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SOME RESULTS ON LIE IDEALS WITH SYMMETRIC REVERSE BI-DERIVATIONS IN SEMIPRIME RINGS II

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Abstract. Let R be a semiprime ring, U a square-closed Lie ideal of R and $D : R\times R \to R$ a symmetric reverse bi-derivation and d be the trace of D. In the present paper, we shall prove that R contains a nonzero central ideal or $D = 0$ if any one of the following holds: i) $d(U) = (0)$, ii)$d(xy) + d(x)d(y) \pm xy \in Z$, iii)$d(xy) + d(x)d(y) \pm yx \in Z$, iv)$d(xy) - d(yx) \pm [x, y] \in Z$, v)$D$ acts as left or right homomorphism on U, vi)$D(d(x), x) = 0$ vii)$d(d(x)) = g(x)$, for all $x, y \in U$, where $G : R \times R \to R$ is symmetric reverse bi-derivations such that g is the trace of G.

1. Introduction

Throughout R will represent an assosiative ring with center Z. A ring R is said to be prime if $xRy = (0)$ implies that either $x = 0$ or $y = 0$ and semiprime if $xRx = (0)$ implies that $x = 0$, where $x, y \in R$. A prime ring is obviously semiprime. However, the opposite is not always true. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy-yx$ and the symbol xoy stands for the commutator $xy+yx$. An additive subgroup U of R is said to be a Lie ideal of R if $[u, r] \in U$, for all $u \in U$, $r \in R$. U is called a square-closed Lie ideal of R if U is a Lie ideal and $u^2 \in U$ for all $u \in U$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. Also, an additive mapping $d : R \to R$ is said to be a reverse derivation if $d(xy) = d(y)x + yd(x)$ holds for all $x, y \in R$. A mapping $D(.,.) : R \times R \to R$ is said to be symmetric if $D(x, y) = D(y, x)$ for all $x, y \in R$. A mapping $d : R \to R$ is called the trace of $D(.,.)$ if $d(x) = D(x, x)$ for all $x \in R$. It is obvious that if $D(.,.)$ is bi-additive (i.e., additive in both arguments),

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then the trace d of $D(.,.)$ satisfies the identity $d(x+y)=d(x)+d(y)+2D(x,y)$, for all $x,y\in R$. If $D(.,.)$ is bi-additive and satisfies the identities

$$D(xy,z)=D(x,z)y+xD(y,z)$$

and

$$D(x,yz)=D(x,y)z+yD(x,z),$$

for all $x,y,z\in R$. Then $D(.,.)$ is called a symmetric bi-derivation. If $D(.,.)$ is reverse bi-additive and satisfies the identity

$$D(xy,z)=D(y,z)x+yD(x,z)$$

and

$$D(x,yz)=D(x,z)y+zD(x,y).$$

Then $D(.,.)$ is called a symmetric reverse bi-derivation.

In 1980, Maksa [**7**] introduced the concept of a symmetric biderivation on a ring R. It was shown in [**8**] that symmetric biderivations are related to general solution of some functional equations. Some results on a symmetric biderivation in prime and semiprime rings can be found in [**11**] and [**12**]. Typical examples are mappings of the form $(x,y)\longmapsto\lambda[x,y]$ where $\lambda\in C$. We shall call such maps inner biderivations. It was shown in [**4**] that every bi-derivation D of a noncommutative prime ring R is of the form $D(x,y)=\lambda[x,y]$ for some $\lambda\in C$. Moreover, in [**5**], Bresar extended this result to semiprime rings.

We shall say that a mapping $D(.,.):R\times R\to R$ acts as a right (resp. left) R homomorphism on I if $D(rx,y)=D(x,y)r$ and $D(x,ry)=D(x,y)r$ (resp. $D(xr,y)=rD(x,y)$ and $D(x,yr)=rD(x,y)$) for all $x,y,z\in R$. In [**13**], Yeşilgül and Argaç investigated that a prime ring and semiprime ring with D acts homomorphism and symmetric bi-derivation on R. In [**6**], Daif and Bell showed that if a semiprime ring R admits a derivation d such that $xy\pm d(xy)=yx\pm d(yx)$, for all $x,y\in R$, then R is commutative ring. In [**2**], Ashraf showed that commutativity of a prime ring R which admits a symmetric bi-derivation and Reddy et al. generalized this for the semiprime ring in [**9**]. Also, in [**1**], Ashraf and Rehman showed that a prime ring R with a nonzero ideal I must be commutative if it admits a derivation d satisfying either of the properties $d(xy)-xy\in Z$ or $d(xy)-yx\in Z$ for all $x,y\in R$. Many authors have investigated these conditions for different derivations. On the other hand, in [**12**], Vukman proved that if a semiprime ring R with symmetric bi-derivation D and d be the trace of D such that $D(d(x),x)=0$ and $d(d(x))=g(x)$, for all $x\in R$, then $D=0$ and Reddy et al. studied symmetric reverse bi-derivations this theorem in [**10**].

In this paper, we shall extend the above results for a square-closed Lie ideal of semiprime rings with symmetric reverse bi-derivations. Throughout the present paper, we shall make use of the following basic identities without any specific mention:

i) $[x,yz]=y[x,z]+[x,y]z$
ii) $[xy,z]=[x,z]y+x[y,z]$
iii) $xyoz=(xoz)y+x[y,z]=x(yoz)-[x,z]y$
iv) $xoyz=y(xoz)+[x,y]z=(xoy)z+y[z,x]$.

2. Results

LEMMA 2.1. [**3**, Theorem 1.3] *Let R be a 2− torsion free semiprime ring and U a noncentral square-closed Lie ideal of R. Then there exist a nonzero ideal I of R such that $I \subseteq U$.*

LEMMA 2.2. [**6**, Lemma 2 (b)] *If R is a semiprime ring, then the center of a nonzero ideal of R is contained in the center of R.*

LEMMA 2.3. *Let R be a 2−torsion free semiprime ring and I a nonzero ideal of R. If $[I, I] \subset Z$, then R contains a nonzero central ideal.*

PROOF. By the hypothesis, we get

$$[x, y] \in Z, \text{ for all } x, y \in I.$$

Replacing y by yx in above expression, we have

$$[x, y]x \in Z, \text{ for all } x, y \in I.$$

Commuting this term with r, $r \in R$, we obtain that

$$[[x, y]x, r] = 0, \text{ for all } x, y \in I, r \in R.$$

Using the hypothesis in the last expression, we get

$$[x, y][x, r] = 0, \text{ for all } x, y \in I, r \in R.$$

Replacing r by ry in the above equation and using this expression, we see that

$$[x, y]R[x, y] = 0, \text{ for all } x, y \in I.$$

Since R is semiprime ring, we get

$$[x, y] = 0, \text{ for all } x, y \in I.$$

That is, $[I, I] = (0)$. By Lemma 2.2, we get $I \subseteq Z$. We conclude that R contains a nonzero central ideal. This completes the proof. □

LEMMA 2.4. *Let R be a 2−torsion free semiprime ring, I an ideal of R, $D : R\times R \to R$ a symmetric reverse bi-derivation, d be the trace of D and $D(R, R) \subset I$. If $d(I) = (0)$, then $D = 0$.*

PROOF. By the hypothesis, we have

$$d(x) = 0, \text{ for all } x \in I.$$

Replacing x by $x + y$, $y \in I$ in this equation and using this equation, we get

$$2D(x, y) = 0, \text{ for all } x, y \in I.$$

Since R is 2−torsion free, we have

$$D(x, y) = 0, \text{ for all } x, y \in I.$$

Taking x by xr, $r \in R$ in the above equation and using this equation, we obtain that

$$D(r, y)x = 0, \text{ for all } x, y \in I, r \in R.$$

Replacing y by $ys, s \in I$, we have

$$D(r, s)yx = 0, \text{ for all } x, y \in I, r, s \in R.$$

Taking x by $tD(r,s)y$, $t \in R$, we have

$$D(r,s)ytD(r,s)y = 0, \text{ for all } y \in I, r, s, t \in R.$$

That is,

$$D(r,s)yRD(r,s)y = (0), \text{ for all } y \in I, r, s \in R.$$

By the semiprimeness of R, we get

$$D(r,s)y = 0, \text{ for all } y \in I, r, s \in R.$$

Replacing y by $tD(r,s)$, $t \in R$, we find that

$$D(r,s)RD(r,s) = 0, \text{ for all } y \in I, r, s \in R.$$

Since R is semiprime, we get $D = 0$. This completes the proof. □

In this section, we examined the above-mentioned commutativity conditions for symmetric reverse bi-derivation on Lie ideal in the semiprime ring.

THEOREM 2.5. *Let R be a $2-$torsion free semiprime ring, U a noncentral square-closed Lie ideal of R and $D : R \times R \to R$ a symmetric reverse bi-derivation and d be the trace of D. If $d(xy) + d(x)d(y) \pm xy \in Z$, for all $x, y \in U$, then R contains a nonzero central ideal.*

PROOF. By Lemma 2.1, there exist a nonzero ideal I of R such that $I \subseteq U$. Thus, using the hypothesis, we get

$$d(xy) + d(x)d(y) \pm xy \in Z, \text{ for all } x, y \in I.$$

Replacing y by $y + z, z \in I$, we have

$$d(xy) + d(xz) + d(x)d(y) + d(x)d(z) + 2D(xy, xz) + 2d(x)D(y,z) \pm xy \pm xz \in Z.$$

Using the hypothesis, we obtain that

$$2(D(xy, xz) + d(x)D(y,z)) \in Z.$$

Since R is $2-$torsion free, we see that

$$D(xy, xz) + d(x)D(y,z) \in Z, \text{ for all } x, y, z \in I.$$

Replacing z by y in this expression, we have

$$D(xy, xy) + d(x)D(y,y) \in Z, \text{ for all } x, y \in I.$$

and so,

$$d(xy) + d(x)d(y) \in Z, \text{ for all } x, y \in I.$$

By the hypothesis, we have

$$xy \in Z, \text{for all } x, y \in I. \tag{2.1}$$

Commuting this term with $r, r \in R$, we get

$$[xy, r] = 0, \text{for all } x, y \in I, r \in R, \tag{2.2}$$

and so

$$[x,r]y + x[y,r] = 0, \text{for all } x, y \in I, r \in R.$$

Replacing y by yz in this equation and using equation (2.2), we have

$$[x,r]yz = 0, \text{for all } x, y, z \in I, r \in R.$$

Writting z by $[x,r]$ in above equation, we arrive at

$$[x,r]y[x,r]=0, \text{for all } x,y\in I, r\in R.$$

That is,

$$[x,r]yR[x,r]y=0, \text{for all } x\in I, r\in R.$$

By the semiprimeness of R, we have

$$[x,r]y=0, \text{for all } x\in I, r\in R.$$

Taking y by $t[x,r]$, $t\in R$ in the last equation, we see that

$$[x,r]R[x,r]=0, \text{for all } x\in I, r\in R.$$

Since R is semiprime, we get

$$[x,r]=0, \text{for all } x\in I, r\in R.$$

That is, $I\subset Z$. By Lemma 2.2, we obtain that R contains a nonzero central ideal. This completes the proof. □

THEOREM 2.6. *Let R be a $2-$torsion free semiprime ring, U a noncentral square-closed Lie ideal of R and $D:R\times R\to R$ a symmetric reverse bi-derivation and d be the trace of D. If $d(xy)+d(x)d(y)\pm yx\in Z$, for all $x,y\in U$, then R contains a nonzero central ideal.*

PROOF. By Lemma 2.1, there exist a nonzero ideal I of R such that $I\subseteq U$. By the hypothesis, we have

$$d(xy)+d(x)d(y)\pm yx\in Z, \text{ for all } x,y\in I.$$

Writting y by $y+z, z\in I$, we have

$$d(xy)+d(xz)+d(x)d(y)+d(x)d(z)+2D(xy,xz)+2d(x)D(y,z)\pm yx\pm zx\in Z.$$

Appliying the hypothesis, we get

$$2(D(xy,xz)+d(x)D(y,z))\in Z.$$

Since R is $2-$torsion free and taking z by y, we see that

$$D(xy,xy)+d(x)D(y,y)\in Z, \text{ for all } x,y\in I.$$

That is,

$$d(xy)+d(x)d(y)\in Z, \text{ for all } x,y\in I.$$

By the hypothesis, we have

$$yx\in Z, \text{for all } x,y\in I.$$

Using the same arguments after (2.1) in the proof of Theorem 2.5, we get the required results. □

THEOREM 2.7. *Let R be a $2-$torsion free semiprime ring, U a noncentral square-closed Lie ideal of R and $D:R\times R\to R$ a symmetric reverse bi-derivation and d be the trace of D. If $d(xy)-d(yx)\pm[x,y]\in Z$, for all $x,y\in U$, then R contains a nonzero central ideal.*

PROOF. By Lemma 2.1, there exist a nonzero ideal I of R such that $I \subseteq U$. We get

$$d(xy) - d(yx) \pm [x, y] \in Z, \text{ for all } x, y \in I.$$

Taking y by $y + z, z \in I$, we have

$$d(xy) + d(xz) + 2D(xy, xz) - d(yx) - d(zx) - 2D(yx, zx) \pm [x, y] \pm [x, z] \in Z.$$

Using the hypothesis, we obtain that

$$2D(xy, xz) - 2D(yx, zx) \in Z.$$

Since R is $2-$torsion free, we get

$$D(xy, xz) - D(yx, zx) \in Z, \text{ for all } x, y \in I.$$

Replacing z by y, we see that

$$D(xy, xy) - D(yx, yx) \in Z, \text{ for all } x, y \in I,$$

and so

$$d(xy) - d(yx) \in Z, \text{ for all } x, y \in I.$$

By the hypothesis, we have

$$[x, y] \in Z, \text{for all } x, y \in I.$$

Using Lemma 2.3, we see that R is commutative ring. □

THEOREM 2.8. *Let R be a $2-$torsion free semiprime ring, I an ideal of R, $D : R \times R \to R$ a symmetric reverse bi-derivation, d be the trace of D and $D(R, R) \subset I$. If D acts as a left (resp. right) homomorphism on I, then $D = 0$.*

PROOF. By our hypothesis, we get D acts as a left homomorphism on I. That is,

$$D(x, yz) = zD(x, y) \text{ for all } x, y, z \in I.$$

On the other hand, since D is reverse bi-derivation, we get

$$D(x, yz) = zD(x, y) + D(x, z)y.$$

Then,

$$D(x, z)y = 0, \text{for all } x, y, z \in I.$$

Taking y by $rD(x, z)$, $r \in R$ in the above equation, we get

$$D(x, z)rD(x, z) = 0, \text{for all } x, z \in I, \ r \in R.$$

That is,

$$D(x, z)RD(x, z) = (0), \text{for all } x, z \in I.$$

By the semiprimeess of R, we obtain that $D(x, z) = 0$, for all $x, z \in I$. Replacing z by x, we get $d(x) = 0$, for all $x \in I$. We conclude that $D = 0$ by Lemma 2.4. If D acts as a right homomorphism on I, it can be proved by using the same techniques. □

THEOREM 2.9. *Let R be a $2-$torsion free semiprime ring, U a noncentral square-closed Lie ideal of R and $D : R \times R \to R$ a symmetric reverse bi-derivation and d be the trace of D. If $D(d(x), x) = 0$, for all $x \in U$, then $D = 0$.*

Proof. By Lemma 2.1, there exist a nonzero ideal I of R such that $I \subseteq U$. We get

$$D(d(x), x) = 0 \text{ for all } x \in I.$$

Replacing x by $x + y$, $y \in I$ in this equation, we have

$$D(d(x), x) + D(d(x), y) + D(d(y), x) + D(d(y), y) + 2(D(x, y), x) + 2(D(x, y), y) = 0$$

By the hypothesis, we get

$$D(d(x), y) + D(d(y), x) + 2D(D(x, y), x) + 2D(D(x, y), y) = 0.$$

Taking x by $-x$ in this equation, we obtain that

$$D(d(x), y) - D(d(y), x) + 2D(D(x, y), x) - 2D(D(x, y), y) = 0$$

We obtained from the last two equations

$$D(d(x), y) + 2D(D(x, y), x) = 0. \tag{2.3}$$

Writtig y by yx in equation (2.3), we obtain that

$$xD(d(x), y) + 2D(x, y)d(x) + 2D(D(x, y), x)x + 2D(y, x)d(x) = 0.$$

Multipliying in equation (2.3) by x on right hand side, we see that

$$D(d(x), y)x + 2(D(x, y), x)x = 0,$$

Combining the last two equations are used, we obtain

$$[x, D(d(x), y)] + 4D(x, y)d(x) = 0. \tag{2.4}$$

Replacing y by yx in equation (2.4), we get

$$[x, xD(d(x), y) + D(d(x), x)y] + 4xD(x, y)d(x) + 4D(x, x)yd(x) = 0$$

By the hypothesis, we have

$$x\{[x, D(d(x), y)] + 4D(x, y)d(x)\} + 4d(x)yd(x) = 0$$

Appliying equation (2.4) and using $2-$torsion free, we find that

$$d(x)yd(x) = 0 \text{ for all } x, y \in I.$$

That is,

$$d(x)yRd(x)y = (0) \text{ for all } x, y \in I.$$

By the semiprimeness of R, we have

$$d(x)y = 0 \text{ for all } x, y \in I.$$

Replacing y by $rd(x), r \in R$ in the last equation, we have

$$d(x)rd(x) = 0 \text{ for all } x \in I,\ r \in R.$$

Since R is semiprime, we get $d(x) = 0$, for all $x \in I$. We conclude that $D = 0$ by Lemma 2.4. This completes proof. □

THEOREM 2.10. *Let R be a $2-$torsion free and $3-$torsion free semiprime ring, I an ideal of R and $D : R \times R \to R$, $G : R \times R \to R$ two symmetric reverse bi-derivations where d is the trace of D and g is the trace of G such that $G(R,R) \subset I$. If $d(d(x)) = g(x)$ for all $x \in I$, then $G = 0$.*

PROOF. By our hypothesis, we have

$$d(d(x)) = g(x) \text{ for all } x \in I.$$

Replacing x by $x + y$, $y \in I$, we get

$$d(d(x)) + d(d(y)) + 2D(d(x), d(y)) + 4d(D(x, y)) + 4D(d(x), D(x, y)) + 4D(d(y), D(x, y))$$
$$= g(x) + g(y) + 2G(x, y)$$

By the hypothesis and since R is $2-$torsion free, we obtain that

$$D((d(x), d(y)) + 2d(D(x, y)) + 2D(d(x), D(x, y)) + 2D(d(y), D(x, y)) - G(x, y) = 0.$$

Taking x by $-x$ in this equation, we see that

$$D((d(x), d(y)) + 2d(D(x, y)) - 2D(d(x), D(x, y)) - 2D(d(y), D(x, y)) + G(x, y) = 0.$$

If the last two equations are used, we obtain

$$4D(d(x), D(x, y)) + 4D(d(y), D(x, y)) = 2G(x, y).$$

Since R is $2-$torsion free, we get

(2.5) $$2D(d(x), D(x, y)) + 2D(d(y), D(x, y)) = G(x, y) \text{ for all } x, y \in I.$$

Taking x by $2x$ in equation (2.5), we see that

$$16D(d(x), D(x, y) + 4D(d(y), D(x, y)) = 2G(2x, y).$$

If the last two equations are used, we obtain

$$12D(d(x), D(x, y)) = 0.$$

Since R is $2-$torsion free and $3-$torsion free, we get

$$D(d(x), D(x, y)) = 0, \text{ for all } x, y \in I.$$

Replacing y by x in this equation, we see that

$$D(d(x), D(x, x)) = 0,$$

and so, $d(d(x)) = 0$ for all $x \in I$. By the hypothesis, we get $g(x) = 0$ for all $x \in I$. By Lemma 2.4, we have $G = 0$. This completes proof. □

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