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ON THE PADOVAN ARRAYS

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Abstract. In the present work, two new recurrences of the Padovan sequence given with delayed initial conditions are defined. Some identities of these sequences which we call the Padovan arrays were examined. Also, generating and series functions of the Padovan arrays are examined.

1. Introduction

The Padovan numbers are an integer sequence named after the Italian architect Richard Padovan. The Padovan sequence is a kind of relative of a better-known Fibonacci sequence and has many interesting properties and applications to almost every field of science, nature, and art. The Padovan family has charming applications to architecture, geometrical shapes, and number theory. Another feature that makes the Fibonacci sequence important is that it has the golden ratio, which maintains its modernity today. The importance of the Padovan sequence is that it is mentioned with the plastic ratio. Hans van der Laan discovered a new and unique architectural proportion system in 1928. This ratio, which he calls the number of plastics, is based on the irrational value, which is one of the roots of the Padovan sequence. In this study, we first are briefly mentioned the Padovan sequence. Then we defined two new iterations of the Padovan sequence and looked at some of their identities. For more information on this sequence, see [**3, 4, 5, 6, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19, 20**].
The Padovan sequence $\{P_n\}_{n\geqslant 0}$ is defined by the delayed initial values $P_0 = 0$, $P_1 = 0$ and $P_2 = 1$ and the recurrence relation

$$P_{n+3} = P_{n+1} + P_n, \qquad n \geqslant 0. \tag{1.1}$$

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The first few terms of this sequence are $0, 0, 1, 0, 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21$. The recurrence (1.1) involves the characteristic equation

$$x^3 - x - 1 = 0.$$

If its roots are denoted by α, β and γ then, the following equalities can be derived

$$\alpha + \beta + \gamma = 0$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = -1$$

$$\alpha\beta\gamma = 1.$$

Moreover, the Binet-like formula for the Padovan sequence is

$$P_n = a\alpha^n + b\beta^n + c\gamma^n \tag{1.2}$$

where,

$$a = \frac{1}{(\alpha-\beta)(\alpha-\gamma)}, \quad b = \frac{1}{(\beta-\alpha)(\beta-\gamma)}, \quad c = \frac{1}{(\gamma-\alpha)(\gamma-\beta)}.$$

In [**16**], the Padovan numbers have the Q_P-matrix

$$Q_P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

such that

$$Q_P^n = \begin{bmatrix} P_{n-3} & P_{n-1} & P_{n-2} \\ P_{n-2} & P_n & P_{n-1} \\ P_{n-1} & P_{n+1} & P_n \end{bmatrix}.$$

2. Recurrences of the Padovan sequence

In [**1**], Carlitz defined a Fibonacci array. Some Remarks on Carlitz's Fibonacci Array are given in [**2**]. Based on this study, we defined the Padovan array as follows. The Padovan array $\{p_{m,n}\}_{m\geqslant 0, n\geqslant 0}$ is defined by the two recurrences

$$p_{m,n} = p_{m,n-2} + p_{m,n-3}, \quad n \geqslant 3 \tag{2.1}$$

$$p_{m,n} = p_{m-2,n} + p_{m-3,n}, \quad m \geqslant 3. \tag{2.2}$$

where defined

$$p_{0,n} = P_n, \quad p_{1,n} = P_{n+3}, \quad p_{2,n} = P_{n+5} \tag{2.3}$$

as the $0-$th, $1-$th and $2-$th row of the Padovan array, respectively.
The following table is readily computed.

$m \diagdown n$	0	1	2	3	4	5	6	7	8	9	10
0	0	0	1	0	1	1	1	2	2	3	4
1	0	1	1	1	2	2	3	4	5	7	9
2	1	1	2	2	3	4	5	7	9	12	16
3	0	1	2	1	3	3	4	6	7	10	13
4	1	2	3	3	5	6	8	11	14	19	25
5	1	2	4	3	6	7	9	13	16	22	29
6	1	3	5	4	8	9	12	17	21	29	38
7	2	4	7	6	11	13	17	24	30	41	54
8	2	5	9	7	14	16	21	30	37	51	67
9	3	7	12	10	19	22	29	41	51	70	92
10	4	9	16	13	25	29	38	54	67	92	121

TABLE 1. The first few members of the Padovan array

As can be seen from the table, the symmetry property

$$p_{m,n} = p_{n,m}$$

is readily proved by making use of equalities (2.1) and (2.2).

PROPOSITION 2.1. *The following equation is valid:*

$$p_{m,n} = P_m P_{n+5} + P_{m+1} P_{n+3} + P_{m-1} P_n. \tag{2.4}$$

PROOF. Considering equalities (2.2) and (2.3), we have

$$\begin{aligned}
p_{3,n} &= p_{1,n} + p_{0,n} = P_{n+3} + P_n \\
p_{4,n} &= p_{2,n} + p_{1,n} = P_{n+5} + P_{n+3} \\
p_{5,n} &= p_{3,n} + p_{2,n} = P_{n+3} + P_n + P_{n+5} \\
p_{6,n} &= p_{4,n} + p_{3,n} = P_{n+5} + 2P_{n+3} + P_n \\
p_{7,n} &= p_{5,n} + p_{4,n} = 2P_{n+5} + 2P_{n+3} + P_n \\
&\dots \\
p_{m,n} &= p_{m-2,n} + p_{m-3,n} = P_m P_{n+5} + P_{m+1} P_{n+3} + P_{m-1} P_n
\end{aligned}$$

□

Some identities given below are proven for the Fibonacci sequence. Look at [**7, 8**].

THEOREM 2.1. *The Binet-like formula for the sequence* $\{p_{m,n}\}$ *is*

$$p_{m,n} = a a_m \alpha^n + b b_m \beta^n + c c_m \gamma^n, \qquad n \geqslant 0$$

where

$$a_m = P_m \alpha^2 + P_{m+3} \alpha + P_{m+4},$$
$$b_m = P_m \beta^2 + P_{m+3} \beta + P_{m+4}$$

and

$$c_m = P_m \gamma^2 + P_{m+3} \gamma + P_{m+4}.$$

PROOF. From the definition of nth Padovan arrays $\{p_{m,n}\}$ in equality (2.2) and the Binet-like formula for nth Padovan sequence $\{P_n\}$ in equality (1.2), we write

$$\begin{aligned}
p_{3,n} &= p_{1,n} + p_{0,n} = a\alpha^n(1+\alpha^3) + b\beta^n(1+\beta^3) + c\gamma^n(1+\gamma^3)\\
p_{4,n} &= p_{2,n} + p_{1,n} = a\alpha^n(\alpha^3+\alpha^5) + b\beta^n(\beta^3+\beta^5) + c\gamma^n(\gamma^3+\gamma^5)\\
p_{5,n} &= p_{3,n} + p_{2,n} = a\alpha^n(1+\alpha^3+\alpha^5) + b\beta^n(1+\beta^3+\beta^5) + c\gamma^n(1+\gamma^3+\gamma^5)\\
p_{6,n} &= p_{4,n} + p_{3,n} = a\alpha^n(1+2\alpha^3+\alpha^5) + b\beta^n(1+2\beta^3+\beta^5) + c\gamma^n(1+2\gamma^3+\gamma^5)\\
p_{7,n} &= p_{5,n} + p_{4,n} = a\alpha^n(1+2\alpha^3+2\alpha^5) + b\beta^n(1+2\beta^3+2\beta^5) + c\gamma^n(1+2\gamma^3+2\gamma^5)\\
&\dots\\
p_{m,n} &= p_{m-2,n} + p_{m-3,n} = a\alpha^n(P_m\alpha^2+P_{m+3}\alpha+P_{m+4}) + b\beta^n(P_m\beta^2+P_{m+3}\beta+P_{m+4})\\
&\quad + c\gamma^n(P_m\gamma^2 + P_{m+3}\gamma + P_{m+4})
\end{aligned}$$

Thus, the proof is completed. □

THEOREM 2.2. *The generating function for the sequence* $\{p_{m,n}\}$ *is*

$$G_p(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} p_{m,n}x^n y^m = \frac{y^2x^2 + 4y^2x + 2y^2 + 3yx + 2y + x + 1}{(1-x^2-x^3)(1-y^2-y^3)}$$

PROOF. Let

$$p_m(x) = \sum_{n=0}^{\infty} p_{m,n}x^n.$$

In particular, it follows from equalities (2.3) that

$$p_0(x) = \sum_{n=0}^{\infty} p_{0,n}x^n = \sum_{n=0}^{\infty} P_n x^n = \frac{x+1}{1-x^2-x^3}, \tag{2.5}$$

$$p_1(x) = \sum_{n=0}^{\infty} p_{1,n}x^n = \sum_{n=0}^{\infty} P_{n+3} x^n = \frac{x^2+2x+2}{1-x^2-x^3}, \tag{2.6}$$

$$p_2(x) = \sum_{n=0}^{\infty} p_{2,n}x^n = \sum_{n=0}^{\infty} P_{n+5} x^n = \frac{2x^2+4x+3}{1-x^2-x^3},$$

and by equality (2.2), we have also

$$p_m(x) = p_{m-2}(x) + p_{m-3}(x). \tag{2.7}$$

Using equalities (2.5) and (2.7), we prove easily that

$$p_m(x) = \sum_{n=0}^{\infty} p_{m,n}x^n = \frac{xP_{m+2} + (x+2)P_{m-1} + P_{m-2} + (x^2+2x)P_{m-4}}{1-x^2-x^3}.$$

So,

$$\begin{aligned}G_p(x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}p_{m,n}x^ny^m &= \frac{x}{1-x^2-x^3}\sum_{m=0}^{\infty}P_{m+2}y^m+\frac{x+2}{1-x^2-x^3}\sum_{m=0}^{\infty}P_{m-1}y^m\\ &+\frac{1}{1-x^2-x^3}\sum_{m=0}^{\infty}P_{m-2}y^m+\frac{x^2+2x}{1-x^2-x^3}\sum_{m=0}^{\infty}P_{m-4}y^m\\ &=\frac{x}{1-x^2-x^3}\frac{y^2+2y+1}{1-y^2-y^3}+\frac{x+2}{1-x^2-x^3}\frac{y^2+y}{1-y^2-y^3}\\ &+\frac{1}{1-x^2-x^3}\frac{1}{1-y^2-y^3}+\frac{x^2+2x}{1-x^2-x^3}\frac{y^2}{1-y^2-y^3}\\ &=\frac{y^2x^2+4y^2x+2y^2+3yx+2y+x+1}{(1-x^2-x^3)(1-y^2-y^3)}.\end{aligned}$$

Thus, the proof is completed. □

THEOREM 2.3. *The exponential generating function for the sequence* $\{p_{m,n}\}$ *is*

$$\sum_{n=0}^{\infty}\frac{p_{m,n}}{n!}x^n=aa_me^{\alpha x}+bb_me^{\beta x}+cc_me^{\gamma x}.$$

PROOF. We know that,

$$e^{\alpha x}=\sum_{n=0}^{\infty}\frac{\alpha^nx^n}{n!},\quad e^{\beta x}=\sum_{n=0}^{\infty}\frac{\beta^nx^n}{n!},\quad e^{\gamma x}=\sum_{n=0}^{\infty}\frac{\gamma^nx^n}{n!}$$

Let's multiply each side of the equalities above by aa_m, bb_m and cc_m and addition the above equalities to side to side, respectively.

$$\begin{aligned}aa_me^{\alpha x}+bb_me^{\beta x}+cc_me^{\gamma x}&=\sum_{n=0}^{\infty}(aa_m\alpha^n+bb_m\beta^n+cc_m\gamma^n)\frac{1}{n!}x^n\\ &=\sum_{n=0}^{\infty}\frac{p_{m,n}}{n!}x^n\end{aligned}$$

Thus, the proof is completed. □

THEOREM 2.4. *The series for the sequence* $\{p_{m,n}\}$ *is*

$$S_p(x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{p_{m,n}}{x^ny^m}=\frac{x^3y^3+x^2y^3+2x^3y^2+2x^3y+2x^2y^2+3x^2y+2xy}{(x^3-x-1)(y^3-y-1)}.$$

PROOF. Let

$$p_m(x)=\sum_{n=0}^{\infty}\frac{p_{m,n}}{x^n}.$$

In particular, it follows from equalities (2.3) that

$$p_0(x) = \sum_{n=0}^{\infty} \frac{p_{0,n}}{x^n} = \sum_{n=0}^{\infty} \frac{P_n}{x^n} = \frac{x^3 + x^2}{x^3 - x - 1}, \tag{2.8}$$

$$p_1(x) = \sum_{n=0}^{\infty} \frac{p_{1,n}}{x^n} = \sum_{n=0}^{\infty} \frac{P_{n+3}}{x^n} = \frac{2x^3 + 2x^2 + x}{x^3 - x - 1}, \tag{2.9}$$

$$p_0(x) = \sum_{n=0}^{\infty} \frac{p_{0,n}}{x^n} = \sum_{n=0}^{\infty} \frac{P_{n+5}}{x^n} = \frac{3x^3 + 4x^2 + 2x}{x^3 - x - 1},$$

and by equality (2.2) we have also

$$p_m(x) = p_{m-2}(x) + p_{m-3}(x). \tag{2.10}$$

Using equalities (2.8) and (2.10), we prove easily that

$$p_m(x) = \sum_{n=0}^{\infty} \frac{p_{m,n}}{x^n} = \frac{x^2 P_{m+1} + x^3 P_m + (x^3 + x^2 + x)P_{m-1} + (x^3 + x^2 + x)P_{m-4}}{x^3 - x - 1}.$$

So,

$$\begin{aligned} S_p(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{p_{m,n}}{x^n y^m} &= \frac{x^2}{x^3 - x - 1} \sum_{m=0}^{\infty} \frac{P_{m+1}}{y^m} + \frac{x^3}{x^3 - x - 1} \sum_{m=0}^{\infty} \frac{P_m}{y^m} \\ &+ \frac{x^3 + x^2 + x}{x^3 - x - 1} \sum_{m=0}^{\infty} \frac{P_{m-1}}{y^m} + \frac{x^3 + x^2 + x}{x^3 - x - 1} \sum_{m=0}^{\infty} \frac{P_{m-4}}{y^m} \\ &= \frac{x^2}{x^3 - x - 1} \frac{y^3 + y^2 + y}{y^3 - y - 1} + \frac{x^3}{x^3 - x - 1} \frac{y^3 + y^2}{y^3 - y - 1} \\ &+ \frac{x^3 + x^2 + x}{x^3 - x - 1} \frac{y^2 + y}{y^3 - y - 1} + \frac{x^3 + x^2 + x}{x^3 - x - 1} \frac{y}{y^3 - y - 1} \\ &= \frac{x^3y^3 + x^2y^3 + 2x^3y^2 + 2x^3y + 2x^2y^2 + 3x^2y + 2xy}{(x^3 - x - 1)(y^3 - y - 1)} \end{aligned}$$

Thus, the proof is completed. □

THEOREM 2.5. *The partial sum for the sequence* $\{p_{m,n}\}$ *is*

$$T_p = \sum_{t=0}^{m}\sum_{k=0}^{n} p_{t,k} = \frac{P_m P_{n+5} + P_{m+1} P_{n+3} + P_{m-1} P_n}{4}.$$

PROOF. Considering equality (2.4), we have

$$\begin{aligned} T_p &= \sum_{t=0}^{m} P_t \sum_{k=0}^{n} P_{k+5} + \sum_{t=0}^{m} P_{t+1} \sum_{k=0}^{n} P_{k+3} + \sum_{t=0}^{m} P_{t-1} \sum_{k=0}^{n} P_k \\ &= (P_{n+9} + P_{n+5} - 3) \sum_{t=0}^{m} P_t + (P_{n+6} + P_{n+5} - 2) \sum_{t=0}^{m} P_{t+1} + (P_{n+5} - 1) \sum_{t=0}^{m} P_{t-1} \\ &= (P_{n+9} + P_{n+5} - 3)(P_{m+5} - 1) + (P_{n+6} + P_{n+5} - 2)(P_{m+6} - 1) + (P_{n+5} - 1)(P_{m+5} - P_{m+1}). \end{aligned}$$

Thus, the proof is completed. □

3. Topics for future study

Above we have given the various identities of the Padovan array. We leave it to the researchers to obtain the various identities of the Padovan matrix array that we have defined below. The Padovan matrix array $\left\{Q_p^{m,n}\right\}_{m,n\geqslant 0}$ is defined by the two recurrences

$$Q_p^{m,n} = Q_p^{m,n-2} + Q_p^{m,n-3}, \quad n \geqslant 3 \tag{3.1}$$

$$Q_p^{m,n} = Q_p^{m-2,n} + Q_p^{m-3,n}, \quad m \geqslant 3. \tag{3.2}$$

where the initial conditions are defined as

$$Q_p^{0,n} = \begin{bmatrix} P_{n-1} & P_{n+1} & P_n \\ P_n & P_{n+2} & P_{n+1} \\ P_{n+1} & P_{n+3} & P_{n+2} \end{bmatrix}, \quad Q_p^{1,n} = \begin{bmatrix} P_{n+2} & P_{n+4} & P_{n+3} \\ P_{n+3} & P_{n+5} & P_{n+4} \\ P_{n+4} & P_{n+6} & P_{n+5} \end{bmatrix}, \tag{3.3}$$

$$Q_p^{2,n} = \begin{bmatrix} P_{n+4} & P_{n+6} & P_{n+5} \\ P_{n+5} & P_{n+7} & P_{n+6} \\ P_{n+6} & P_{n+8} & P_{n+7} \end{bmatrix}.$$

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