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QUASI IDEALS OF NEARNESS SEMIGROUPS

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Abstract. The aim of this paper is to study the notion of quasi-ideals in semigroups on weak nearness approximation spaces and explain some of the concepts and definitions. Also, it is given an example related to the subject. The features described in this study will contribute greatly to the theoretical developments of the nearness semigroup theory.

1. Introduction

Semigroups, as the basic algebraic structure were used in the areas of theoretical computer science as well as in the solutions of graph theory, optimization theory and in particular for studying automata, coding theory and formal languages. The subject of quasi-ideal is generalization of the concept of one sided ideal. In 1956, the concept of quasi ideals for semigroups [**11**] and rings was firstly defined by Otto Steinfeld. Many researchers studied important properties for quasi ideals. Readers can find several paper about quasi-ideals in [**12, 16, 2, 17**].

In 2002, Peters introduced near set theory that is a generalization of rough set theory [**7**]. In this theory, Peters defined an indiscernibility relation by using the features of the objects to determine the nearness of the objects [**8**]. Afterwards, he generalized approach theory of the nearness of non-empty sets resembling each other [**9, 10**]. In 2012, İnan and Öztürk investigated the concept of nearness semigroups [**1**] and other algebraic approaches of near sets in [**4**]. Also, Tekin defined quasi-nearness ideals in semirings on weak nearness approximaion spaces [**13**]. Also, readers can find other studies related to the nearness algebraic structures in [**5, 6, 14, 15**].

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In this paper, we introduced the concept of the quasi ideals of nearness semigroups theory and also studied some properties.

2. Preliminaries

A nearness approximation space is a tuple $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B), \nu_{N_r})$ where the nearness approximation space is defined with a set of perceived objects $\mathcal{O}$, set of probe functions $\mathcal{F}$ representing object features, indiscernibility relation $\sim_{B_r}$ defined relative to $B_r \subseteq B \subseteq \mathcal{F}$, collection of partitions (families of neighbour-hoods) $N_r(B)$, and overlap function ν_{N_r}.

Indiscernibility relation on $\mathcal{O}$ defined as

$$\sim_B = \{(x, x') \in \mathcal{O} \times \mathcal{O} \mid \triangle_{\varphi_i} = 0,\ \forall \varphi_i \in B\, B \subseteq \mathcal{F}\}$$

where description length is $i \leqslant \mid \Phi \mid$. In addition, $\sim_{B_r}$ is also indiscernibility relation determined by utilizing B_r.

Near equivalence class is stated as $[x]_{B_r} = \{x' \in \mathcal{O} \mid x \sim_{B_r} x'\}$. After getting near equivalence classes, quotient set $\mathcal{O}/\sim_{B_r} = \{[x]_{B_r} \mid x \in \mathcal{O}\} = \xi_{\mathcal{O},B_r}$ and set of partitions $N_r(B) = \{\xi_{\mathcal{O},B_r} \mid B_r \subseteq B\}$ can be found. By using near equivalence classes, $N_r(B)^* X = \bigcup\limits_{[x]_{B_r} \cap X \neq \varnothing} [x]_{B_r}$ upper approximation set can be attained.

Definition 2.1. [**1**] *Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B), \nu_{N_r})$ be a nearness approximation space and $S \subseteq \mathcal{O}$. If the following properties are satisfied, then S is called a near semigroup on nearness approximate approximation spaces.*

i) $xy \in N_r(B)^* S$ for all $x, y \in S$,

ii) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ property holds in $N_r(B)^* S$ for all $x, y \in S$.

Definition 2.2. [**1**] *Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B), \nu_{N_r})$ be a nearness approximation space, S be a near semigroup and I a nonempty subset of S. If $N_r(B)^*(I)$ is a left(right,two sided) ideal of S, then I is called a near left (right, two sided) ideal of S.*

Definition 2.3. [**4**] *Let $\mathcal{O}$ be a set of sample objects, $\mathcal{F}$ a set of the probe functions, $\sim_{B_r}$ an indiscernibility relation,and N_r a collection of partitions. Then, $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B))$ is called a weak nearness approximation space.*

Theorem 2.1. [**4**] *Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B))$ be a weak nearness approximation space and $X, Y \subset \mathcal{O}$. Then the following statements hold:*

i) $X \subseteq N_r(B)^* X$,

ii) $N_r(B)^* (X \cup Y) = N_r(B)^* X \cup N_r(B)^* Y$,

iii) $X \subseteq Y$ implies $N_r(B)^* X \subseteq N_r(B)^* Y$,

iv) $N_r(B)^* (X \cap Y) \subseteq N_r(B)^* X \cap N_r(B)^* Y$.

Lemma 2.1. [**3**] *Let S be a nearness semiring. The following properties hold:*

i) If $X, Y \subseteq S$, then $(N_r(B)^* X) + (N_r(B)^* Y) \subseteq N_r(B)^* (X + Y)$.

ii) If $X, Y \subseteq S$, then $(N_r(B)^* X)(N_r(B)^* Y) \subseteq N_r(B)^* (XY)$.

Definition 2.4. [**14**] *Let S be a semigroup and I be a subsemigroup of S on weak nearness approximation space.*

i) I is called a right(left) ideals of S if $IS \subseteq N_r(B)^* I (SI \subseteq N_r(B)^* I)$.

ii) I is called a upper-near right(left) ideals of S if $(N_r(B)^* I)S \subseteq N_r(B)^* I$ $(S(N_r(B)^* I) \subseteq N_r(B)^* I)$.

3. Quasi ideals of nearness semigroups

DEFINITION 3.1. *Let S be a semigroup on weak near approximation spaces, and A be a non-empty subset of S.*

i) A is called a subsemigroup of S if $AA \subseteq N_r(B)^* A$.

ii) A is called a upper-near subsemigroup of S if $(N_r(B)^* A)(N_r(B)^* A) \subseteq N_r(B)^* A$.

DEFINITION 3.2. *Let S be a nearness semigroup and Q be a non-empty subset of S.*

i) Q is said to be quasi-ideal of S if $QS \cap SQ \subseteq N_r(B)^* Q$.

ii) Q is said to be a quasi upper-near ideal of S if $(N_r(B)^* Q)S \cap S(N_r(B)^* Q) \subseteq N_r(B)^* Q$.

iii) Q is said to be an (m, n) quasi-ideal of S, if $QS^m \cap S^n Q \subseteq N_r(B)^* Q$.

LEMMA 3.1. *Let S be a nearness semigroup. If S is commutative, then each quasi ideal of S is two-sided ideal of S.*

PROOF. Let S be a commutative nearness semigroup and Q be a quasi ideal of S. In this case, $QS \cap SQ \subseteq N_r(B)^* Q$. Since S is commutative and $Q \subseteq S$, then $SQ = QS$. Thus, $SQ \subseteq N_r(B)^* Q$ and Q is a left ideal of S. Similarly, we can show that Q is a right-nearness ideal of S. In this way, each quasi-nearness ideal of S is a two-sided nearness ideal of S. □

EXAMPLE 3.1. *Let $\mathcal{O} = \{a, b, c, d, e, f, g, h, i, j, k\}$ be a set of perceptual objects where $B = \{\chi_1, \chi_2, \chi_3\} \subseteq \mathcal{F}$ is a set of probe functions and $S = \{d, f, g\} \subset \mathcal{O}$. For $r = 1$, values of the probe functions*

$$\chi_1 : \mathcal{O} \to V_1 = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5\},$$
$$\chi_2 : \mathcal{O} \to V_2 = \{\beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \},$$
$$\chi_3 : \mathcal{O} \to V_3 = \{\beta_2, \beta_3, \beta_4, \beta_5\}$$

are given in following table.

	a	b	c	d	e	f	g	h	i	j	k
χ_1	β_1	β_2	β_3	β_2	β_4	β_3	β_4	β_3	β_2	β_5	β_1
χ_2	β_2	β_3	β_4	β_5	β_4	β_4	β_6	β_5	β_2	β_5	β_3
χ_3	β_2	β_2	β_2	β_3	β_4	β_3	β_3	β_4	β_4	β_5	β_5

Now, we find the near equivalence classes according to the indiscernibility relation $\sim_{B_r}$ for $\mathcal{O}$.

$$\begin{aligned}
[a]_{\chi_1} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_1(a) = \beta_1\} = \{a, k\} \\
&= [k]_{\chi_1}, \\
[b]_{\chi_1} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_1(b) = \beta_2\} = \{b, d, i\} \\
&= [d]_{\chi_1} = [i]_{\chi_1}, \\
[c]_{\chi_1} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_1(c) = \beta_3\} = \{c, f, h\} \\
&= [f]_{\chi_1} = [h]_{\chi_1}, \\
[e]_{\chi_1} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_1(e) = \beta_4\} = \{e, g\} \\
&= [g]_{\chi_1}, \\
[j]_{\chi_1} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_1(j) = \beta_5\} = \{j\}.
\end{aligned}$$

Then, we get $\xi_{\chi_1} = \left\{[a]_{\chi_1}, [b]_{\chi_1}, [c]_{\chi_1}, [e]_{\chi_1}, [j]_{\chi_1}\right\}$.

$$\begin{aligned}
[a]_{\chi_2} &= \{x \in \mathcal{O} \mid \chi_2(x) = \chi_2(a) = \beta_2\} = \{a, i\} \\
&= [i]_{\chi_2}, \\
[b]_{\chi_2} &= \{x \in \mathcal{O} \mid \chi_2(x) = \chi_2(b) = \beta_3\} = \{b, k\} \\
&= [k]_{\chi_2}, \\
[c]_{\chi_2} &= \{x \in \mathcal{O} \mid \chi_2(x) = \chi_2(c) = \beta_4\} = \{c, e, f\} \\
&= [e]_{\chi_2} = [f]_{\chi_2}, \\
[d]_{\chi_2} &= \{x \in \mathcal{O} \mid \chi_2(x) = \chi_2(d) = \beta_5\} = \{d, h, j\} \\
&= [h]_{\chi_2} = [j]_{\chi_2}, \\
[g]_{\chi_2} &= \{x \in \mathcal{O} \mid \chi_2(x) = \chi_2(g) = \beta_6\} = \{g\}.
\end{aligned}$$

Thus, we have $\xi_{\chi_2} = \left\{[a]_{\chi_2}, [b]_{\chi_2}, [c]_{\chi_2}, [d]_{\chi_2}, [g]_{\chi_2}\right\}$.

$$\begin{aligned}
[a]_{\chi_3} &= \{x \in \mathcal{O} \mid \chi_3(x) = \chi_3(a) = \beta_2\} = \{a, b, c\} \\
&= [b]_{\chi_3} = [c]_{\chi_3}, \\
[d]_{\chi_3} &= \{x \in \mathcal{O} \mid \chi_3(x) = \chi_3(d) = \beta_3\} = \{d, f, g\} \\
&= [f]_{\chi_3} = [g]_{\chi_3}, \\
[e]_{\chi_3} &= \{x \in \mathcal{O} \mid \chi_3(x) = \chi_3(e) = \beta_4\} = \{e, h, i\} \\
&= [h]_{\chi_3} = [i]_{\chi_3}, \\
[j]_{\chi_3} &= \{x \in \mathcal{O} \mid \chi_3(x) = \chi_3(j) = \beta_5\} = \{j, k\} \\
&= [k]_{\chi_3}.
\end{aligned}$$

Hence we obtain $\xi_{\chi_3} = \left\{[a]_{\chi_3}, [d]_{\chi_3}, [e]_{\chi_3}, [j]_{\chi_3}\right\}$. *Therefore, for* $r = 1$, *a set of partitions of* $\mathcal{O}$ *is* $N_1(B) = \{\xi_{\chi_1}, \xi_{\chi_2}, \xi_{\chi_3}\}$. *Then we can write*

$$\begin{aligned} N_1(B)^* S &= \bigcup_{[x]_{\chi_i} \cap S \neq \varnothing} [x]_{\chi_i} \\ &= [b]_{\chi_1} \cup [c]_{\chi_1} \cup [e]_{\chi_1} \cup [c]_{\chi_2} \cup [d]_{\chi_2} \cup [g]_{\chi_2} \cup [d]_{\chi_3} \\ &= \{b, c, d, e, f, g, h, i, j\}. \end{aligned}$$

Considering the following table of operation:

$\circ$	d	f	g
d	e	g	f
f	g	d	e
g	f	e	d

In that case, $(S, \circ)$ *is a semigroup on* $\mathcal{O}$*. Next, we take* $Q = \{f, g\} \subseteq S$*.*

$$\begin{aligned} N_1(B)^* Q &= \bigcup_{[x]_{\chi_i} \cap Q \neq \varnothing} [x]_{\chi_i} \\ &= [c]_{\chi_1} \cup [e]_{\chi_1} \cup [c]_{\chi_2} \cup [g]_{\chi_2} \cup [d]_{\chi_3} \\ &= \{c, d, e, f, g, h\}. \end{aligned}$$

In this case, Q *satisfies the condition* $QS \cap SQ \subseteq N_r(B)^* Q$*. Thus,* Q *is a quasi-nearness ideal of* S*.*

Lemma 3.2. *Let* S *be a nearness semigroup. Each one or two-sided nearness ideal of* S *is a quasi-nearness ideal of* S*.*

Proof. Let Q be a left nearness ideal of S. In this case, $SQ \subseteq N_r(B)^* Q$ by Definition 2.4.(i). From here, we have $QS \cap SQ \subseteq SQ \subseteq N_r(B)^* Q$. Thus, $QS \cap SQ \subseteq N_r(B)^* Q$ and Q is a quasi-nearness ideal of S. It can be easily shown that if Q is a right nearness ideal of S, namely $QS \subseteq N_r(B)^* Q$, then Q is a quasi-nearness ideal of S. Hence, each one or two-sided ideal of S is a quasi-nearness ideal of S. $\square$

Theorem 3.1. *Let* S *be a nearness semigroup and* $\{Q_i | i \in I\}$ *be set of quasi-nearness ideals of* S *with index set* I*. If* $N_r(B)^* (\bigcap_{i \in I} Q_i) = \bigcap_{i \in I} N_r(B)^* Q_i$*, then* $\bigcap_{i \in I} Q_i = \emptyset$ *or* $\bigcap_{i \in I} Q_i$ *is a quasi-nearness ideal of* S*.*

Proof. Let $\bigcap_{i \in I} Q_i = Q$. Now, we demonstrate Q is either empty or a quasi-nearness ideal of S. Assume that Q is non-empty. Since Q_i is quasi-nearness ideals of S for $i \in I$, we have that $Q_i S \cap SQ_i \subseteq N_r(B)^* Q_i$ for all $i \in I$.

$$SQ = S(\bigcap_{i \in I} Q_i) = \bigcap_{i \in I} (SQ_i) \subseteq SQ_i$$

and

$$QS = (\bigcap_{i \in I} Q_i)S = \bigcap_{i \in I} (Q_i S) \subseteq Q_i S.$$

In this case, we get that

$$QS \cap SQ \subseteq Q_i S \cap SQ_i \subseteq N_r(B)^* Q_i, \forall i \in I.$$

Furthermore, $QS \cap SQ \subseteq N_r(B)^* Q$. Thus, Q is a quasi-nearness ideal of S. □

EXAMPLE 3.2. *Let $\mathcal{O} = \{a, b, c, d, e, f, g, h, i\}$ be a set of perceptual objects where $B = \{\chi_1, \chi_2, \chi_3\} \subseteq \mathcal{F}$ is a set of probe functions and $S = \{d, e, f\} \subset \mathcal{O}$. For $r = 1$, values of the probe functions*

$$\chi_1 : \mathcal{O} \to V_1 = \{\beta_1, \beta_2, \beta_3, \beta_4\},$$
$$\chi_2 : \mathcal{O} \to V_2 = \{\beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \},$$
$$\chi_3 : \mathcal{O} \to V_3 = \{\beta_1, \beta_2, \beta_3, \beta_4\}$$

are given in the following table.

	a	b	c	d	e	f	g	h	i
χ_1	β_1	β_2	β_4	β_3	β_1	β_2	β_1	β_4	β_4
χ_2	β_2	β_3	β_4	β_4	β_5	β_6	β_5	β_2	β_3
χ_3	β_1	β_2	β_1	β_3	β_4	β_4	β_3	β_1	β_2

Next, it can be found the near equivalence classes according to the indiscernibility relation $\sim_{B_r}$ for $\mathcal{O}$.

$$\begin{aligned}
[a]_{\chi_1} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_1(a) = \beta_1\} = \{a, e\} \\
&= [e]_{\chi_1}, \\
[b]_{\chi_1} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_1(b) = \beta_2\} = \{b, f\} \\
&= [f]_{\chi_1}, \\
[d]_{\chi_1} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_1(d) = \beta_3\} = \{d\}, \\
[h]_{\chi_1} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_1(h) = \beta_4\} = \{c, h, i\} \\
&= [h]_{\chi_1} = [i]_{\chi_1}.
\end{aligned}$$

Hence, we have the near equivalence classes $\xi_{\chi_1} = \left\{[a]_{\chi_1}, [b]_{\chi_1}, [d]_{\chi_1}, [h]_{\chi_1}\right\}$ for χ_1.

$$
\begin{aligned}
[a]_{\chi_2} &= \{x \in \mathcal{O} \mid \chi_2(x) = \chi_2(a) = \beta_2\} = \{a, h\} \\
&= [h]_{\chi_2}, \\
[b]_{\chi_2} &= \{x \in \mathcal{O} \mid \chi_2(x) = \chi_2(b) = \beta_3\} = \{b, i\} \\
&= [i]_{\chi_2}, \\
[c]_{\chi_2} &= \{x \in \mathcal{O} \mid \chi_2(x) = \chi_2(c) = \beta_4\} = \{c, d\} \\
&= [d]_{\chi_2}, \\
[e]_{\chi_2} &= \{x \in \mathcal{O} \mid \chi_2(x) = \chi_2(e) = \beta_5\} = \{e, g\} \\
&= [g]_{\chi_2}, \\
[f]_{\chi_2} &= \{x \in \mathcal{O} \mid \chi_2(x) = \chi_2(f) = \beta_6\} = \{f\}.
\end{aligned}
$$

Therefore, we get the near equivalence classes $\xi_{\chi_2} = \left\{ [a]_{\chi_2}, [b]_{\chi_2}, [c]_{\chi_2}, [e]_{\chi_2}, [f]_{\chi_2} \right\}$ *for* χ_2.

$$
\begin{aligned}
[a]_{\chi_3} &= \{x \in \mathcal{O} \mid \chi_3(x) = \chi_3(a) = \beta_1\} = \{a, c, h\} \\
&= [c]_{\chi_3} = [h]_{\chi_3}, \\
[b]_{\chi_3} &= \{x \in \mathcal{O} \mid \chi_3(x) = \chi_3(b) = \beta_2\} = \{b, i\} \\
&= [i]_{\chi_3}, \\
[d]_{\chi_3} &= \{x \in \mathcal{O} \mid \chi_3(x) = \chi_3(d) = \beta_3\} = \{d, g\} \\
&= [g]_{\chi_3}, \\
[e]_{\chi_3} &= \{x \in \mathcal{O} \mid \chi_3(x) = \chi_3(e) = \beta_4\} = \{e, f\} \\
&= [f]_{\chi_3}.
\end{aligned}
$$

Thereby, we obtain the near equivalence classes $\xi_{\chi_3} = \left\{ [a]_{\chi_3}, [b]_{\chi_3}, [d]_{\chi_3}, [e]_{\chi_3} \right\}$ *for* χ_3*, and so for* $r = 1$*, a set of partitions of* $\mathcal{O}$ *is* $N_1(B) = \{\xi_{\chi_1}, \xi_{\chi_2}, \xi_{\chi_3}\}$*. Then we can find*

$$
\begin{aligned}
N_1(B)^* S &= \bigcup_{[x]_{\chi_i} \cap S \neq \varnothing} [x]_{\chi_i} \\
&= [a]_{\chi_1} \cup [b]_{\chi_1} \cup [d]_{\chi_1} \cup [c]_{\chi_2} \cup [e]_{\chi_2} \cup [f]_{\chi_2} \cup [d]_{\chi_3} \cup [e]_{\chi_3} \\
&= \{a, b, c, d, e, f, g\}.
\end{aligned}
$$

Considering the following table of operation:

$\cdot$	d	e	f
d	e	d	c
e	d	e	f
f	c	f	e

In that case, $(S, \cdot)$ *is a semigroup on* $\mathcal{O}$*, and we take* $Q = \{e, f\} \subseteq S$*.*

$$N_1(B)^* Q = \bigcup_{[x]_{\chi_i} \cap Q \neq \varnothing} [x]_{\chi_i}$$
$$= [a]_{\chi_1} \cup [b]_{\chi_1} \cup [e]_{\chi_2} \cup [f]_{\chi_2} \cup [e]_{\chi_3}$$
$$= \{a, b, e, f, g\}.$$

It is seen for $d \in S$ *and* $f \in Q$, $\{d \cdot f\} \cap \{f \cdot d\} = c$ *and* $c \in QS \cap SQ$, *but* $c \notin N_r(B)^* Q$. *Therefore,* Q *does not satisfy the condition* $SQ \cap QS \subseteq N_r(B)^* Q$, *and* Q *is not a quasi-nearness ideal of* S.

Theorem 3.2. *Let* S *be a nearness semigroup,* L *be a left nearness ideal and* R *be a right nearness ideal of* S. *If* $N_r(B)^* L \cap N_r(B)^* R \subseteq N_r(B)^* (L \cap R)$, *then*
i) $RL \subseteq N_r(B)^* (L \cap R)$,
ii) $Q = L \cap R$ *is a quasi-nearness ideal of* S.

Proof. *i)* Let L be a left nearness ideal and R be a right nearness ideal of S. Since $R \subseteq S$ and L is a left nearness ideal of S, $RL \subseteq SL \subseteq N_r(B)^* L$. Similarly, since $L \subseteq S$ and R is a right nearness ideal of S, $RL \subseteq RS \subseteq N_r(B)^* R$. In this case, we get that $RL \subseteq N_r(B)^* L \cap N_r(B)^* R \subseteq N_r(B)^* (L \cap R)$ by hypothesis. Hence, $RL \subseteq N_r(B)^* (L \cap R)$.

ii) Let L be a left nearness ideal of S, R be a right nearness ideal of S and $Q = L \cap R$. We show that $QS \cap SQ \subseteq N_r(B)^* Q$. Since L is a left nearness ideal,

$$SQ = S(L \cap R) = SL \cap SR \subseteq SL \subseteq N_r(B)^* L.$$

Similarly, since R is a right nearness ideal of S,

$$QS = (L \cap R)S = LS \cap RS \subseteq RS \subseteq N_r(B)^* R.$$

From here, $SQ \cap QS \subseteq N_r(B)^* L \cap N_r(B)^* R \subseteq N_r(B)^* (L \cap R)$ by hypothesis. Hence, $SQ \cap QS \subseteq N_r(B)^* Q$.

□

Theorem 3.3. *Let* S *be a nearness semigroup and* A_i *be an* (m, n) *quasi-nearness ideals of* S. *If* $N_r(B)^* (\bigcap_{i \in I} A_i) = \bigcap_{i \in I} N_r(B)^* A_i$, *then the intersection of any set of* (m, n) *quasi-nearness ideals of* S *is a* (m, n) *quasi-nearness ideal of* S.

Proof. Let $\bigcap_{i \in I} A_i = A$. Now, we demonstrate A is either empty or a (m, n) quasi-nearness ideal of S. Suppose that A is non-empty. Since A_i is (m, n) quasi-nearness ideals of S for $i \in I$, we have that $A_i S^m \cap S^n A_i \subseteq N_r(B)^* A_i$ for all $i \in I$.

$$AS^m = (\bigcap_{i \in I} A_i) S^m = \bigcap_{i \in I} (A_i S^m) \subseteq A_i S^m.$$

and

$$S^n A = S^n (\bigcap_{i \in I} A_i) = \bigcap_{i \in I} (S^n A_i) \subseteq S^n A_i$$

In this case, we get that

$$AS^m \cap S^n A \subseteq A_i S^m \cap S^n A_i \subseteq N_r(B)^* A_i, \forall i \in I.$$

Furthermore, $AS^m \cap S^n A \subseteq N_r(B)^* A$. Thus, A is a (m,n) quasi-nearness ideal of S. □

Example 3.3. *Let $\mathcal{O} = \{a,b,c,d,e,f,g,h\}$ be a set of perceptual objects where $B = \{\chi_1,\chi_2,\chi_3\} \subseteq \mathcal{F}$ is a set of probe functions and $S = \{c,d,e\} \subset \mathcal{O}$. For $r = 1$, values of the probe functions*

$$\begin{aligned}
\chi_1 &: \mathcal{O} \to V_1 = \{\beta_1,\beta_2,\beta_3,\beta_4\},\\
\chi_2 &: \mathcal{O} \to V_2 = \{\beta_2,\beta_3,\beta_4,\beta_5,\beta_6,\},\\
\chi_3 &: \mathcal{O} \to V_3 = \{\beta_3,\beta_4,\beta_5,\beta_6\}
\end{aligned}$$

are given in the following table:

	a	b	c	d	e	f	g	h	i
χ_1	β_1	β_2	β_3	β_1	β_3	β_3	β_2	β_2	β_4
χ_2	β_2	β_3	β_4	β_5	β_3	β_6	β_5	β_5	β_2
χ_3	β_3	β_3	β_4	β_4	β_4	β_5	β_6	β_6	β_5

Now, we find the near equivalence classes according to the indiscernibility relation $\sim_{B_r}$ for $\mathcal{O}$.

$$\begin{aligned}
[a]_{\chi_1} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_1(a) = \beta_1\} = \{a,d\}\\
&= [d]_{\chi_1},\\
[b]_{\chi_1} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_1(b) = \beta_2\} = \{b,g,h\}\\
&= [g]_{\chi_1} = [h]_{\chi_1},\\
[c]_{\chi_1} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_1(c) = \beta_3\} = \{c,e,f\}\\
&= [e]_{\chi_1} = [f]_{\chi_1},\\
[i]_{\chi_1} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_1(i) = \beta_4\} = \{i\}.
\end{aligned}$$

Then, we get $\xi_{\chi_1} = \left\{[a]_{\chi_1}, [b]_{\chi_1}, [c]_{\chi_1}, [i]_{\chi_1}\right\}$.

$$\begin{aligned}
[a]_{\chi_2} &= \{x \in \mathcal{O} \mid \chi_2(x) = \chi_2(a) = \beta_2\} = \{a,i\}\\
&= [i]_{\chi_2},\\
[b]_{\chi_2} &= \{x \in \mathcal{O} \mid \chi_2(x) = \chi_2(b) = \beta_3\} = \{b,e\}\\
&= [e]_{\chi_2},\\
[c]_{\chi_2} &= \{x \in \mathcal{O} \mid \chi_2(x) = \chi_2(c) = \beta_4\} = \{c\},\\
[d]_{\chi_2} &= \{x \in \mathcal{O} \mid \chi_2(x) = \chi_2(d) = \beta_5\} = \{d,g,h\}\\
&= [g]_{\chi_2} = [h]_{\chi_2},\\
[f]_{\chi_2} &= \{x \in \mathcal{O} \mid \chi_2(x) = \chi_2(f) = \beta_6\} = \{f\}.
\end{aligned}$$

Thus, we have $\xi_{\chi_2} = \left\{ [a]_{\chi_2}, [b]_{\chi_2}, [c]_{\chi_2}, [d]_{\chi_2}, [f]_{\chi_2} \right\}$.

$$\begin{aligned}
[a]_{\chi_3} &= \{x \in \mathcal{O} \mid \chi_3(x) = \chi_3(a) = \beta_3\} = \{a, b\} \\
&= [b]_{\chi_3}, \\
[c]_{\chi_3} &= \{x \in \mathcal{O} \mid \chi_3(x) = \chi_3(c) = \beta_4\} = \{c, d, e\} \\
&= [d]_{\chi_3} = [e]_{\chi_3}, \\
[f]_{\chi_3} &= \{x \in \mathcal{O} \mid \chi_3(x) = \chi_3(f) = \beta_5\} = \{f, i\} \\
&= [i]_{\chi_3}, \\
[g]_{\chi_3} &= \{x \in \mathcal{O} \mid \chi_3(x) = \chi_3(g) = \beta_6\} = \{g, h\} \\
&= [h]_{\chi_3}.
\end{aligned}$$

Hence we obtain $\xi_{\chi_3} = \left\{ [a]_{\chi_3}, [c]_{\chi_3}, [f]_{\chi_3}, [g]_{\chi_3} \right\}$. *Therefore, for* $r = 1$, *a set of partitions of* $\mathcal{O}$ *is* $N_1(B) = \{\xi_{\chi_1}, \xi_{\chi_2}, \xi_{\chi_3}\}$. *Then we can write*

$$\begin{aligned}
N_1(B)^* S &= \bigcup_{[x]_{\chi_i} \cap S \neq \varnothing} [x]_{\chi_i} \\
&= [a]_{\chi_1} \cup [c]_{\chi_1} \cup [b]_{\chi_2} \cup [c]_{\chi_2} \cup [d]_{\chi_2} \cup [c]_{\chi_3} \\
&= \{a, b, c, d, e, f, g, h\}.
\end{aligned}$$

Considering the following table of operation:

$\bullet$	c	d	e
c	c	d	e
d	d	c	f
e	e	f	c

In that case, $(S, \bullet)$ *is a semigroup on* $\mathcal{O}$. *Next, we take* $A = \{c, e\} \subseteq S$.

$$\begin{aligned}
N_1(B)^* A &= \bigcup_{[x]_{\chi_i} \cap A \neq \varnothing} [x]_{\chi_i} \\
&= [c]_{\chi_1} \cup [b]_{\chi_2} \cup [c]_{\chi_2} \cup [c]_{\chi_3} \\
&= \{b, c, d, e, f\}.
\end{aligned}$$

In this case, A *satisfies the condition* $AS^m \cap S^n A \subseteq N_r(B)^* A$. *Thus,* A *is a* (m, n) *quasi-nearness ideal of* S.

THEOREM 3.4. *Let* S *be a nearness semigroup and* Q *be a quasi-nearness ideal of* S. *If* S *is commutative and* $N_r(B)^*(N_r(B)^* Q) = N_r(B)^* Q$, *then* Q *is a quasi upper-near ideal of* S.

PROOF. We show that $(N_r(B)^* Q)S \cap S(N_r(B)^* Q) \subseteq N_r(B)^* Q$. $(N_r(B)^* Q)S \cap S(N_r(B)^* Q) \subseteq (N_r(B)^* Q)(N_r(B)^* S) \cap (N_r(B)^* S)(N_r(B)^* Q)$ by Theorem 2.1.(i). Afterward,

$(N_r(B)^* Q)(N_r(B)^* S) \cap (N_r(B)^* S)(N_r(B)^* Q) \subseteq N_r(B)^*(QS) \cap N_r(B)^*(SQ)$

by Lemma 2.1.(ii). From here, we have that

$$N_r(B)^*(QS) \cap N_r(B)^*(SQ) \subseteq N_r(B)^*(N_r(B)^*Q) \cap N_r(B)^*(N_r(B)^*Q)$$

by Lemma 3.1 since each quasi-nearness ideal of S is two-sided nearness ideal of S. Since $N_r(B)^*(N_r(B)^*Q) = N_r(B)^*Q$ by hypothesis, hence, $(N_r(B)^*Q)S \cap S(N_r(B)^*Q) \subseteq N_r(B)^*Q$ and Q is a quasi upper-near ideal of S. □

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