let us define a self-mapping $f:\mathbb{R}\to\mathbb{R}$ as

$$fx=\left\{\begin{array}{ccc}1 & if & x=2\\ x & & otherwise\end{array}\right.,$$

for all $x\in\mathbb{R}$. We get

$$r=\inf\{d(x,fx):x=2\}=1.$$

Then f satisfies the conditions of Theorem 2.3 and Corollary 2.2 with $x_0 = 0$. Consequently, f fixes the circle $C_{0,1} = \{-1, 1\}$ and the disc $D_{0,1} = [-1, 1]$. Also, the number of fixed points is infinite.

Now we recall the following function family to obtain a new fixed-circle result.

DEFINITION 2.1. [**21**] *Let $\mathbb{F}$ be the family of all functions $F : (0, \infty) \longrightarrow \mathbb{R}$ such that*

(F_1) F is strictly increasing

(F_2) For each sequence $\{\alpha_n\}$ in $(0, \infty)$ the following holds

$$\lim_{n\to\infty} \alpha_n = 0 \text{ if and only if } \lim_{n\to\infty} F(\alpha_n) = -\infty,$$

(F_3) There exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Some examples of functions that satisfies the conditions (F_1), (F_2) and (F_3) are as follows:

(i) $F(x) = \ln(x)$,

(ii) $F(x) = \ln(x) + x$,

(iii) $F(x) = -\frac{1}{\sqrt{x}}$,

(iv) $F(x) = \ln(x^2 + x)$ (see [**21**] for more details).

THEOREM 2.4. *Let (X, d) be a metric space, $f : X \longrightarrow X$ be a self-mapping, r be defined as in* (2.4) *and φ be defined as in* (2.5)*. If there exists $x_0 \in X$, $t > 0$, and $F \in \mathbb{F}$ such that*

1. *$d(x_0, fx) \leqslant r$ and $0 \leqslant \varphi(x) \leqslant 1$ for all $x \in C_{x_0,r}$,*
2. *For all $x \in X$,*

$$\varphi(x) > 0 \Longrightarrow t + F(\varphi(x)) \leqslant F\left([\varphi(x) - \varphi(x_0)]m(x, x_0)\right),$$

then $fx_0 = x_0$ and the circle $C_{x_0,r}$ is a fixed circle of f.

PROOF. Let $r = 0$. Then we have $C_{x_0,r} = \{x_0\}$. As an immediate consequence of the condition (2), we get $fx_0 = x_0$. Now suppose that $r > 0$ and $x \in C_{x_0,r}$ be any point $x \neq fx$. Then using the conditions (1), (2) and the strictly increasing property of F, we find

$$\begin{aligned} t + F(\varphi(x)) &= t + F(d(x, fx)) \leqslant F([\varphi(x) - \varphi(x_0)]m(x, x_0)) \\ &= F\left(d(x, fx) \max\left\{\begin{array}{c} d(x, x_0), ad(x, fx) + (1-a)d(x_0, fx_0), \\ (1-a)d(x, fx) + ad(x_0, fx_0), \\ \frac{d(x, fx_0) + d(x_0, fx)}{2} \end{array}\right\}\right) \\ &= F\left(d(x, fx) \max\left\{r, ad(x, fx), (1-a)d(x, fx), \frac{r + d(x_0, fx)}{2}\right\}\right) \\ &\leqslant F\left(d(x, fx) \max\left\{r, ad(x, fx), (1-a)d(x, fx)\right\}\right). \end{aligned} \tag{2.8}$$

Case 1: Let $\max\{r, ad(x, fx), (1-a)d(x, fx)\} = r$. Using the inequality (2.8), we get

$$t + F(d(x, fx)) \leqslant F(rd(x, fx)) < F\left([d(x, fx)]^2\right),$$

a contradiction. Hence it should be $fx = x$.

Case 2: Let $\max\{r, ad(x,fx), (1-a)d(x,fx)\} = ad(x,fx)$. Using the inequality (2.8), we get

$$t + F(d(x,fx)) \leqslant F(d(x,fx)ad(x,fx)) < F\left(d\left[(x,fx)\right]^2\right),$$

a contradiction. Hence it should be $fx = x$.

Case 3: Let $\max\{r, ad(x,fx), (1-a)d(x,fx)\} = (1-a)d(x,fx)$. Using the inequality (2.8), we get

$$t + F(d(x,fx)) \leqslant F(d(x,fx)(1-a)d(x,fx)) < F\left(d\left[(x,fx)\right]^2\right),$$

a contradiction. Hence it should be $fx = x$.

Consequently, the circle $C_{x_0,r}$ is a fixed circle of f. □

Corollary 2.3. *Let (X,d) be a metric space, $f : X \longrightarrow X$ be a self-mapping, r be defined as in* (2.4) *and φ be defined as in* (2.5). *If there exist $x_0 \in X$, $t > 0$, and $F \in \mathbb{F}$ such that*

1. *$d(x_0, fx) \leqslant r$ and $0 \leqslant \varphi(x) \leqslant 1$ for all $x \in D_{x_0,r}$,*
2. *For all $x \in X$,*

$$\varphi(x) > 0 \Longrightarrow t + F(\varphi(x)) \leqslant F([\varphi(x) - \varphi(x_0)]m(x,x_0)),$$

then $fx_0 = x_0$ and the disc $D_{x_0,r}$ is a fixed disc of f.

Example 2.2. *Let us consider Example 2.1. Then f satisfies the conditions of Theorem 2.4 and Corollary 2.3 with $x_0 = 0$ and $F = \ln x$. Consequently, f fixes the circle $C_{0,1} = \{-1,1\}$ and the disc $D_{0,1} = [-1,1]$.*

3. An application to ReLU

Recently, the investigation to some applications of fixed-point theory is important to show the importance of the obtained theoretical results. For example, activation functions can be used in some applications. These functions are important in neural network. In this section, we give an example to fixed-circle and fixed-disc results using the Rectified Linear Unit Activation Functions (ReLU) defined as follows (see, [**8**] for more details):

$$ReLU(x) = \begin{cases} 0 & \text{if} \quad x \leqslant 0 \\ x & \text{if} \quad x > 0 \end{cases} = \max\{0, x\}.$$

Let us consider $X = [0,\infty) \cup \{-1\}$ with the usual metric. Then, we get

$$r = \inf\{|x - ReLU(x)| : x = -1\} = 1,$$

$$m(-1,1) = \left\{2, a, (1-a), \frac{3}{2}\right\} = 2.$$

The function $ReLU$ satisfies the conditions of Theorem 2.3 and 2.4 with $x_0 = 1$ and $F = \ln x$. Consequently, $C_{1,1} = \{0,2\}$ is a fixed circle of $ReLU$.

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