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ON TOPOLOGICAL INDICES OF JUMP GRAPHS

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Abstract. Combinatorial expressions are obtained for the jump graph of the first and second Zagreb indices, the forgotten index, and their coindices, valid for any simple connected graph.

1. Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The order and size of G are $V(G)| = n$ and $E(G)| = m$, respectively. The degree $d(u)$ of the vertex $u \in V(G)$ is the number of vertices adjacent to u. The edge connecting the vertices u and v will be denoted by uv.

By $L(G)$ is denoted the line graph of the graph G, in which the vertices pertain to the edges of G, and two vertices of $L(G)$ are adjacent if and only if the respective edges of G are incident. By $\overline{G}$ we denote the complement of the graph G, i.e., $V(\overline{G}) = V(G)$ and $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$. For other graph-theoretical terminology and notation see in respective textbooks [**2, 5**].

The jump graph $J(G)$ of G is the graph whose vertices are the edges of G, and two vertices of $J(G)$ are adjacent if and only if they are not adjacent in G.

Equivalently, the jump graph $J(G)$ of G is the complement of the line graph $L(G)$ of G. Hevia et al. [**14**] studied the planarity of jump graphs. Wu and Meng [**19**] characterized hamiltonian jump graphs and settled two conjectures posed by Chartrand et al. [**4**].

In the contemporary literature, a large number of vertex-degree-based graph invariants (usually referred to as"topological indices") are being considered [**9, 15, 18**], many of which found applications in chemistry. Among them we consider here

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here the first Zagreb and second indices M_1 and M_2 [**13, 12, 10, 3, 17**] and the forgotten index F [**8, 6, 16**], defined as

$$M_1(G) = \sum_{u\in V(G)} d(u)^2 = \sum_{uv\in E(G)} \big[d(u)+d(v)\big] \quad , \quad M_2(G) = \sum_{uv\in E(G)} d(u)\,d(v)$$

and

$$F(G) = \sum_{u\in V(G)} d(u)^3 = \sum_{uv\in E(G)} \big[d(u)^2+d(v)^2\big]\,.$$

The coindex of a vertex-degree-based topological index is defined by replacing the summation $\sum_{uv\in E(G)}$ by $\sum_{uv\notin E(G)}$, assuming that $u \neq y$ [**7, 1**]. For the two Zagreb indices, the following identities are known [**11**]:

$$\overline{M}_1(G) = 2m(n-1) - M_1(G) \tag{1.1}$$

$$\overline{M}_2(G) = 2m^2 - \frac{1}{2}\,M_1(G) - M_2(G)\,. \tag{1.2}$$

2. Main results

In this section, we derive explicit formulas for first and second zagreb indices and their coindices, forgotten index and its coindex of the jump graph of a graph G.

THEOREM 2.1. *Let G be a graph of order n and size m respectively. Then*

$$M_1[J(G)] = m(m-1)(m+3) - 2(m-1)M_1(G) + M_1[L(G)]\,. \tag{2.1}$$

PROOF.

$$\begin{aligned} M_1[J(G)] &= \sum_{u\prime,v\prime\in V[J(G)]} d(u\prime v\prime) = \sum_{uv\in V[L(G)]} \big[(m-1)d(uv)\big]^2 \\ &= \sum_{uv\in V[L(G)]} \Big[(m-1)^2 + d(uv)^2 - 2(m-1)d(uv)\Big] \\ &= m(m-1)^2 + M_1[L(G)] - 2(m-1)\left[\frac{1}{2}\sum_{v\in V(G)} d(v)^2 - m\right] \\ &= m(m-1)^2 + M_1[L(G)] - 2(m-1)[M_1(G)-2m] \end{aligned}$$

from which Eq. (2.1) directly follows. □

THEOREM 2.2. *Let G be graph of order n and size m respectively. Then*

$$M_2[J(G)] = \frac{1}{2}[m(m+1) - M_1(G)](m-1)^2 - (m-1)\overline{M}_1[L(G)] + \overline{M}_2[L(G)]\,. \tag{2.2}$$

Proof.

$$M_2[J(G)] = \sum_{u\prime v\prime, w\prime x\prime \in E[J(G)]} d(u\prime v\prime)\, d(w\prime x\prime)$$

$$= \sum_{uv, wx \notin E[L(G)]} [(m-1) - d(u,v)][(m-1) - d(w,x)]$$

$$= \sum_{uv, wx \notin E[L(G)]} [(m-1)^2 - (m-1)[(d(u,v) + d(w,x)) + d(u,v)d(w,x)]$$

$$= \frac{m(m-1)}{2} - \left[\frac{1}{2}\sum_{v \in V(G)} d(v)^2 - m\right](m-1)^2 - (m-1)\overline{M}_1[L(G)] + \overline{M}_2[L(G)]$$

$$= \frac{m(m-1)}{2} - \left[\frac{1}{2}\, M_1(G) + m\right](m-1)^2 - (m-1)\,\overline{M}_1[L(G)] + \overline{M}_2[L(G)]$$

from which Eq. (2.2) directly follows. □

Theorem 2.3. *For any (m,n)-graph G,*

$$\overline{M}_1[J(G)] = (m-1)\big(M_1(G) - 2m\big) - M_1[L(G)]\,. \tag{2.3}$$

Proof. Recall that $\overline{M}_1(G) = \sum_{uv \notin G}[d(u) + d(v)]$. The number of edges in the jump graph, $J(G)$ is $\frac{1}{2}\sum_{u \in V(G)} d(u)^2 - m$. Using Eq. (1.1), we have

$$\overline{M}_1[J(G)] = 2\left[\frac{m(m-1)}{2} - \left(\frac{1}{2}\sum_{u \in V(G)} d(u)^2 - m)\right)\right](m-1) - M_1[J(G)]$$

$$= \big[m(m-1) - M_1(G) + 2m\big](m-1) - M_1[J(G)]$$

$$= m^3 - 2m^2 + m - (m-1)M_1(G) + 2m^2 - 2m - M_1[J(G)]$$

$$= m^3 - m - (m-1)M_1(G) - m^3 - 2m^2 - 3m + 2(m-1)M_1(G) - M_1[L(G)]$$

$$= 2m - 2m^2 + (m-1)M_1(G) - M_1[L(G)]$$

from which Eq. (2.3) directly follows. □

Theorem 2.4. *For any (m,n)-graph G,*

$$\begin{aligned} \overline{M}_2[J(G)] &= m(m^2+1) - \frac{M_1(G)}{2}(m^2 + 2m + 1) \\ &+ \frac{1}{2}\big(M_1(G)^2 - M_1[L(G)]\big) + (m-1)\overline{M}_1[L(G)] - \overline{M}_2[L(G)]\,. \end{aligned} \tag{2.4}$$

Proof. By Eq. (1.2),

$$\overline{M}_2[J(G)] = 2\left(\frac{m(m-1)}{2} - \frac{1}{2}M_1(G) + m\right)^2 - \frac{1}{2}M_1[J(G)] - M_2[J(G)]\,.$$

Applying Theorems 2.1 and 2.2, we get

$$
\begin{aligned}
M_2[J(G)] = \frac{1}{2}\Big[& m^2(m^2+2m+1) - (m^2-m)(m+3) - (m^2+m)(m^2-2m+1)\Big] \\
& - \frac{M_1(G)}{2}(2m^2+2m-2m+2-m^2+2m-1) \\
& + \frac{1}{2}\, M_1(G)^2 - M_1[L(G)] + (m-1)\overline{M}_1[L(G)] - \overline{M}_2[L(G)]
\end{aligned}
$$

from which Eq. (2.4) follows. □

THEOREM 2.5. *For any (m,n)-graph G,*

$$
\begin{aligned}
F[J(G)] &= m(m+5)(m-1)^2 \\
&- 3(m-1)\Big[M_1(G)(m-1) - M_1[L(G)]\Big] - F[L(G)]\,. \qquad (2.5)
\end{aligned}
$$

PROOF.

$$
\begin{aligned}
F[J(G)] &= \sum_{u\prime v\prime \in V[J(G)]} d(u\prime v\prime)^3 = \sum_{uv\in V[L(G)]} ((m-1)-d(uv))^3 \\
&= \sum_{uv\in V[L(G)]} \Big[(m-1)^3 - 3(m-1)^2\, d(uv) + 3d(uv)^2\,(m-1) - d(uv)^3\Big] \\
&= m(m-1)^3 - 3(m-1)^2 \left[\frac{1}{2}\sum_{u\in V(G)} d(u)^2 - m\right] + 3(m-1)M_1[L(G)] - F[L(G)] \\
&= m(m-1)^3 - 3(m-1)^2\big(M_1(G)-2m\big) + 3(m-1)M_1[L(G)] - F[L(G)]
\end{aligned}
$$

from which Eq. (2.5) directly follows. □

THEOREM 2.6. *For any (m,n)-graph G,*

$$
\overline{F}[J(G)] = \big[M_1(G) - 2m\big](m-1)^2 - 2(m-1)M_1[L(G)] + F[L(G)]\,. \qquad (2.6)
$$

PROOF. Note first that

$$
\overline{F}[J(G)] = \sum_{u\prime v\prime \notin E[J(G)]} \big[d(u\prime)^2 + d(v\prime)^2\big]\,.
$$

Therefore

$$\overline{F}[J(G)] = \sum_{uv\in E[L(G)]} \Big[(m-1)-d(u)\Big]^2 + \Big[(m-1)-d(v)\Big]^2$$

$$= \sum_{uv\in E[L(G)]} \Big[(m-1)^2-2(m-1)d(u)+d(u)^2+(m-1)^2-2(m-1)d(v)+d(v)^2\Big]$$

$$= \sum_{uv\in E[L(G)]} \Big[2(m-1)^2-2(m-1)\big[d(u)+d(v)\big]+d(u)^2+d(v)^2\Big]$$

$$= 2\left[\frac{1}{2}\sum_{x\in V(G)} d(x)^2 - m\right](m-1)^2 - 2(m-1)M_1[L(G)] + F[L(G)]$$

and Eq. (2.6) follows. □

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