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WEIGHTED MEAN VALUE INEQUALITIES AND THEIR APPLICATION TO MONOTONICITY PROPERTIES

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Abstract. It has been shown by Feng Qi [1] that using some generalizations of weighted mean values of two parameters, some monotonicity results and inequalities can be derived for the gamma function and the incomplete gamma function. In this paper we are going to derive analogues monotonicity results for the extended incomplete gamma function which is derived from the k-gamma function.

1. Introduction

1.1. The k-gamma function $\Gamma_k(z)$, introduced by Diaz *et. al* [2] is the extension of the ordinary gamma function and is defined as follows for $\Re s > 0$and $k > 0$

$$\Gamma_k(z) = \int_0^\infty t^{z-1} e^{-\frac{t^k}{k}} dt. \tag{1.1}$$

The k-gamma function is related to the ordinary gamma function as follows:

$$\Gamma_k(z) = k^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right). \tag{1.2}$$

Using Eqn. (1.1), we can define the following k-incomplete gamma functions:

$$\Gamma_k(z,x) = \int_x^\infty t^{z-1} e^{-\frac{t^k}{k}} dt, \ \ \gamma_k(z,x) = \int_0^x t^{z-1} e^{-\frac{t^k}{k}} dt \tag{1.3}$$

In [3], Feng Qi established the following generalized weighted mean values:

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$M_{p,f}(r,s;x,y)$ of a positive function f with two parameters $x,y\in \mathrm{R}$ and non-negative weight $p\neq 0$ for $x,y\in \mathrm{R}$ by

i)

$$M_{p,f}(r,s;x,y)=\left(\frac{\int_x^y p(u)\,f^s(u)\,du}{\int_x^y p(u)\,f^r(u)\,du}\right)^{\frac{1}{s-r}}, \tag{1.4}$$

valid for $(r-s)(x-y)\neq 0$.

ii)

$$M_{p,f}(r,r;x,y)=\exp\left(\frac{\int_x^y p(u)\,f^r(u)\ln f(u)\,du}{\int_x^y p(u)\,f^r(u)\,du}\right), \tag{1.5}$$

valid for $x-y\neq 0$ and

iii)

$$M_{p,f}(r,s;x,x)=f(x)\,,x=y. \tag{1.6}$$

1.2. Throughout the sequel we will employ the notation$M_{p,f}(r,s;x,y)=M_{p,f}(r,s)=M_{p,f}(x,y)=M_{p,f}$. If $p(u)=1$,$f(u)=u$and $x,y>0$then the generalized mean values are reduced to the extended mean values $E(r,s;x,y)$defined as

i)

$$E(r,s;x,y)=\left[\frac{r}{s}\cdot\frac{y^s-x^s}{y^r-x^r}\right]^{\frac{1}{s-r}},rs(r-s)(x-y)\neq 0 \tag{1.7}$$

ii)

$$E(r,0;x,y)=\left[\frac{1}{r}\cdot\frac{y^r-x^r}{\ln y-\ln x}\right]^{\frac{1}{r}},r(x-y)\neq 0 \tag{1.8}$$

iii)

$$E(r,r;x,y)=e^{-\frac{1}{r}}\left(\frac{x^{x^r}}{y^{y^r}}\right)^{\frac{1}{x^r-y^r}},r(x-y)\neq 0 \tag{1.9}$$

iv)

$$E(0,0;x,y)=\sqrt{xy},x\neq y \tag{1.10}$$

and

v)

$$E(r,s;x,x)=x,x=y. \tag{1.11}$$

Now, using the Lemmas from [1] presented in appendix, we derive the main results of our paper. Throughout the sequel we assume that $k>0$.

2. Main results I

THEOREM 2.1. *For any $x > 0$, the function $\frac{s\gamma_k(s,x)}{x^s}$ is decreasing in $s > 0$.*

PROOF. Set $p(t) = e^{-\frac{t^k}{k}}$ and $f(t) = t$ in Lemma 4.1 for $t \in (0,\infty)$ and $s > r > 0$ we get

$$\left(\frac{\int_0^x t^{s-1}e^{-\frac{t^k}{k}}dt}{\int_0^x t^{r-1}e^{-\frac{t^k}{k}}dt}\right)^{\frac{1}{s-r}} \leqslant \left(\frac{r}{s}\cdot\frac{x^s}{x^r}\right)^{\frac{1}{s-r}}. \tag{2.1}$$

after further simplifying, we get

$$\frac{s\gamma_k(s,x)}{x^s} \leqslant \frac{r\gamma_k(r,x)}{x^r} \tag{2.2}$$

□

THEOREM 2.2. *The function $\left(\frac{\Gamma_k(s)}{\Gamma_k(r)}\right)^{\frac{1}{s-r}}$ is increasing with $r > 0$ and $s > 0$*

PROOF. Let $p(u) = e^{-\frac{u^k}{k}}$ and $f(u) = u$, $u \in (0,\infty)$ in Lemma 4.2 and the desired result readily follows. □

THEOREM 2.3. *For $s > r > 0$, we have*

$$\exp\left[(s-r)\psi_k(s)\right] > \frac{\Gamma_k(s)}{\Gamma_k(sr)} > \exp\left[(s-r)\psi_k(r)\right] \tag{2.3}$$

PROOF. The result follows from the standard argument that [**1**] $M_{p,f}(s,s) > M_{p,f}(r,s) > M_{p,f}(r,r)$. □

THEOREM 2.4. *For $s > r > 0$ and $x > 0$, $\left(\frac{\gamma_k(s,x)}{\gamma_k(r,x)}\right)^{\frac{1}{s-r}}$ and $\left(\frac{\Gamma_k(s,x)}{\Gamma_k(r,x)}\right)^{\frac{1}{s-r}}$ increases with either x or r and s. Therefore $\frac{\gamma_k(s,x)}{x^{s-1}}$ decreases and $\frac{\Gamma_k(s,x)}{x^{s-1}}$ increases with $s > 0$ respectively.*

Proof of above theorem follows from Lemma 4.2 and 4.3.

THEOREM 2.5. *Let $g(t)$ be an integrable positive function such that $e^{\frac{x^k}{k}}g(t)$ is decreasing, then*

$$\frac{\gamma_k(s,x)}{\gamma_k(r,x)} \geqslant \frac{\int_0^x t^{s-1}g(t)\,dt}{\int_0^x t^{r-1}g(t)\,dt}, \tag{2.4}$$

$$\frac{\Gamma_k(s,x)}{\Gamma_k(r,x)} \geqslant \frac{\int_x^\infty t^{s-1}g(t)\,dt}{\int_x^\infty t^{r-1}g(t)\,dt} \tag{2.5}$$

and

$$\frac{\Gamma_k(s)}{\Gamma_k(r)} \geqslant \frac{\int\limits_0^\infty t^{s-1} g(t)\, dt}{\int\limits_0^\infty t^{r-1} g(t)\, dt} \tag{2.6}$$

valid for $s > r > 0$ *and* $x > 0$. *On the other hand, if* $e^{\frac{x^k}{k}} g(t)$ *is increasing then the inequalities are reverses.*

Proof. Above inequalities are consequences of Lemma 4.4 when applied to $f(t) = t$, $p_1(t) = e^{-\frac{t^k}{k}}$ and $p_2(t) = g(t)$. □

Theorem 2.6. *Let* $f(u)$ *be positive and integrable function on* $(0,\infty)$. *If* $\frac{f(u)}{u} > 1$ *is increasing, then for* $s > r > 0$ *and* $x > 0$, *we have*

$$\frac{\gamma_k(s+1,x)}{\gamma_k(r+1,x)} \geqslant \frac{\int\limits_0^x f^s(u)\, e^{\frac{-u^k}{k}} du}{\int\limits_0^x f^r(u)\, e^{\frac{-u^k}{k}} du}, \tag{2.7}$$

$$\frac{\Gamma_k(s+1,x)}{\Gamma_k(r+1,x)} \geqslant \frac{\int\limits_x^\infty f^s(u)\, e^{\frac{-u^k}{k}} du}{\int\limits_x^\infty f^r(u)\, e^{\frac{-u^k}{k}} du} \tag{2.8}$$

and

$$\frac{\Gamma_k(s+1)}{\Gamma_k(r+1)} \geqslant \frac{\int\limits_0^\infty f^r(u)\, e^{\frac{-u^k}{k}} du}{\int\limits_0^\infty f^r(u)\, e^{\frac{-u^k}{k}} du}. \tag{2.9}$$

The inequalities are reversed if $\frac{f(u)}{u} < 1$ *is decreasing and* $s > r > 0$ *and* $x > 0$.

Proof. Above inequalities are consequences of Lemma 4.5 when applied to $f_1(u) = f(u)$, $f_2(u) = f(u)$ and $p(u) = e^{-\frac{u^k}{k}}$. □

3. Main results II

Gehlot [**2**] introduced the following two parameter gamma function also known as the $p,k-$gamma function which is a slight modification of the $k-$ gamma function defined for $\Re s > 0$ and $p, k > 0$

$$_p\Gamma_k(z) = \int_0^\infty t^{z-1} e^{-\frac{t^k}{p}} dt. \tag{3.1}$$

The $p,k-$gamma function is related to the ordinary gamma function as follows

$$_p\Gamma_k(x) = \frac{p^{\frac{x}{k}}}{k} \Gamma\left(\frac{x}{k}\right). \tag{3.2}$$

Using Eqn. (3.1), we get the following corresponding incomplete $p,k-$gamma function

$$ {}_p\Gamma_k\left(z,x\right)=\int_x^{\infty}t^{z-1}e^{-\frac{t^k}{p}}dt, \tag{3.3}$$

$$ {}_p\gamma_k\left(z,x\right)=\int_0^{x}t^{z-1}e^{-\frac{t^k}{p}}dt. \tag{3.4}$$

Using Lemma 4.1-4.5 from appendix, we can readily get the analogues of above theorems for the $p,k-$gamma function. Throughout the sequel, we assume that $p,k>0$.

Theorem 3.1. *For any $x>0$, the function $\frac{s({}_p\gamma_k(s,x))}{x^s}$ is decreasing in $s>0$.*

Theorem 3.2. *The function $\left(\frac{{}_p\Gamma_k(s)}{{}_p\Gamma_k(r)}\right)^{\frac{1}{s-r}}$ is increasing with $r>0$ and $s>0$*

Theorem 3.3. *For $s>r>0$, we have*

$$\exp\left[(s-r)\,{}_p\psi_k\left(s\right)\right]>\frac{{}_p\Gamma_k\left(s\right)}{{}_p\Gamma_k\left(sr\right)}>\exp\left[(s-r)\,{}_p\psi_k\left(r\right)\right] \tag{3.5}$$

Theorem 3.4. *For $s>r>0$ and $x>0$, $\left(\frac{{}_p\gamma_k(s,x)}{{}_p\gamma_k(r,x)}\right)^{\frac{1}{s-r}}$ and $\left(\frac{{}_p\Gamma_k(s,x)}{{}_p\Gamma_k(r,x)}\right)^{\frac{1}{s-r}}$ increases with either x or r and s. Therefore $\frac{{}_p\gamma_k(s,x)}{x^{s-1}}$ decreases and $\frac{{}_p\Gamma_k(s,x)}{x^{s-1}}$ increases with $s>0$ respectively.*

Theorem 3.5. *Let $g(t)$ be an integrable positive function such that $e^{\frac{x^k}{p}}g(t)$ is decreasing, then*

$$\frac{{}_p\gamma_k\left(s,x\right)}{{}_p\gamma_k\left(r,x\right)}\geqslant\frac{\int_0^x t^{s-1}g\left(t\right)dt}{\int_0^x t^{r-1}g\left(t\right)dt}, \tag{3.6}$$

$$\frac{{}_p\Gamma_k\left(s,x\right)}{{}_p\Gamma_k\left(r,x\right)}\geqslant\frac{\int_x^\infty t^{s-1}g\left(t\right)dt}{\int_x^\infty t^{r-1}g\left(t\right)dt} \tag{3.7}$$

and

$$\frac{{}_p\Gamma_k\left(s\right)}{{}_p\Gamma_k\left(r\right)}\geqslant\frac{\int_0^\infty t^{s-1}g\left(t\right)dt}{\int_0^\infty t^{r-1}g\left(t\right)dt} \tag{3.8}$$

valid for $s>r>0$ and $x>0$. On the other hand, if $e^{\frac{x^k}{p}}g(t)$ is increasing then the inequalities are reverses.

THEOREM 3.6. *Let $f(u)$ be positive and integrable function on $(0,\infty)$. If $\frac{f(u)}{u} > 1$ is increasing, then for $s > r > 0$ and $x > 0$, we have*

$$\frac{{}_p\gamma_k\,(s+1,x)}{{}_p\gamma_k\,(r+1,x)} \geqslant \frac{\int\limits_0^x f^s\,(u)\,e^{\frac{-u^k}{p}}\,du}{\int\limits_0^x f^r\,(u)\,e^{\frac{-u^k}{p}}\,du}, \tag{3.9}$$

$$\frac{{}_p\Gamma_k\,(s+1,x)}{{}_p\Gamma_k\,(r+1,x)} \geqslant \frac{\int\limits_x^\infty f^s\,(u)\,e^{\frac{-u^k}{p}}\,du}{\int\limits_x^\infty f^r\,(u)\,e^{\frac{-u^k}{p}}\,du} \tag{3.10}$$

and

$$\frac{{}_p\Gamma_k\,(s+1)}{{}_p\Gamma_k\,(r+1)} \geqslant \frac{\int\limits_0^\infty f^r\,(u)\,e^{\frac{-u^k}{p}}\,du}{\int\limits_0^\infty f^r\,(u)\,e^{\frac{-u^k}{p}}\,du}. \tag{3.11}$$

The inequalities are reversed if $\frac{f(u)}{u} < 1$ is decreasing and $s > r > 0$ and $x > 0$.

4. Appendix

LEMMA 4.1. *Suppose $f(t)$ is a positive differentiable function and $p\,(t) \neq 0$ an integrable non-negative weight on the interval $[a,b]$, if $f'(t)$ and $\frac{f'(t)}{p(t)}$ are integrable and both increasing or both decreasing then $\forall r,s \in \mathbb{R}$ we have*

$$M_{p,f}(r,s;a,b) < E(r+1,s+1;f(a),f(b)). \tag{4.1}$$

If one of the functions $f'(t)$ or $\frac{f'(t)}{p(t)}$ is non-decreasing and the other is non-increasing, then

$$M_{p,f}(r,s;a,b) > E(r+1,s+1;f(a),f(b)). \tag{4.2}$$

LEMMA 4.2. *Let $p(u) \neq 0$ be a non-negative and continuous function. Then $M_{p,f}(r,s)$ increases with both r and s.*

LEMMA 4.3. *Let $p(u) \neq 0$ be a non-negative and continuous function, $f(u)$ a positive increasing (or decreasing) continuous function. Then $M_{p,f}(x,y)$ increases (or decreases) with respect to either x or y.*

LEMMA 4.4. *Let $p_1(u) \neq 0$ and $p_2(u) \neq 0$ be a non-negative and integrable functions on the interval between x and y. Let $f(u)$ a positive and integrable function on the interval x and y, the ratio $\frac{p_1(u)}{p_2(u)}$ an integrable function, $\frac{p_1(u)}{p_2(u)}$ and $f_(u)$ both increasing or both decreasing then*

$$M_{p_1,f}(r,s;x;y) \geqslant M_{p_2,f}(r,s;x,y) \tag{4.3}$$

if one of the functions between $f(u)$ or $\frac{p_1(u)}{p_2(u)}$ is non-increasing and the other non-decreasing, the the inequality is reversed.

LEMMA 4.5. *Let $p(u) \neq 0$ be a non-negative and integrable function, and $f_1(u)$ and $f_2(u)$ positive and integrable functions on the interval x and y. If the ratio $\frac{f_1(u)}{f_2(u)}$ and $f_2(u)$ are integrable and both increasing or both decreasing then*

$$M_{p,f_1}(r,s;x;y) \geqslant M_{p,f_2}(r,s;x,y) \tag{4.1}$$

holds for $r, s \geqslant 0$ or $r \geqslant 0 \geqslant s$ and $\frac{f_1(u)}{f_2(u)} \geqslant 1$.

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