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IDEAL ELEMENTS IN ORDERED SEMIRINGS

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Abstract. The aim of this paper is to study the structures of some ordered semirings not only with the ideal elements but also with the generalization of ideal elements. The ideal elements play an important and necessary role in studying the structure of ordered semirings. We introduce the notion of ideal elements, interior ideal elements, quasi ideal elements, bi-ideal elements, bi-interior ideal elements, quasi interior ideal elements and weak interior ideal elements of ordered semirings. We study the properties of ideal elements and relations between them. We characterize the ordered semirings, regular ordered semirings and simple ordered semirings using ideal elements. We prove that if M is a simple ordered semiring, then every element of M is an ideal element of M.

1. Introduction

We know that the notion of a one sided ideal of any algebraic structure is a generalization of the notion of an ideal. The quasi ideals are generalization of left ideals and right ideals whereas the bi-ideals are generalization of quasi ideals. In 1952, the concept of bi-ideals was introduced by Good and Hughes [2] for semigroups. The notion of bi-ideals in rings and semirings were introduced by Lajos and Szasz [9,10]. In 1956, Steinfeld [23] first introduced the notion of quasi ideals for semigroups and then for rings. Iseki [4,5,6] introduced the concept of quasi ideal for a semiring. Henriksen [3] studied ideals in semirings. Quasi ideals in $\Gamma-$semirings studied by Jagtap and Pawar. Marapureddy Murali Krishna Rao [12-21] introduced and studied bi-quasi-ideals in semirings, bi-quasi-ideals and fuzzy bi-quasi ideals in $\Gamma-$semigroups

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In this paper, we introduce the notion of ideal elements, interior ideal elements, quasi ideal elements, bi-ideal elements, bi- interior ideal elements, quasi interior ideal elements and weak interior ideal elements of ordered semirings. We study the properties of ideal elements and relations between them. We characterize the ordered semirings, regular ordered semirings and simple ordered semirings using ideal elements.

2. Preliminaries

In this section, we recall some of the fundamental concepts and definitions which are necessary for this paper.

Definition 2.1. A set M together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$ respectively) will be called semiring provided

(i) addition is a commutative operation.
(ii) multiplication distributes over addition both from the left and from the right.
(iii) there exists $0 \in M$ such that $x + 0 = x$ and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in M$.

Example 2.1. Let M be the set of all natural numbers. Then (M, max, min) is a semiring.

Definition 2.2. Let M be a semiring. If there exists $1 \in M$ such that $a \cdot 1 = 1 \cdot a = a$, for all $a \in M$, is called an unity element of M then M is said to be semiring with unity.

Definition 2.3. An element a of a semiring M is called a regular element if there exists an element b of M such that $a = aba$.

Definition 2.4. A semiring M is called a regular semiring if every element of M is a regular element.

Definition 2.5. An element a of a semiring M is called a multiplicatively idempotent (an additively idempotent) element if $aa = a(a + a = a)$.

Definition 2.6. An element b of a semiring M is called an inverse element of a of M if $ab = ba = 1$.

Definition 2.7. A semiring M is called a division semiring if for each non-zero element of M has multiplication inverse.

Definition 2.8. A non-empty subset A of a semiring M is called

(i) a subsemiring of M if $(A, +)$ is a subsemigroup of $(M, +)$ and $AA \subseteq A$.
(ii) a quasi ideal of M if A is a subsemiring of M and $AM \cap MA \subseteq A$.
(iii) a bi-ideal of M if A is a subsemiring of M and $AMA \subseteq A$.
(iv) an interior ideal of M if A is a subsemiring of M and $MAM \subseteq A$.
(v) a left (right) ideal of M if A is a subsemiring of M and $MA \subseteq A(AM \subseteq A)$.
(vi) an ideal if A is a subsemiring of $M, AM \subseteq A$ and $MA \subseteq A$.

(vii) a bi-interior ideal of M if A is a subsemiring of M and $MAM \cap AMA \subseteq A$.
(viii) a left bi-quasi ideal (right bi-quasi ideal) of M if A is a $-$subsemiring of M and $MA \cap AMA \subseteq A$ $(AM \cap AMA \subseteq A)$.
(ix) a bi-quasi ideal of M if A is a subsemiring of M and A is a left bi-quasi ideal and a right bi-quasi ideal of M.
(x) a left quasi-interior ideal (right quasi-interior ideal) of M if A is asubsemiring of M and $MAMA \subseteq A$ $(AMAM \subseteq A)$.
(xi) a quasi-interior ideal of M if A is asubsemiring of M and $A$is a left quasi-interior ideal and a right quasi-interior ideal of M.
(xii) a bi-quasi-interior ideal of M if A is a subsemiring of M and $AMAMA \subseteq A$.
(xiii) a left (right) tri- ideal of M if A is a subsemiring of M and $AMAA \subseteq A$ $(AAMA \subseteq A)$.
(xiv) a tri- ideal of M if A is a subsemiring of M and $AMAA \subseteq A$ and $AAMA \subseteq A$.
(xv) a left(right) weak-interior ideal of M if A is a $-$subsemiring of M and $MAA \subseteq A(AAM \subseteq A)$.A weak-interior ideal of M if A is a $-$subsemiring of M and A is a left weak-interior ideal and a right weak-interior ideal)of M.
(xvi) a $k-$ideal of M if A is an ideal of M and $x \in M,\ x + y \in A, y \in A$ then $x \in A$.
(xvii) a $m - k-$ideal of M if A is an ideal of M and $x \in A,\ xy \in A, 1 \neq y \in M$then $y \in A$.

Definition 2.9. A semiring M is called an ordered semiring if it admits a compatible relation $\leqslant$. i.e. $\leqslant$ is a partial ordering on M satisfies the following conditions. If $a \leqslant b$ and $c \leqslant d$ then

(i) $a + c \leqslant b + d, c + a \leqslant d + b$
(ii) $ac \leqslant bd$
(iii) $ca \leqslant db$, for all $a, b, c, d \in M$

Example 2.2. Let $M = [0, 1]$. A binary operation $+$ is defined as $a + b = \max\{a, b\}$, for all $a, b \in M$ and $x \cdot y = \min\{x, y\}$, for all $x, y \in M$. Then M is an ordered semiring M with usual ordering . All ideals of M are closed intervals, $[0, a]$ for some $a \in M$.

Definition 2.10. An ordered semiring M is said to have zero element if there exists an element $0 \in M$ such that $0+x = x = x+0$ and $0x = x0 = 0,$ for all $x \in M$.

Definition 2.11. An ordered semiring M is said to be commutative semiring if $xy = yx$, for all $x, y \in M$.

Definition 2.12. A non-zero element a in an ordered semiring M is said to be a zero divisor if there exists non zero element $b \in M$, such that $ab = ba = 0$.

Definition 2.13. An ordered semiring M with unity 1 and zero element 0 is called an integral ordered semiring if it has no zero divisors.

Definition 2.14. A non-empty subset A of an ordered semiring M is called a subsemiring M if $(A,+)$ is a subsemigroup of $(M,+)$ and $ab \in A$ for all $a, b \in A$.

Definition 2.15. Let M and N be ordered semirings. A mapping $f : M \to N$ is called a homomorphism if

(i) $f(a+b) = f(a) + f(b)$
(ii) $f(ab) = f(a)f(b)$, for all $a, b \in M$.

Definition 2.16. A subsemiring I of an ordered semiring M is called an ideal if for any $x \in I, y \in M$ and $y \leqslant x \Rightarrow y \in I$.

3. A study on ideal elements in ordered semirings

In this section, we introduce the notion of ideal elements, interior ideal elements, bi-ideal elements, quasi interior ideal elements and weak interior ideal elements of ordered semirings. We study the properties of ideal elements and relations between them. We characterize the ordered semirings using ideal elements.

Definition 3.1. An element a of an ordered semiring M is called a subsemiring element if $aa \leqslant a$.

Definition 3.2. An element a of an ordered semiring M is called a left(right) ideal element of M, if $xa \leqslant a(ax \leqslant a)$, for all $x \in M$.

Definition 3.3. An element of an ordered semiring M is called an ideal element of M, if it is both a left ideal element and a right ideal element of M.

Definition 3.4. Let M be an ordered semiring. An element a of M is said to be bi-ideal element of M if $aa \leqslant a$, $axa \leqslant a$, for all $x \in M$.

Definition 3.5. An element a of an ordered semiring M is called a quasi ideal element of M, if $aa \leqslant a$, there exist elements $x, y \in M$, such that $xa = ay \leqslant a$.

Definition 3.6. Let M be an ordered semiring. An element a of M is said to be quasi interior ideal element of M if $aa \leqslant a$, $axay \leqslant a$ and $xaya \leqslant a$, for all $x, y \in M$.

Definition 3.7. A bi-ideal element b is said to be minimal if for every bi-ideal element a of an ordered semiring M, $a \leqslant b \Rightarrow a = b$.

Definition 3.8. Let M be an ordered semiring. An element a of M is said to be interior ideal element if $aa \leqslant a$, $xay \leqslant a$, for all $x, y \in M$.

Definition 3.9. An element a of an ordered semiring M is called a left(right) weak interior ideal of M if $aa \leqslant a$, $xaa \leqslant a(aax \leqslant a)$, for all $x \in M$.

Definition 3.10. An element of an ordered semiring M is called a weak interior ideal element if $aa \leqslant a$, $aax \leqslant a$ and $xaa \leqslant a$, for all $x \in M$.

Definition 3.11. An element a of an ordered semiring M is called a regular if there exists element $x \in M$, such that $a \leqslant axa$.

Theorem 3.1. *If a is an ideal element of an ordered semiring M, then a is an interior ideal element of M.*

Proof. Suppose a is an ideal element of the semiring M.
Then $ax \leqslant a$ and $ya \leqslant a$, for all $x, y \in M$.
That implies $xay \leqslant xa \leqslant a$ and hence a is an interior ideal element of M. □

Theorem 3.2. *If a is a left ideal element of an ordered semiring M, then a is a quasi ideal element of M.*

Proof. Suppose that a is a left ideal element of M.
Then $xa \leqslant a$, for all $\in$, $x \in M$.
Let $x, y \in M$, such that $xa = ay$.
Then $xa \leqslant a$, Therefore $xa = ay \leqslant a$. □

Theorem 3.3. *If a is is a quasi ideal element of an ordered semiring M, then a is a bi- ideal element of M.*

Proof. Suppose that a is a quasi ideal element of M.Then there exist elements $x, y \in M$, such that $xa = ay \leqslant a$. Then $ay \leqslant a$, and $aya \leqslant aa \leqslant a$,
That implies $aya \leqslant a$. Hence a is a bi- ideal element of M. □

Theorem 3.4. *If a is an ideal element of an ordered semiring M, then a is a bi- ideal element of M.*

Proof. Suppose that a is an ideal element of M.
Then $ax \leqslant a$, and $xa \leqslant a$, for all $x \in M$.
That implies $axa \leqslant xa \leqslant a$. Hence a is a bi- ideal element of M. □

Theorem 3.5. *If a and b are minimal left ideal elements of an ordered semiring M, then ab is a minimal left ideal element of an ordered semiring M.*

Proof. Suppose a and b are minimal left ideals of the ordered semiring M.
Then $xa \leqslant a$ and $xb \leqslant b$,for al $x \in M$.
That implies $xab \leqslant ab$, for all $x \in M$.
Therefore ab is the left ideal element of M.
Suppose c is any left ideal element of M and $c \leqslant ab$.
Then $c \leqslant ab \leqslant a$
Therefore $ab = c$.
Hence ab is a minimal left ideal of the ordered semiring M. □

Theorem 3.6. *Let M be an ordered semiring M. If a is a maximal element of M then a is an ideal element of M.*

Proof. Suppose a is a maximal element of the orderedsemiring M.
Then $ax \leqslant a$ and $xa \leqslant a$, for all $x \in M$.
Therefore a is an ideal element of M. □

Theorem 3.7. *If a is a minimal element of an ordered semiring M, then a is a not an ideal element of M.*

Proof. Suppose a is a minimal element of M.
Then $a \leqslant x$, for all $x \in M$.
Therefore $a \leqslant ax$, for all $x \in M$.Hence the theorem. □

Theorem 3.8. *If a is an interior ideal element and idempotent element of ordered semiring M, then a is an ideal element of M.*

Proof. Suppose a is an interior ideal element and a is an $-$idempotent of the ordered semiring M.
Then $xay \leqslant a$, for all and $x, y \in M$.
That implies $xaa \leqslant a$ and hence $xa \leqslant a$, for all $x \in M$.
Similarly we can prove that $ax \leqslant a$.
Hence a is an ideal element of M. □

Theorem 3.9. *Let a be a regular element of an ordered semiring M. Then a is an interior ideal element if and only if a is an ideal element of M.*

Proof. Assume that a be an interior ideal element of M.
Then $xay \leqslant a$, for all $x, y \in M$.
We have that $a \leqslant axa$, for some $x \in M$.
Then $ax \leqslant axax \leqslant a$, by the definition of interior ideal element of M.
Similarly we can prove that $xa \leqslant a$.
Hence a is an ideal element of M. Conversely, assume that a is an ideal element of M.
Suppose $y, z \in M$.
Then $yaz \leqslant az \leqslant a$. Hence a is an interior ideal element of the ordered semiring M. □

Theorem 3.10. *Let M be an ordered regular semiring. If a is a bi-ideal element of M and a commutes with every element of M then a is an ideal element.*

Proof. Let M be an ordered semiring, a be a bi-ideal element of M and a commutes with every element of M.
Then $a \leqslant axa$, for some $x \in M$.
That implies $a \leqslant axa \leqslant a$.
Therefore $axa = a$.
Suppose $y \in M$.
Then $ay = axay = axya \leqslant a$, for all $y \in M$.
Hence a is an ideal element of M. □

Theorem 3.11. *Every interior ideal element of an ordered semiring M is a quasi interior ideal element.*

Proof. Suppose a is an interior ideal element of M.
Then $xay \leqslant a$, for $x, y \in M$.
That implies $axya \leqslant a$ and $xaaa \leqslant a$.
Hence every interior ideal element is a quasi interior ideal element. □

Theorem 3.12. *Every left ideal element of an ordered semiring M is a left quasi interior element of M.*

PROOF. Suppose x is a left ideal element of M.
Then $xa \leqslant a$, for $x \in M$. That implies $xyya \leqslant a$.
Hence a is left quasi ideal element of M. □

THEOREM 3.13. *If a is a quasi interior ideal element of a regular ordered semiring M then a is an ideal element of M.*

PROOF. Suppose a is a quasi interior ideal element of M.
Then $axay \leqslant a$, $xaya \leqslant a$, for all $x, y \in M$ and $a \leqslant aba$, for some $b \in M$.
Suppose $x \in M$.
Then $ax \leqslant abax \leqslant a$ and $xa \leqslant xaba \leqslant a$.
Hence a is an ideal element of M. □

THEOREM 3.14. *Every interior ideal element of an ordered semiring M is a quasi interior ideal element.*

PROOF. Suppose a is an interior ideal element of M.
Then $xay \leqslant a$, for all $x, y \in M$
That implies $axay \leqslant a$ and $xaaa \leqslant a$.
Hence every interior ideal element is a quasi interior ideal element. □

THEOREM 3.15. *Every left ideal element of an ordered semiring M is a left quasi interior element of M.*

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PROOF. Suppose x is a left ideal element of M.
Then $xa \leqslant a$, for all $x \in M$.
That implies $xyya \leqslant a$.
Hence a is left quasi ideal element of M. □

THEOREM 3.16. *If a is a quasi interior ideal element of a regular ordered semiring M then a is an ideal element of M.*

PROOF. Suppose a is a quasi interior ideal element of M.
Then $axay \leqslant a$, $xaya \leqslant a$, for all $x, y \in M$.
and $a \leqslant aba$, for some $b \in M$. Suppose $x \in M$.
Then $ax \leqslant abax \leqslant a$ and $xa \leqslant xaba \leqslant a$.
Hence a is an ideal element of M. □

COROLLARY 3.1. *Let a be an interior ideal element of an ordered semiring M. Then a is a weak interior ideal element of M.*

THEOREM 3.17. *If a is an ideal element of an ordered semiring M then a is a left weak interior ideal element of M.*

PROOF. Suppose a is an ideal element of M.
Then $ax \leqslant a$ and $xa \leqslant a$, for all $x \in M$.
That implies $xaa \leqslant aa \leqslant a$ and $ax \leqslant a$.
Therefore $aax \leqslant aa \leqslant a$.
Hence ideal element of M is a weak interior ideal element of M. □

Theorem 3.18. *Let M be an ordered semiring. If a is a weak interior ideal element and idempotent element of M then a is an interior ideal element of M.*

Proof. Let a be an interior ideal and idempotent element of M.
Then $aa = a$.
Suppose $x, y \in M$.
Now $xay = xaaaay \leqslant aa = a$. □

Corollary 3.2. *Let a be an interior ideal element of an ordered semiring M. Then a is a weak interior ideal element of M.*

Theorem 3.19. *Let M be a simple ordered semiring. Then every element of M is an ideal element of M.*

Proof. Let $I = \{a \mid ax \leqslant a \text{ and } xa \leqslant a, \text{ for all } x \in M, \}$and $a, b \in I$.
Then $(a+b)x = ax + bx \leqslant a + b$ and $x(a+b) = xa + xb \leqslant a + b$.
Therefore $a + b \in I$.
For $abx = a(bx) \leqslant ab$ and $x(ab) = (xa)b \leqslant ab$.
Therefore $ab \in I$. Hence I is a subsemiring of M.
Suppose that $x \in M,\ a \in I$ and $x \leqslant a$,
Then $yx \leqslant ya \leqslant a$ and $xy \leqslant ay \leqslant a$. Hence $x \in I$.
Therefore I is an ideal of the ordered semiring M. Hence $I = M$. □

Theorem 3.20. *Let M be a left (right) simple ordered semiring. Then every element of M is a left (right) ideal element of M.*

Theorem 3.21. *If a is an ideal element of an ordered semiring M, then a is a bi-ideal element of M.*

Proof. Suppose a is an ideal element of M.
Then $ax \leqslant a$ and $xa \leqslant a$, for all $x \in M$.
That implies $xay \leqslant xa \leqslant a$ and hence a is an interior ideal element of M. □

Theorem 3.22. *If a is an interior ideal element of an ordered semiring M, then a is an ideal element of M.*

Proof. Suppose a is an interior ideal element and a is an idempotent.
Then $xay \leqslant a$, and $x, y \in M$.
That implies $xaa \leqslant a$ and hence $xa \leqslant a$, for all $x \in M$.
Similarly we can prove that $ax \leqslant a$.
Hence a is an ideal element of M. □

4. Conclusion

In this paper, we studied the structures of some ordered semirings with the generalization of ideal elements.The ideal elements play an important and necessary role in studying the structure of ordered semirings. We introduced the notion of ideal elements, interior ideal elements, quasi ideal elements, bi-ideal elements, bi-interior ideal elements, quasi interior ideal elements and weak interior ideal elements of ordered semirings. We studied the properties of ideal elements and relations between them. We characterize the ordered semirings, regular ordered semirings and simple ordered semirings using ideal elements.

References

1. P. J. Allen, *A fundamental theorem of homomorphism for semirings*, Proc. Amer. Math. Sco., **21** (1969), 412–416.
2. R. A. Good and D. R. Hughes, *Associated groups for a semigroup*, Bull. Amer. Math. Soc., **58** (1952), 624-625.
3. M. Henriksen, *Ideals in semirings with commutative addition*, Amer. Math. Soc. Noti,, **5** (1958), 321.
4. K. Iseki, *Quasi-ideals in semirings without zero*, Proc. Japan Acad., **34** (1958), 79-84.
5. K. Iseki, *Ideal theory of semiring*, Proc. Japan Acad., **32** (1956), 554-559.
6. K. Iseki, *Ideal in semirings*, Proc. Japan Acad., **34** (1958), 29-31.
7. K. Izuka, *On the Jacobson radical of a semiring*, Tohoku, Math. J., **11(2)** (1959), 409-421.
8. N.Kehayopulu, *Interior ideals and interior ideal elements in ordered semigroups*, Pure Math.and Appl.,2(3)(1999), 407-409.
9. S. Lajos and F. A. Szasz, *On the bi-ideals in associative ring*, Proc. Japan Acad., **46** (1970), 505-507.
10. S. Lajos, *On the bi-ideals in semigroups*, Proc. Japan Acad., **45** (1969), 710-712.
11. S. Lajos, *(m; k; n)-ideals in semigroups.* In: Notes on Semigroups II, Karl Marx Univ. Econ., Dept. Math. Budapest (1976), No. 1, 12-19.
12. M. Murali Krishna Rao, *bi-interior Ideals in semigroups*, Disc. Math. General Alg. and Appli., **38** (2018), 69-78.
13. M. Murali Krishna Rao, *bi-interior Ideals in* $\Gamma-$ *semirings*, Disc. Math. General Alg. and Appli. , **38(2)** (2018), 239-254.
14. M. Murali Krishna Rao, *Ideals in ordered* $\Gamma-$*semirings*, Disc. Math. General Alg. and Appli., **38** (2018), 47-68.
15. M.Murali Krishna Rao, *A study of generalization of bi- ideal, quasi- ideal and interior ideal of semigroup*, Mathematica Morovica, **22(2)** (2018), 103-115.
16. M. Murali Krishna Rao, *A study of bi-quasi-interior ideal as a new generalization of ideal of generalization of semiring*, *Bull. Int. Math. Virtual Inst.,* **8** (2018), 519-535.
17. M.Murali Krishna Rao, *A study of quasi-interior ideal of semiring*, *Bull. Int. Math. Virtual Inst.,* **2** (2019), 287-300.
18. M. Murali Krishna Rao, *Left bi-quasi ideals of semirings*, Bull. Int. Math. Virtual Inst., **8** (2018),45-53.
19. M. Murali Krishna Rao, B. Venkateswarlu and N.Rafi, *Left bi-quasi-ideals of* $\Gamma-$*semirings*, Asia Pacific J. of Math., **4(2)** (2017), 144-153.
20. M. Murali Krishna Rao, *Bi-quasi-ideals and fuzzy bi-quasi ideals of* $\Gamma-$ *semigroups*, Bull. Int. Math. Virtual Inst., **7(2)** (2017), 231-242.
21. A. M. Shabir, A. Batod, *A note on quasi ideal in semirings*, Southeast Asian Bull. Math., **7** (2004), 923-928.
22. O. Steinfeld, *Uher die quasi ideals*, Von halbgruppend, Publ. Math., Debrecen, **4** (1956), 262–275.
23. H. S. Vandiver, *Note on a simple type of algebra in which cancellation law of addition does not hold*, Bull. Amer. Math. Soc.(N.S.), **40** (1934), 914–920.

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