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V-MODULES, SSI-MODULES, AND IDEMPOTENT OF HEREDITARY PRETORSION CLASSES IN $\sigma[M]$

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ABSTRACT. A module M is called a V-module (cosemisimple module) if every simple module in $\sigma[M]$ is M-injective. A module M is an SSI-module if every semisimple module in $\sigma[M]$ is M-injective. The main objective of this paper is to prove that a module M is an SSI-module if and only if M is a locally noetherian V-module. We also prove that an R-module M over a commutative ring R is semisimple if and only if every hereditary pretorsion class in ptors-M is an idempotent radical.

1. Introduction

By a ring R we mean an associative ring with unity unless otherwise stated. By the word R-module we mean unitary right R-module unless stated otherwise, and the category of unital right R-modules shall be denoted by Mod-R. The symbol $\subseteq$ denotes containment and $\subset$ proper containment for sets. When we write $L \leqslant_e M$, we mean L is an essential submodule of M or M is an essential extension of L.
For a functor τ, σ: Mod-$R\to$ Mod-R, τ is a subfunctor of σ if for any morphism f : $M \to N$, $\tau(M) \subseteq \sigma(M)$ and $\tau(f) = \sigma(f)|_{\tau(M)}$ is the restriction of $\sigma(f)$ to $\tau(f)$. A preradical τ is a subfunctor of the identity functor id: Mod-$R \to$ Mod-R. For a preradical τ and a module M, M is τ-torsion if $\tau(M) = M$ and τ-trosion free if $\tau(M) = 0$. A preradiacl τ is idempotent (resp. radical) if τ^2=τ, i.e., $\tau(\tau(M))$= $\tau(M)$ (resp. $\tau(M/\tau(M)) = 0$, i.e.,$\tau(M) = M$) for all modules M in Mod-R. A preradical τ is left exact if the sequence $0 \to \tau(A) \to \tau(B) \to \tau(C)$ is exact for every short exact sequence $0 \to A \to B \to C \to 0$. A preradical class τ is

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idempotent radical if it is both idempotent and radical. A class of modules is called hereditary if it is closed under submodules and a preradical class τ is herediatry if τ is idempotent and the class of modules $T_\tau = \{M \in \text{Mod-}R : \tau(M) = M\}$ is hereditary. We call a class of modules pretorsion (resp. hereditary pretorsion) class if it is closed under factor module and (arbitrary) direct sums (resp. factor modules, (arbitrary) direct sums, and submodules). A hereditary pretorsion class which is closed under extension is called herediraty torsion class. Associated to a preradical τ there are two classes of modules in Mod-R, namely $T_\tau = \{N \in \text{Mod-}R : \tau(N) = N\}$ and $F_\tau = \{N \in \text{Mod-}R : \tau(N) = 0\}$. Note that T_τ is a pretorsion class and F_τ is a pretorsion free class. When we say an idempotent of a herediatry pretorsion classes in $\sigma[M]$, we mean an idempotent hereditary preradical class τ for which the class of modules $T_\tau = \{N \in \sigma[M] : \tau(N) = N\}$ is pretorsion in $\sigma[M]$ see [**2, 4**].

A subcategory S of a category C is called full subcategory if for each pair of objects N and M of S, Mor_S(N,M)= Mor_C(N,M) (where Mor starnd for morphism). In the above definition if we take $S = \sigma[M]$ and C= Mod-R, then we say $\sigma[M]$ is a full subcategory of the category Mod-R if for ecah pair of modules N and M in $\sigma[M]$, $\text{Hom}_{\sigma[M]}$(N,M)=Hom_{Mod-R}(N,M) see [**9**].

If R is a ring, we denote the set of all hereditary pretorsion classes of right R-modules by ptor-R_R and the set of all hereditary torsion classes by tors-R_R. If N, M are right R-modules, then we say that N is subgenerated by M, or M is a subgenerator for N if N is isomorphic to a submodule of an M-generated module. We denote by $\sigma[M]$, the full subcategory of Mod-R whose objects are all R-modules subgenerated by M see [**9**]. Let M and N be R-modules. We call N, M-injective if for every submodule K of M every R-homomorphism from K to N can be lifted to an R-homomorphism from M to N. A module M is called a V-module (cosemisimple module in [**5**]) if every simple module in $\sigma[M]$ is M-injective. A module M is SSI-module if every semisimple module in $\sigma[M]$ is M-injective. A module N in $\sigma[M]$ is said to be uniform if any two non-zero submodules of N has nonzero intersection in $\sigma[M]$. For $M \in \text{Mod-}R$ we denote the set of all hereditary pretorsion classes of the full subcategory $\sigma[M]$ of Mod-R by ptors-M and we denote the set of all hereditary torsion classes by tors-M. For any R-module M its injective hull is denoted by $E(M)$ and if $N \in \sigma[M]$, then the M-injective hull of N is denoted by the symbol $E_M(N)$. If $\tau \in$ tors-R_R and $M \in$ Mod-R, then there is a (unique) largest submodule of M belonging to τ, denoted by $\tau(M)$, and called the τ-torsion submodule of M. If $\tau(M) = M$, or equivalently $M \in \tau$, we say that M is τ-torsion and if $\tau(M) = 0$ we say that M is τ-torsion free. We call $\tau \in$ ptors-M splitting if $\tau(N)$ is a direct summand of N for each $N \in \sigma[M]$. We call a module M locally noetherian (artinian) if every finitely generated submodule of M is noetherian (artinian). This is equivalent to the requirement that all finitely generated (cyclic) modules in $\sigma[M]$ are noetherian(artinian) in $\sigma[M]$ see [**5**].

Let M be an R-module. A socle of M (= Soc(M), Soc M) we denote the sum of all simple (minimal) submodules of M [**9**], i.e.,

$$\text{Soc } M = \sum\{N \leqslant M \mid N \text{ is simple}\}.$$

The socle of M is the largest submodule of M generated by simple modules or equivalently, it is the largest semisimple module of M. Dually the radical of M is the submodule which is the intersection of all maxiaml submodle of M

$$\text{Rad } M = \bigcap\{N \leqslant M \mid N \text{ is maximal in M}\}.$$

It is proved in [**1**, Theorem 2.5] that the direct sum of any family of M-injective modules is M-injective if and only if every cyclic submodule of M is noetherian, that is, M is locally noetherian. It is also shown in [**9**] that any M-injective module U is also N-injective whenever N is a submodule, a factor module or direct sum of copies of M. It follows immediately that a module is injective in $\sigma[M]$ if it is M-injective. This is a Baer's like criterion. Thus, there are enough injectives in $\sigma[M]$ [**9**]. It is proved in [4, Proposition 1, p.236] that R is a right SSI-ring if and only if R is a right noetherian right V-ring. We prove in Theorem 1.3 that an R-module M is an SSI-module if and only if M is a locally noetherian V-module. Viola-Prioli [7, Theorem 1, p. 545] has proved that for a commutative ring R, every hereditary pretorsion class is a hereditary torsion class if and only if R is semisimple. This result is generalized in Theorem 1.5 in which it is proven that if M is a module over a commutative ring R, then ptors-M = tors-M if and only if M is semisimple.

1.1. V-modules and SSI-modules. The following result generalizes the V-rings of Villamayor [**3**] to V-modules.

THEOREM 1.1. *The following statements are equivalent for $M \in$ Mod-R:*

(a) M is a V-module;
(b) $Rad(N) = 0$ for all $N \in \sigma[M]$ where $Rad(N)$ is the Jacobson Radical;
(c) Every proper submodule of M is an intersection of maximal submodules.

PROOF. (a) $\Rightarrow$ (b) Suppose M is a V-module and $0 \neq N \in \sigma[M]$. Take $0 \neq x \in N$ and let B be maximal among the submodules of N which exclude x. Since every nonzero submodule of N/B contains $(xR + B)/B$, we infer that N/B is uniform and $(xR + B)/B$ is simple. By hypothesis, $(xR + B)/B$ is M-injective and thus a direct summand of N/B. Since N/B is uniform, this means $N/B = (xR + B)/B$, so B is a maximal submodule of N. Since $\text{Rad}\,(N) \subseteq B$ and $x \notin B, x \notin \text{Rad}\,(N)$. Therefore, $\text{Rad}\,(N) = 0$, as required.
(b) $\Rightarrow$ (c) Suppose (b) holds. Let L be a proper submodule of M. Since $M/L \in \sigma[M]$, $\text{Rad}\,(M/L) = 0$ from which we infer that L is an intersection of maximal submodules of M.

(c) $\Rightarrow$ (a) Assume (c) holds. Let S be any simple module in $\sigma[M]$, L a submodule of M and
$\rho : L \to S$ be an R-homomorphism. We need to extend ρ to a mapping of M to S. We may assume ρ is epimorphism and replace ρ by the canonical epimorphism $\alpha : L \to L/K$, $\alpha(x) = x + K$ where $K = ker\rho$, $L/K \cong S$. It is enough to extend α to M. By (b) we have $\text{Rad}\,(M/K) = 0$ and since L/K is simple, there exists a maximal submodule N/K of M/K such that $L/K \bigoplus N/K = M/K$. Consider the canonical epimorphism $\pi : M \to M/K$ and the projection $\beta : M/K \to L/K$. Then $\beta \circ \pi : M \to L/K (\cong S)$ extends α. This implies S is M-injective. This shows

that every simple module in $\sigma[M]$ is M-injective and hence M is a V-module as desired. □

THEOREM 1.2. *If* Soc *splits in* $\sigma[M]$*, then* M *is an SSI-module.*

PROOF. Assume L is semisimple in $\sigma[M]$. Obviously $L \leqslant_e E_M(L)$ (where $E_M(L)$ is the M-injective hull of L) so that $L \leqslant_e Soc(E_M(L))$. Since a semisimple module possesses no proper essential submodule, this entails $L = \text{Soc}(E_M(L))$. By hypothesis, $L = \text{Soc}(E_M(L))$ is a direct summand of $E_M(L)$, whence $L = \text{Soc}(E_M(L)) = E_M(L)$. We conclude that L is M-injective. This shows that M is an SSI-module. □

Let $\{L_\delta : \delta \in \Delta\}$ be a family of right R-modules. If $x = \{x_\delta\}_{\delta\in\Delta} \in \prod_{\delta\in\Delta} L_\delta$, then the support of x abbreviated by supp x is defined by

$$\text{supp } x = \{\delta \in \Delta : x_\delta \neq 0\}.$$

If $X \subseteq \prod_{\delta\in\Delta} L_\delta$, we define

$$\text{supp } X = \bigcup_{x\in X} \text{supp } x.$$

THEOREM 1.3. *A right R-module M is an SSI- module if and only if M is a locally noetherian V-module.*

PROOF. ($\Rightarrow$) Assume that $M \in$ Mod-R is an SSI-module. Since M is clearly a V-module, it remains to show that M is locally noetherian. Suppose, on the contrary, that M is not locally noetherian. This entails the existence of a cyclic submodule xR of M and a strictly ascending chain of submodules

$$N_0 \subset N_1 \subset N_2 \subset \cdots$$

of xR. Take $k \in \mathbb{N}$. Because M is a V-module and $N_k/N_{k-1} \in \sigma[M]$, it follows from Theorem 1.1(b) that $\text{Rad}(N_k/N_{k-1}) = 0$, so that N_k/N_{k-1} has a maximal proper submodule M_k/N_{k-1} say, where $N_{k-1} \subseteq M_k \subseteq N_k$. Put $N = \bigcup_{k\in\mathbb{N}} N_k = \bigcup_{k\in\mathbb{N}} M_k$.

Consider the canonical map $\lambda : N \to \prod_{k\in\mathbb{N}} N/M_k$ defined by:

$$\lambda(t) = \{t + M_k\}_{k\in\mathbb{N}}.$$

Since $N = \bigcup_{k\in\mathbb{N}} N_k$, $\lambda(t)$ has finite support for each $t \in \mathbb{N}$, so we may interpret λ as a mapping with codomain $\bigoplus_{k\in\mathbb{N}} N_k/M_k$. Take $j \in \mathbb{N}$. Since M is a V-module, the simple module $N_j/M_j \in \sigma[M]$ is M-injective so that N_j/M_j is a direct summand of N/M_j. If $\kappa_j : N_j/M_j \to N/M_j$ denotes the inclusion map, there exists a projection map $\rho_j : N/M_j \to N_j/M_j$ such that $\rho_j\kappa_j$ coincides with the identity map on N_j/M_j.

Consider the diagram

$$\begin{array}{ccccc}
 & & \mathfrak{F} \longmapsto \{\hat{\varphi}_{S_P}\}_P & & \\
xR & \xrightarrow{\Theta} & \bigoplus_{k\in\mathbb{N}} N_k/M_k & \xrightarrow{\psi_j} & \bigoplus N_j/M_j \\
\uparrow{\scriptstyle \iota} & & \uparrow{\scriptstyle \rho} & & {\scriptstyle \kappa_j}\updownarrow{\scriptstyle \rho_j} \\
N & \xrightarrow{\lambda} & \bigoplus_{k\in\mathbb{N}} N/M_k & \xrightarrow{\varphi_j} & N/M_j \\
 & & I \longmapsto \{I_P\}_P & &
\end{array}$$

where φ_j and ψ_j are the canonical projection maps, ρ is the epimorphism induced by the family of epimorphisms $\{\rho_k : k \in \mathbb{N}\}$ and ι is the inclusion map.

Inasmuch as $\bigoplus_{k\in\mathbb{N}} N_k/M_k$ is semisimple, it is, by hypothesis, M-injective so that $\rho\lambda$ can be extended to an R-homomorphism Θ from xR to $\bigoplus_{k\in\mathbb{N}} N_k/M_k$. Observe that since xR is finitely generated, $\operatorname{Im}(\Theta)$, and hence $\operatorname{Im}\rho\lambda$, has finite support. We now argue that this conclusion yields a contradiction. For each $j \in \mathbb{N}$, pick $t_j \in N_j \backslash M_j$. Then

$$\begin{aligned}
(\psi_j\rho\lambda)(t_j) &= (\rho_j\varphi_j\lambda)(t_j) \\
&= \rho_j\varphi_j(\{t_j + M_k\}_{k\in\mathbb{N}}) \\
&= \rho_j(t_j + M_j) \\
&= \rho_j\kappa_j(t_j + M_j) \quad \text{[because } t_j \in N_j \text{ and } \kappa_j \text{ is an inclusion map]} \\
&= t_j + M_j \\
&\neq 0 \ \text{[because } t_j \notin M_j\text{]}.
\end{aligned}$$

It follows from the above that $j \in \operatorname{supp}((\rho\lambda)(t_j))$, whence $\operatorname{supp}(\operatorname{Im}(\rho\lambda)) = \mathbb{N}$. This contradicts the fact that $\operatorname{Im}(\rho\lambda)$ has a finite support.

($\Leftarrow$) This is an immediate consequence of the fact that if M is locally noetherian then every direct sum of M-injective modules is M-injective [**9**]. $\square$

The following result generalizes the results in [**3**] on V-rings and SSI-rings.

Corollary 1.1. *(a) If M is a V-module, then any locally artinian module is locally noetherian.*

(b) If M is an SSI-module, then a module N in $\sigma[M]$ is locally artinian if and only if it is semisimple.

Proof. (a) If M is a V-module and $0 \neq N \in \sigma[M]$, we can find a maximal proper submodule N_1. Since N_1 is nonzero, similarly we can find a maximal proper submodule N_2 of N_1. Repeating this process we get a chain $N \supset N_1 \supseteq N_2 \supset \cdots$ which terminates, if N is locally artinian, in a composition series for N. Thus it follows that N is locally noetherian.

(b) ($\Rightarrow$) Assume M is an SSI module and $N \in \sigma[M]$ is locally artinain. Take a cyclic submodule xR of $N(x \in N)$. Since xR is artinian $\mathrm{Soc}\,(xR)$ is essential in xR. Since xR is essential in its injective hull $E_M(xR)$ in $\sigma[M]$, we have $E_M(\mathrm{Soc}\,(xR)) \cong E_M(xR)$. But $\mathrm{Soc}\,(xR)$ is M-injective. Then $E_M(\mathrm{Soc}\,(xR)) \cong \mathrm{Soc}\,(xR)$. Hence $E_M(xR)$ is semisimple. Thus xR is semisimple. It follows that N is semisimple. The converse easily follows from that fact that any semisimple module is locally artinian.

$\square$

It is easy to infer from [**7**, Lemma 5] that $\mathcal{P} = \{N \in \sigma[\mathrm{M}]$: every proper submodule of N is an intersection of maximal submodules of $N\}$ is a preradical class. It is also proved in [**7**, Proposition 6] that a right noetherian ring R for which all preradical classes in Mod-R are hereditary torsion classes is a right V-ring.
The following theorem generalizes this result to the full subcategory $\sigma[M]$.

Theorem 1.4. *Let M be a locally noetherian module such that all preradical classes in the subcategory $\sigma[M]$ of Mod-R are hereditary torsion classes. Then M is a V-module.*

Proof. Consider

$$\mathcal{P} = \{N \in \sigma[M]\colon\ \mathrm{Rad}\,(N/N') = 0 \text{ for all } N' \leqslant N\}.$$

An argument similar to [**7**, Lemma 5] shows that $\mathcal{P}$ is a preradical class. By hypothesis $\mathcal{P}$ is a hereditary torsion class.

Let L be an arbitrary finitely generated module belonging to $\sigma[M]$. We claim that $L \in \mathcal{P}$. Suppose, on the contrary, that $L \notin \mathcal{P}$.

Define

$$\mathcal{S} = \{U \leqslant L : L/U \notin \mathcal{P}\}.$$

Observe that $\mathcal{S} \neq \emptyset$ since $0 \in \mathcal{S}$. Since L is noetherian, $\mathcal{S}$ has maximal member, N say. Since $L/N \notin \mathcal{P}$, there exists a submodule $\hat{N}$ of L such that $\hat{N} \supseteq N$ and $Rad(L/N) = \hat{N}/N \neq 0$.

Observe that if K is any submodule of L satisfying $N \subseteq K \subseteq \hat{N}$, then $\mathrm{Rad}\,(L/K) = \hat{N}/K$.

Since $\hat{N}$ is noetherian, it has a maximal proper submodule, say K. Clearly $\hat{N}/K$ is simple and thus a member of $\mathcal{P}$.

It follows from the maximality of N that $L/\hat{N} \in \mathcal{P}$. Since $\mathcal{P}$ is closed under module extensions, this entails $L/K \in \mathcal{P}$, so $\mathrm{Rad}\,(L/K) = 0$ which contradicts the fact that $\mathrm{Rad}\,(L/N) = \hat{N}/N \neq 0$.

We conclude that $L \in \mathcal{P}$. We have thus shown that $\mathcal{P}$ contains every finitely generated module belonging to $\sigma[M]$. Since $\sigma[M]$ is generated by its cyclic (and thus finitely generated) members and $\mathcal{P}$ is closed under direct sums and homomorphic images, we conclude that $\mathcal{P} = \sigma[M]$, whence $\mathrm{Rad}\,(N) = 0$ for all $N \in \sigma[M]$. The result follows from Theorem 1.1((b) $\Rightarrow$ (a)).

□

THEOREM 1.5. *Let R be a commutative ring. The following statements are equivalent for a right R-module M:*

(a) M is an SSI-module;
(b) M is a locally noetherian V-module;
(c) M is semisimple;
(d) ptors-M = tors-M.

PROOF. (a) and (b) are equivalent by Theorem 1.3 without the commutativity assumption on R.

Clearly (c) implies (a) and (d), again, without the commutativity assumption on R.

(b) $\Rightarrow$ (c) Pick $0 \neq x \in M$ and put $I = \{r \in R : xr = 0\}$. Observe that Mod-R/I $=\{N \in$ Mod-$R : NI = 0\} \subseteq \sigma[M]$. Take arbitrary $N \in$ Mod-R/I. It follows from Theorem 1.1, that $\mathrm{Rad}\,(N) = 0$ (N considered as a right R-module). Since the R-submodules and R/I-submodules of N coincide, we may infer that $\mathrm{Rad}\,(N) = 0$ (N considered as a right R/I-module). It follows that R/I is a V-ring. We know from theory [**8**, Lemma 5] that a commutative V-ring is Von Neumann Regular. Since R/I is also noetherian (because M is locally noetherian), we conclude that R/I is a semisimple ring (and thus a finite product of fields).

We infer from the above that xR is a semisimple right R-module for every $x \in M$, so M is semisimple.

(d) $\Rightarrow$ (c) Pick $0 \neq x \in M$ and put $I = \{r \in R : xr = 0\}$ and consider the commutative ring R/I. It is easily seen that each member of ptors-R/I belongs to ptors-M. Since, by hypothesis, ptors-M = tors-M, it follows that ptors-R/I = tors-R/I. It follows from [**6**, Theorem 1, p. 545], [the Viola-Prioli paper] that R/I is a semisimple ring. As argued in the proof of (b) $\Rightarrow$ (c), this entails M is semisimple, as required.

□

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