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REMARK ON THE REVERSE SOMBOR (δ–SOMBOR) INDICES

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Abstract. Let $G=(V,E)$, $V=\{v_1,v_2,\dots,v_n\}$, be a simple graph of order n, size m with vertex–degree sequence $\Delta=d_1\geqslant d_2\geqslant\cdots\geqslant d_n=\delta$, $d_i=d(v_i)$. Sombor, reverse Sombor and δ–Sombor indices are respectively defined as $SO(G)=\sum_{i\sim j}\sqrt{d_i^2+d_j^2}$, $RSO(G)=\sum_{i\sim j}\sqrt{c_i^2+c_j^2}$ and $\delta SO(G)=\sum_{i\sim j}\sqrt{\delta_i^2+\delta_j^2}$, where $c_i=\Delta-d_i+1$ and $\delta_i=d_i-\delta+1$, $i=1,2,\dots,n$. A relationship between $RSO(G)$ and $\delta SO(G)$ as well as some new bounds on $RSO(G)$ and $\delta SO(G)$ are derived.

1. Introduction and preliminaries

Let $G=(V,E)$, $V=\{v_1,v_2,\dots,v_n\}$, be a simple graph with n vertices, m edges with vertex–degree sequence $\Delta=d_1\geqslant d_2\geqslant\cdots\geqslant d_n=\delta>0$, $d_i=d(v_i)$. If vertices v_i and v_j are adjacent in G, we write $i\sim j$.

A topological index is a number related to graph which is invariant under graph isomorphism. In theoretical chemistry, topological indices (or, chemical indices or graphical indices) play an important role in studying the properties of molecules [**10**].

A great number of topological indices are the so-called degree-based graph invariants. These indices can be commonly represented as [**11**],

$$TI(G)=\sum_{i\sim j}F(d_i,d_j),$$

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where $F(x,y)$ is an appropriately chosen function with the property $F(x,y) = F(y,x)$.

Let f be a mapping from the set $D = \{d_1, d_2, \dots, d_n\}$ into the set of positive real numbers. The general vertex–degree–based topological indices, $TI_f(G)$, are defined as

$$TI_f(G) = \sum_{i \sim j} F(f(d_i), f(d_j)). \tag{1.1}$$

If F is an additive function, $F(x,y) = x + y$, then the following is valid [**7**]

$$TI_f(G) = \sum_{i \sim j} (f(d_i) + f(d_j)) = \sum_{i=1}^{n} d_i f(d_i). \tag{1.2}$$

The first Zagreb index, $M_1(G)$, is vertex–degree–based index introduced in [**12**] as

$$M_1(G) = \sum_{i=1}^{n} d_i^2.$$

For $f(x) = x$ from (1.2) it follows [**6**]

$$M_1(G) = \sum_{i \sim j} (d_i + d_j).$$

The first Zagreb index became one of the most popular and most extensively studied graph–based molecular structure descriptors. More on its applications and mathematical properties can be found in [**2, 3, 13, 14**] and in the references cited therein.

Using the function $f(x) = x - \delta + 1$, a δ–set $\delta D = \{\delta_1, \delta_2, \dots, \delta_n\}$ can be associated to the set $D = \{d_1, d_2, \dots, d_n\}$ in the following way [**17**]:

$$\delta_i = \delta_i(G) = d_i - \delta + 1,$$

for $i = 1, 2, \dots, n$. Now, by analogy with the first Zagreb index $M_1(G)$, two δ–first Zagreb indices, $\delta M_1^{\alpha}(G)$ and $\delta M_1^{\beta}(G)$, can be defined as

$$\delta M_1^{\alpha}(G) = \sum_{i=1}^{n} \delta_i^2 = \sum_{i=1}^{n} (d_i - \delta + 1)^2 = M_1(G) - 4m(\delta - 1) + n(\delta - 1)^2$$

and

$$\delta M_1^{\beta}(G) = \sum_{i \sim j} (\delta_i + \delta_j) = \sum_{i=1}^{n} d_i \delta_i = \sum_{i=1}^{n} d_i (d_i - \delta + 1) = M_1(G) - 2m(\delta - 1).$$

It can be easily verified that for a graph with the property $\delta = 1$, we have $M_1(G) = \delta M_1^{\alpha}(G) = \delta M_1^{\beta}(G)$. Of course, this is not true in general.

In [**8**], using the function $f(x) = \Delta - x + 1$, a set $\Delta D = \{c_1, c_2, \dots, c_n\}$ is associated to the set $D = \{d_1, d_2, \dots, d_n\}$ in the following way

$$c_i = c_i(G) = \Delta - d_i + 1,$$

for $i = 1, 2, \ldots, n$. Having this in mind, two new indices resembling on $M_1(G)$, called reverse Zagreb indices can be defined as

$$(1.3)\quad RM_1^{\alpha}(G) = \sum_{i=1}^{n} c_i^2 = \sum_{i=1}^{n} (\Delta - d_i + 1)^2 = M_1(G) - 4m(\Delta + 1) + n(\Delta + 1)^2$$

and

$$(1.4)\quad RM_1^{\beta}(G) = \sum_{i\sim j} (c_i + c_j) = \sum_{i=1}^{n} d_i c_i = \sum_{i=1}^{n} d_i(\Delta - d_i + 1) = 2m(\Delta+1) - M_1(G).$$

The Sombor index is another vertex-degree index, recently conceived in [**15**] as

$$SO(G) = \sum_{i\sim j} \sqrt{d_i^2 + d_j^2}.$$

More on its mathematical properties can be found in [**4, 16, 18, 19, 22**].

The so called δ–Sombor index, $\delta SO(G)$, was introduced in [**17**] as

$$\delta SO(G) = \sum_{i\sim j} \sqrt{\delta_i^2 + \delta_j^2},$$

and the reverse Sombor index, $RSO(G)$, in [**23**] as

$$RSO(G) = \sum_{i\sim j} \sqrt{c_i^2 + c_j^2}.$$

By definition, a topological invariant $TI(G)$ is called an irregularity measure of a graph G if $TI(G) \geqslant 0$ and $TI(G) = 0$ if and only if G is a regular graph. The majority of irregularity measures are the so-called degree-based graph invariants. The Albertson irregularity measure is defined as [**1**]

$$irr(G) = \sum_{i\sim j} |d_i - d_j|.$$

One can easily see that the following equalities hold

$$irr(G) = \sum_{i\sim j} |d_i - d_j| = \sum_{i\sim j} |\delta_i - \delta_j| = \sum_{i\sim j} |c_i - c_j|.$$

In the present paper we investigate a relationship between $\delta SO(G)$ and $RSO(G)$ and determine new bounds for $RSO(G)$, i.e. $\delta SO(G)$, in terms of invariants $M_1(G)$ and $irr(G)$ and prove Nordhaus–Gadumm type inequalities (see [**20**]).

2. Main results

At the beginning let us recall one analytical inequality for real number sequences proven in [**21**] which will be used in proofs of theorems.

LEMMA 2.1. *Let $x = (x_i)$ and $a = (a_i)$, $i = 1, 2, \dots, n$, be positive real number sequences. Then for any $r \geqslant 0$ holds*

$$\sum_{i=1}^{n} \frac{x_i^{r+1}}{a_i^r} \geqslant \frac{\left(\sum_{i=1}^{n} x_i\right)^{r+1}}{\left(\sum_{i=1}^{n} a_i\right)^{r}}. \tag{2.1}$$

Equality holds if and only if $r = 0$, or $\frac{x_1}{a_1} = \frac{x_2}{a_2} = \cdots = \frac{x_n}{a_n}$.

REMARK 2.1. The inequality (2.1) is given in its original form as proven in [**21**]. But it is not difficult to see that it is valid for any real r such that $r \leqslant -1$ or $r \geqslant 0$, and when $-1 \leqslant r \leqslant 0$ the opposite inequality holds. The equality is also attained when $r = -1$. The inequality (2.1) is known in the literature as Radon's inequality.

In the next theorem we determine a relationship between $RSO(G)$ and $M_1(G)$ and $irr(G)$.

THEOREM 2.1. *Let G be a connected graph with $m \geqslant 1$ edges. Then*

$$RSO(G) \geqslant \frac{\sqrt{2}}{2} \sqrt{irr(G)^2 + (2m(\Delta + 1) - M_1(G))^2}. \tag{2.2}$$

Equality holds if and only if G is an edge–regular graph.

PROOF. The following identities are valid

$$RSO(G) - \sum_{i \sim j} \frac{2c_i c_j}{\sqrt{c_i^2 + c_j^2}} = \sum_{i \sim j} \frac{(c_i - c_j)^2}{\sqrt{c_i^2 + c_j^2}}$$

and

$$RSO(G) + \sum_{i \sim j} \frac{2c_i c_j}{\sqrt{c_i^2 + c_j^2}} = \sum_{i \sim j} \frac{(c_i + c_j)^2}{\sqrt{c_i^2 + c_j^2}}.$$

After summing the above equalities we get

$$2RSO(G) = \sum_{i \sim j} \frac{(c_i - c_j)^2}{\sqrt{c_i^2 + c_j^2}} + \sum_{i \sim j} \frac{(c_i + c_j)^2}{\sqrt{c_i^2 + c_j^2}}. \tag{2.3}$$

For $r = 1$, $x_i := |c_i - c_j|$, $a_i := \sqrt{c_i^2 + c_j^2}$, with summation performed over all adjacent vertices v_i and v_j in graph G, the inequality (2.1) becomes

$$\sum_{i \sim j} \frac{(c_i - c_j)^2}{\sqrt{c_i^2 + c_j^2}} \geqslant \frac{\left(\sum_{i \sim j} |c_i - c_j|\right)^2}{\sum_{i \sim j} \sqrt{c_i^2 + c_j^2}},$$

that is

$$\sum_{i\sim j}\frac{(c_i-c_j)^2}{\sqrt{c_i^2+c_j^2}}\geqslant\frac{irr(G)^2}{RSO(G)}. \tag{2.4}$$

On the other hand, for $r=1$, $x_i:=c_i+c_j$, $a_i:=\sqrt{c_i^2+c_j^2}$, with summation performed over all adjacent vertices v_i and v_j in G, the inequality (2.1) becomes

$$\sum_{i\sim j}\frac{(c_i+c_j)^2}{\sqrt{c_i^2+c_j^2}}\geqslant\frac{\left(\sum_{i\sim j}(c_i+c_j)\right)^2}{\sum_{i\sim j}\sqrt{c_i^2+c_j^2}}=\frac{RM_1^{\beta}(G)^2}{RSO(G)}.$$

From the above and (1.4) it follows

$$\sum_{i\sim j}\frac{(c_i+c_j)^2}{\sqrt{c_i^2+c_j^2}}\geqslant\frac{(2m(\Delta+1)-M_1(G))^2}{RSO(G)}. \tag{2.5}$$

According to (2.3), (2.4) and (2.5) we have

$$2RSO(G)\geqslant\frac{irr(G)^2+(2m(\Delta+1)-M_1(G))^2}{RSO(G)},$$

from which (2.2) is obtained.

Equality in (2.5) holds if and only if $\frac{c_i+c_j}{\sqrt{c_i^2+c_j^2}}=const.$ for any pair of adjacent vertices v_i and v_j in G. Let v_j and v_k be two vertices adjacent to v_i. Then

$$\frac{c_i+c_k}{\sqrt{c_i^2+c_k^2}}=\frac{c_i+c_j}{\sqrt{c_i^2+c_j^2}},$$

i.e.

$$c_i(c_i^2-c_jc_k)(c_j-c_k)=0.$$

This means that equality in (2.5) holds if and only if $c_j=c_k$, i.e. if and only if G is an edge–regular graph. The equality in (2.4) holds under same condition, which implies that equality in (2.2) holds if and only if G is an edge–regular graph. □

Corollary 2.1. *Let G be a connected graph with $m\geqslant 1$ edges. Then*

$$RSO(G)\geqslant\frac{\sqrt{2}}{2}(2m(\Delta+1)-M_1(G)). \tag{2.6}$$

Equality holds if and only if G is a regular graph.

Proof. For any graph G we have that $irr(G)\geqslant 0$. Based on this, from (2.2) we arrive at (2.6). □

Corollary 2.2. *Let G be a connected graph with $n\geqslant 2$ vertices and m edges. Then*

$$RSO(G)\geqslant\frac{\sqrt{2}}{2}(n\Delta\delta-2m(\delta-1)).$$

Equality holds if and only if G is a regular graph.

PROOF. In [**5**] the following inequality was proven

$$M_1(G) \leqslant 2m(\Delta+\delta) - n\Delta\delta. \tag{2.7}$$

From the above and (2.6) we get the desired result. □

The proof of the next theorem is analogous to that of Theorem 2.1, hence omitted.

THEOREM 2.2. *Let G be a connected graph with $m \geqslant 1$ edges. Then*

$$\delta SO(G) \geqslant \frac{\sqrt{2}}{2}\sqrt{irr(G)^2 + (M_1(G) - 2m(\delta-1))^2}.$$

Equality holds if and only if G is an edge–regular graph.

COROLLARY 2.3. *Let G be a connected graph with $m \geqslant 1$ edges. Then*

$$\delta SO(G) \geqslant \frac{\sqrt{2}}{2}(M_1(G) - 2m(\delta-1)). \tag{2.8}$$

Equality holds if and only if G is a regular graph.

COROLLARY 2.4. *Let G be a connected graph with $n \geqslant 2$ vertices and m edges. Then*

$$\delta SO(G) \geqslant \frac{\sqrt{2}m}{n}(2m - n(\delta-1)) \geqslant \sqrt{2}m.$$

Equality holds if and only if G is a regular graph.

PROOF. In [**9**] the following was proven

$$M_1(G) \geqslant \frac{4m^2}{n}.$$

From the above and (2.8) we get the desired result. □

COROLLARY 2.5. *Let G be a connected graph with $m \geqslant 1$ edges. Then*

$$RSO(G) + RSO(\overline{G}) \geqslant \sqrt{2}m(\Delta - \delta + 2). \tag{2.9}$$

Equality holds if and only if G is a regular graph.

PROOF. Since

$$\begin{aligned} \delta_i &= \delta_i(G) = d_i - \delta + 1 = d_i + (n-1) - (n-1) - \delta + 1 \\ &= \overline{\Delta} - d_i\left(\overline{G}\right) + 1 = c_i\left(\overline{G}\right) = \overline{c_i} \end{aligned}$$

we have that

$$RSO(\overline{G}) = \sum_{i\sim j}\sqrt{\overline{c_i}^2 + \overline{c_j}^2} = \sum_{i\sim j}\sqrt{\delta_i^2 + \delta_j^2} = \delta SO(G),$$

i.e.

$$RSO(G) + RSO(\overline{G}) = RSO(G) + \delta SO(G).$$

From the above and (2.6) and (2.8) we arrive at (2.9). □

COROLLARY 2.6. *Let G be a connected graph with $n \geqslant 2$ vertices and m edges. Then*

$$RSO(G).RSO(\overline{G}) \geqslant m(n\Delta\delta - 2m(\delta - 1)).$$

Equality holds if and only if G is a regular graph.

In the next theorem we determine an upper bound for $RSO(G)$ in terms of topological indices $M_1(G)$ and $irr(G)$ and parameters m and Δ.

THEOREM 2.3. *Let G be a connected graph with $m \geqslant 1$ edges. Then*

$$RSO(G) \leqslant \frac{\sqrt{2}}{2}(irr(G) + 2m(\Delta + 1) - M_1(G)). \tag{2.10}$$

Equality holds if and only if G is a regular graph.

PROOF. For any two nonnegative real numbers a and b holds

$$\sqrt{a + b} \leqslant \sqrt{a} + \sqrt{b}, \tag{2.11}$$

with equality if and only if $ab = 0$.

For $a_i := \frac{1}{2}(c_i + c_j)^2$, $b_i := \frac{1}{2}(c_i - c_j)^2$, the inequality (2.11) transforms into

$$\sqrt{c_i^2 + c_j^2} \leqslant \frac{\sqrt{2}}{2}(c_i + c_j + |c_i - c_j|). \tag{2.12}$$

After summation the above inequality over all adjacent vertices v_i and v_j in graph G, we get

$$\sum_{i\sim j} \sqrt{c_i^2 + c_j^2} \leqslant \frac{\sqrt{2}}{2} \left(\sum_{i\sim j}(c_i + c_j) + \sum_{i\sim j} |c_i - c_j| \right),$$

that is

$$RSO(G) \leqslant \frac{\sqrt{2}}{2} \left(irr(G) + RM_1^{\beta}(G) \right).$$

From the above and (1.4) we get (2.10).

Equality in (2.12) holds if and only if $(c_i + c_j)(c_i - c_j) = 0$, i.e. $c_i = c_j$ for any two adjacent vertices v_i and v_j in G. Consequently the equality in (2.10) holds if and only if G is a regular graph. □

COROLLARY 2.7. *Let G be a connected graph with $n \geqslant 2$ vertices and m edges. Then*

$$RSO(G) \leqslant \frac{\sqrt{2}}{2}(irr(G) + \frac{2m}{n}(n(\Delta + 1) - 2m)).$$

Equality holds if and only if G is a regular graph.

By a similar procedure as in the case of Theorem 2.3, we get the following result.

THEOREM 2.4. *Let G be a connected graph with $m \geqslant 1$ edges. Then*

$$\delta SO(G) \leqslant \frac{\sqrt{2}}{2}(irr(G) + M_1(G) - 2m(\delta - 1)).$$

Equality holds if and only if G is a regular graph.

Corollary 2.8. *Let G be a connected graph with $n \geqslant 2$ vertices and m edges. Then*

$$\delta SO(G) \leqslant \frac{\sqrt{2}}{2}(irr(G) + 2m(\Delta + 1) - n\Delta\delta).$$

Equality holds if and only if G is a regular graph.

Corollary 2.9. *Let G be a connected graph with $m \geqslant 1$ edges. Then*

$$RSO(G) + RSO(\overline{G}) \leqslant \sqrt{2}(irr(G) + m(\Delta - \delta + 2)).$$

Equality holds if and only if G is a regular graph.

Corollary 2.10. *Let G be a connected graph with $n \geqslant 2$ vertices and m edges. Then*

$$RSO(G).RSO(\overline{G}) \leqslant \frac{1}{2}(irr(G) + \frac{2m}{n}(n(\Delta+1) - 2m))(irr(G) + 2m(\Delta+1) - n\Delta\delta).$$

Equality holds if and only if G is a regular graph.

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