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HUB-INTEGRITY OF TOTAL TRANSFORMATION GRAPHS

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Abstract. The *hub-integrity* of a connected graph $G = (V(G), E(G))$ is denoted as $HI(G)$ and defined by $HI(G) = min\{|S| + m(G - S)\}$, where S is hub set and $m(G-S)$ is the order of a maximum component of $G-S$. In this paper, we give results for the hub-integrity of total transformation graphs.

The vulnerability of network have been studied in various contexts including road transportation system, information security, structural engineering and communication network. The vulnerability of a graph is a determination that includes certain properties of the graph not to be damaged after the deletion of a number of vertices or edges. In the theory of graphs, the vulnerability implies a lack of resistance of graph network arising from deletion of vertices or edges or both. Communication networks must be so designed that they do not easily get disrupted under external attack and even if they get disturbed then should be easily reconstructible. The concept of integrity of a graph is in [**2**], which is defined as follows.

Definition 0.1. [**2**] The integrity of a graph G is denoted by $I(G)$ and defined as $I(G) = min\{|S| + m(G-S) : S \subseteq V(G)\}$, where $m(G-S)$ denotes the order of a maximum components of $G-S$.

In 2006, Walsh [**18**] have defined hub number of a graph to study a network related problem, which is defined as follows.

Let G be a graph with vertex set $V(G)$ and S be a subset in a graph G such that $S \subseteq V(G)$ and let $x, y \in V(G)$. An S-path between x and y is a path where

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all intermediate vertices are from S. A set $S \subseteq V(G)$ is a hub set of G if it has the property that, for any $x, y \in V(G) \setminus S$ there is an S-path in G between x and y. The minimum cardinality of hub set is called *hub number* and is denoted by $h(G)$ [**11, 18**]. The concept of hub-integrity of a graph was introduced by Sultan et al. [**15**], is defined as follows.

DEFINITION 0.2. [**15**] The hub-integrity of a graph G is denoted as $HI(G)$ and defined as $HI(G) = min\{|S| + m(G-S)\}$, where S is a hub set and $m(G-S)$ is the order of a maximum components of $G-S$.

Barefoot et al. [**2, 3**] are studied integrity and edge-integrity. Clark et al. [**6**] obtained the computational complexity of integrity. Goddard et al. [**8, 10, 9**] discussed relation between integrity and other graph parameters. Mamut et al. [**14**] obtained the integrity of middle graphs. Grauman et al. [**11**] obtained the relationship between hub number, connected hub number and connected domination number of a graph. For more on hub-integrity refer [**16, 17**]. In this paper, we compute hub-integrity of total transformation graphs.

1. Preliminaries

All graphs considered in this paper are nontrivial, connected, simple and undirected graphs. Let G be a graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and edge set $E(G) = \{e_1, e_2, ..., e_m\}$. Thus $|V(G)| = n$ and $|E(G)| = m$ where, n and m are called *order* and *size* of graph G respectively. The *complement* of a graph G [**12**] is denoted by $\overline{G}$ whose vertex set is $V(G)$ and two vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G. The *line graph* $L(G)$ of a graph G [**19**] is the graph with vertex set as the edge set of G and two vertices of $L(G)$ are adjacent whenever the corresponding edges in G have a vertex in common. The *subdivision graph* $S(G)$ of a graph G [**12**] whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if and only if one is a vertex of G and other is an edge of G incident with it. The *partial complement of subdivision graph* $\overline{S}(G)$ of a graph G [**13**] whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if and only if one is a vertex of G and the other is an edge of G not incident with it. A vertex v is called leaf vertex of G if $d_G(v) = 1$. For undefined terminology and notations refer [**5, 12**].

The *total transformation graphs* G^{xyz}, introduced by Baoyindureg et al. [**1**] is defined as follows. Let $G = (V(G), E(G))$ be a graph, and x, y, z be three variables taking values $+$ or $-$. The transformation graph G^{xyz} is the graph having $V(G) \cup E(G)$ as the vertex set, and for α, $\beta \in V(G) \cup E(G)$, α and β are adjacent in G^{xyz} if and only if one of the following holds:
(i) α, $\beta \in V(G)$. α and β are adjacent in G if $x = +$; α and β are nonadjacent in G if $x = -$.
(ii) α, $\beta \in E(G)$. α and β are adjacent in G if $y = +$; α and β are nonadjacent in G if $y = -$.
(iii) $\alpha \in V(G)$, $\beta \in E(G)$. α and β are incident in G if $z = +$; α and β are nonincident in G if $z = -$.

Thus, one can obtain eight kinds of transformation graphs, in which G^{+++} is the total graph [**4**] of G, and G^{---} is its complement. Also, G^{--+}, G^{-+-}, and G^{-++} are the complements of G^{++-}, G^{+-+}, and G^{+--} respectively, are depicted in Figure 1. The vertex v of G^{xyz} corresponding to a vertex v of G is referred to as a *point vertex*. The vertex e of G^{xyz} corresponding to an edge e of G is referred to as a *line vertex*.

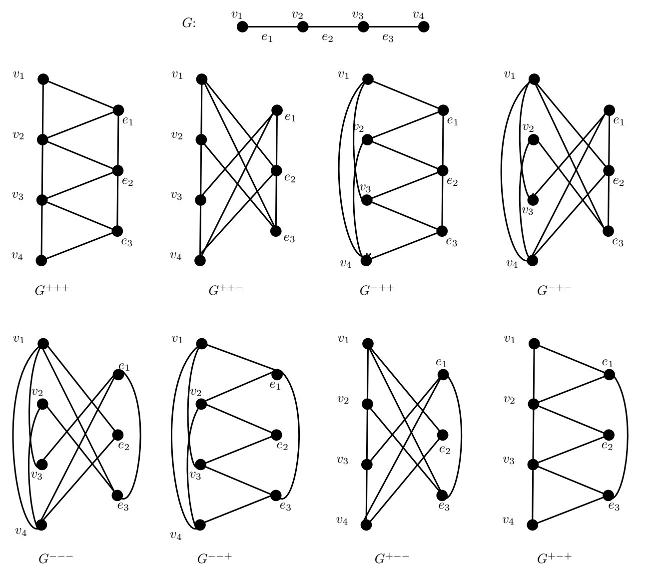

FIGURE 1. Graph G and its total transformation graphs G^{xyz}.

PROPOSITION 1.1. [**15**] *The hub-integrity of*
i) The complete graph K_n, $HI(K_n) = n$,
ii) The path P_n *with* $n \geqslant 3$, $HI(P_n) = n-1$,
iii) The cycle C_n,

$$HI(C_n) = \begin{cases} n-1 & if\ n = 4,\ 5, \\ n-2 & if\ n \geqslant 6. \end{cases}$$

2. Hub-integrity of total transformation graphs

THEOREM 2.1. *Let G be any graph of order n, size m and l be a leaves. Then*

$$HI(G^{+++}) = \begin{cases} 3 & if\ G = P_2, \\ 4 & if\ G = P_3, \\ 5 & if\ G = C_3, \\ \lfloor \frac{n}{2} \rfloor + m & path,\ if\ n \geqslant 4, \\ m+2 & star,\ if\ n \geqslant 4, \\ n - l + \lceil \frac{m}{2} \rceil + 2 & tree,\ if\ n \geqslant 4, \\ \lceil \frac{n}{2} \rceil + \lceil \frac{m}{2} \rceil + 2 & cycle,\ if\ n \geqslant 4, \\ n+m-1 & otherwise,\ if\ n \geqslant 4. \end{cases}$$

Proof. Let $G = P_2$, then P_2^{+++} is a complete graph with three vertices. Therefore, $HI(P_2^{+++}) = 3$ (since hub-integrity of complete graph K_n is n [**15**]). Let $G = P_3$. Then choose nonpendent vertex v_i and line vertex e_i in P_3^{+++}. Therefore, $|S| = 2$ and $m(G-S) = 2$. Hence, $HI(P_3^{+++}) = 4$. Let $G = C_3$, then (C_3^{+++}) is a 4-regular graph with 6 vertices. Choose any two point vertices v_i and v_j in C_3^{+++}. Next, choose two line vertices e_i and e_j which is adjacent to one of point vertices v_i and v_j in C_3^{+++}. If we remove all 4 vertices in C_3^{+++} then we get a totally disconnected graph. Hence, $HI(C_3^{+++}) = 5$. Let G be a path, each point vertex is adjacent to adjacent point vertices and are adjacent to incident line vertices in G^{+++}. Each line vertices are adjacent to adjacent line vertices in G^{+++}. Thus, choose $\lfloor \frac{n}{2} \rfloor$ point vertices and $m-2$ line vertices in G^{+++}. Therefore, $|S| = \lfloor \frac{n}{2} \rfloor + m - 2$ and $m(G-S) = 2$. Hence, $HI(G^{+++}) = \lfloor \frac{n}{2} \rfloor + m$. Let G be a star. Then, choose a point vertex v_i which has maximum degree and m line vertices in G^{+++}. Therefore, $|S| = m+1$ and $m(G-S) = 1$. Hence, $HI(G^{+++}) = m+2$. Let G be a tree. Then, choose nonpendant point vertices v_i in G^{+++} and m line vertices in G^{+++}. Therefore, $|S| = n-l+\lceil \frac{m}{2} \rceil$ and $m(G-S) = 2$. Hence, $HI(G^{+++}) = n - l + \lceil \frac{m}{2} \rceil + 2$. Let G be a cycle. Similarly, choose $\lceil \frac{n}{2} \rceil$ point vertices and $\lceil \frac{m}{2} \rceil$ line vertices in G^{+++}. Therefore, $|S| = \lceil \frac{n}{2} \rceil + \lceil \frac{m}{2} \rceil$ and $m(G-S) = 2$. Hence, $HI(G^{+++}) = \lceil \frac{n}{2} \rceil + \lceil \frac{m}{2} \rceil + 2$. Otherwise, each point vertex is adjacent to adjacent point vertices and are adjacent to incident line vertices in G^{+++}. Each line vertices are adjacent to adjacent line vertices in G^{+++}. Then, choose all $(n-1)$ point vertices and $(m-1)$ line vertices in G^{+++}, are required to forms a hub set. If we remove $(n-1)$ point vertices and $(m-1)$ line vertices in G^{+++} then we get a totally disconnected graph. Therefore, hub set in G^{+++} is $S = \{v_1, v_2, v_3, ...v_{n-1}, e_1, e_2, e_3, ..., e_{m-1}\}$ that is $|S| = n+m-2$ and $m(G-S) = 1$. Hence, $HI(G^{+++}) = n + m - 1$. □

Theorem 2.2. *Let G be any graph of order n and size m. Then*

$$HI(G^{++-}) = \begin{cases} 3 & if\ G = P_2, \\ 4 & if\ G = P_3, \\ 5 & if\ G = P_4,\ C_3, \\ \lfloor \frac{n}{2} \rfloor + m + 1 & path,\ if\ n \geqslant 5, \\ m+2 & star,\ if\ n \geqslant 4, \\ n+m-2 & otherwise\ if\ n \geqslant 5. \end{cases}$$

Proof. Let $G = P_2$. Then P_2^{++-} is a disconnected graph with two components, K_2 and K_1. Therefore, $HI(P_2^{++-}) = 3$ (since hub-integrity of complete graph K_n is n [**15**]). Let $G = P_3$, then $P_3^{++-} \cong C_5$. Therefore, $HI(P_3^{++-}) = 4$ (since hub-integrity of cycle C_n is $(n-1)$ if $n = 4, 5$ [**15**]). Let $G = P_4$, then choose nonpendant vertices v_1 and v_2 and a line vertex e_1 in P_4^{++-}, is adjacent to point vertices v_1 and v_2. Thus, $|S| = 3$ and $m(G-S) = 2$. Hence, $HI(P_4^{++-}) = 5$. Let $G = C_3$, then each point vertex is adjacent to all other point vertices and nonincident line vertices in C_3^{++-}. Choose two point vertices v_1, v_2 and a line vertex e_1 which is incident to point vertices v_1 and v_2 in C_3^{++-}. Thus, $|S| = 3$ and $m(G-S) = 2$. Therefore, $HI(C_3^{++-}) = 5$. Let G be a path, each point vertex is adjacent to adjacent point vertices and are adjacent to nonincident line vertices in G^{++-}. Each line vertices are adjacent to adjacent line vertices and adjacent to nonincident point vertices in G^{++-}. Thus, choose $\lfloor \frac{n}{2} \rfloor$ point vertices and m line vertices in G^{++-}. Therefore, $|S| = \lfloor \frac{n}{2} \rfloor + m$ and $m(G-S) = 1$. Hence, $HI(G^{++-}) = \lfloor \frac{n}{2} \rfloor + m + 1$. Let G be a star. Then, choose a point vertex v_i which has maximum degree and m line vertices in G^{++-}. Therefore, $|S| = m+1$ and $m(G-S) = 1$. Hence, $HI(G^{++-}) = m+2$. Otherwise, the graph G^{++-} by definition has adjacent point vertices of G as adjacent point vertices in G^{++-}, adjacent edges of G as adjacent line vertices in G^{++-}. Then, choose all $(n-2)$ point vertices and $(m-2)$ line vertices are required to forms a hub set in G^{++-}. If we remove $(n-2)$ point vertices and $(m-2)$ line vertices in G^{++-} then we get a disconnected graph with K_2 components. Therefore, hub set in G^{++-} is $S = \{v_1, v_2, v_3, ...v_{n-2}, e_1, e_2, e_3, ..., e_{m-2}\}$ that is $|S| = n+m-4$ and $m(G-S) = 2$. Hence, $HI(G^{++-}) = n+m-2$. □

Theorem 2.3. *Let G be any graph of order n and size m. Then*

$$HI(G^{-++}) = \begin{cases} 2 & if\ G = P_2, \\ 4 & if\ G = P_3,\ C_3, \\ n + \lceil \frac{m}{2} \rceil & path,\ if\ n \geqslant 4, \\ m+1 & complete\ graph,\ if\ n \geqslant 4, \\ n+m-2 & otherwise,\ if\ n \geqslant 4. \end{cases}$$

Proof. Let $G = P_2$, then $P_2^{-++} \cong P_3$. Therefore, $HI(P_2^{-++}) = 2$ (since hub-integrity of any path P_n is $(n-1)$ if $n \geqslant 3$ [**15**]). Let $G = P_3$, then each line vertex is adjacent to incident point vertices in P_3^{-++}. Choosing both line vertices e_1 and e_2 forms a hub set in P_3^{-++}. Thus, $|S| = 2$ and $m(G-S) = 2$. Therefore, $HI(P_3^{-++}) = 4$. Let $G = C_3$, then each line vertex is adjacent to incident point vertices and adjacent to adjacent line vertices in C_3^{-++}. Choose all line vertices in C_3^{-++}. Thus, $S = \{e_1, e_2, e_3\}$ that is $|S| = 3$ and $m(G-S) = 1$. Therefore, $HI(C_3^{-++}) = 4$. Let G be a Path, each point vertex is adjacent to nonadjacent point vertices and incident line vertices in G^{-++} and each line vertices are adjacent to adjacent line vertices in G^{-++}. Then, choose $(n-1)$ point vertices and $\lceil \frac{m}{2} \rceil$ line vertices, which forms a hub set in G^{-++}. Therefore, $|S| = n-1+\lceil \frac{m}{2} \rceil$ and $m(G-S) = 1$. Hence, $HI(G^{-++}) = n + \lceil \frac{m}{2} \rceil$. Let G be a complete graph, then each point vertices are adjacent to incident line vertices and each line vertices are adjacent to adjacent line vertices in G^{-++}. Then, choose all line vertices in G^{-++}, which forms a hub set. Thus, $S = \{e_1, e_2, e_3, ..., e_m\}$ that is $|S| = m$ and $m(G-S) = 1$. Therefore, $HI(G^{-++}) = m+1$. Otherwise, each line vertex is adjacent to incident point vertices and adjacent line vertices in G^{-++}. Then, choose all $(n-1)$ point vertices and $(m-2)$ line vertices are required to forms a hub set in G^{-++}. If we remove $(n-1)$ point vertices and $(m-2)$ line vertices in G^{-++}, then we get a completely disconnected graph. Therefore, hub set in G^{-++} is $S = \{v_1, v_2, v_3, ...v_{n-1}, e_1, e_2, e_3, ..., e_{m-2}\}$ that is $|S| = n+m-3$ and $m(G-S) = 1$. Hence, $HI(G^{-++}) = n+m-2$. □

THEOREM 2.4. *Let G be any graph of order n and size m. Then*

$$HI(G^{-+-}) = \begin{cases} 3 & if\ G = P_2, \\ 5 & if\ G = P_4, \\ 4 & if\ G = C_3, \\ n+m-1 & star,\ if\ n \geqslant 4, \\ m+1 & complete\ graph,\ if\ n \geqslant 4, \\ n+m-2 & otherwise,\ if\ n \geqslant 5. \end{cases}$$

PROOF. Let $G = P_2$, then P_2^{-+-} is a completely disconnected graph with three vertices. Therefore, hub-integrity of P_2^{-+-} is 3. Let $G = P_4$, each point vertices are adjacent to nonadjacent point vertices and adjacent to nonincident line vertices in P_4^{-+-}. Each line vertices are adjacent to adjacent line vertices in P_4^{-+-}. Choose any two adjacent point vertices v_1 and v_2 which has maximum degree in P_4^{-+-}. Next choose two line vertices e_1 and e_2 which are adjacent to point vertices v_1 and v_2, which is required to forms a hub set in P_4^{-+-}. Thus, $|S| = 4$ and $m(G-S) = 1$. Therefore, $HI(P_4^{-+-}) = 5$. Similarly, let $G = C_3$. Then, line vertices forms a complete graph with three vertices and each point vertex is adjacent to a line vertex in C_3^{-+-}. Choose all the three line vertices $S = \{e_1, e_2, e_3\}$ which forms a hub set. If we remove all line vertices in C_3^{-+-} then we get totally disconnected graph. Thus, $|S| = 3$ and $m(G - S) = 1$. Therefore, $HI(C_3^{-+-}) = 4$. Let G be a star graph. Then G^{-+-} is disconnected graph with two components. Each point vertex is adjacent to nonincident line vertices and each line vertices are adjacent to adjacent line vertices in G^{-+-}. Choose $(n-2)$ point vertices and $(m-1)$ line vertices in G^{-+-} which forms a hub set. Thus, $S = \{v_1, v_2, v_3, ..., v_{n-2}, e_1, e_2, e_3, ..., e_{m-1}\}$ that is $|S| = n + m - 3$ and $m(G - S) = 2$. Therefore, $HI(G^{-+-}) = n + m - 1$. Let G be a complete graph. Then, choose all line vertices in G^{-+-} which forms a hub set. Thus, $S = \{e_1, e_2, e_3, ..., e_m\}$ that is $|S| = m$ and $m(G - S) = 1$. Therefore, $HI(G^{-+-}) = m + 1$. Otherwise, by definition of graph G^{-+-}, point vertices are adjacent to nonadjacent point vertices and nonincident line vertices in G^{-+-}. Then, choose all $(n-2)$ point vertices and $(m-1)$ line vertices are required to forms a hub set in G^{-+-}. If we remove $(n-2)$ point vertices and $(m-1)$ line vertices in G^{-+-}, then we get a completely disconnected graph. Therefore, hub set in G^{-+-} is $S = \{v_1, v_2, v_3, ...v_{n-2}, e_1, e_2, e_3, ..., e_{m-1}\}$ that is $|S| = n + m - 3$ and $m(G - S) = 1$. Hence, $HI(G^{-+-}) = n + m - 2$. □

THEOREM 2.5. *Let G be any graph of order n and size m, then*

$$HI(G^{---}) = \begin{cases} 3 & if\ G = P_2, \\ 4 & if\ G = P_3, \\ 6 & if\ G = C_3,\ K_4, \\ n+1 & star\ if\ n \geqslant 4, \\ m+1 & complete\ graph,\ if\ n \geqslant 5, \\ n+m-2 & otherwise,\ if\ n \geqslant 4. \end{cases}$$

PROOF. Let $G = P_2$, then P_2^{---} is a completely disconnected graph with three vertices. Therefore, hub-integrity of P_2^{---} is 3. Let $G = P_3$, then P_3^{---} is a disconnected graph with two components P_4 and K_1. Therefore, $HI(P_3^{---}) = 4$ (since hub-integrity of any path P_n is $n-1$ if $n \geqslant 3$ [**15**]). Let $G = C_3$, then C_3^{---} is a disconnected graph with three components of K_2. Therefore, $HI(C_3^{---}) = 6$ (since hub-integrity of any complete graph K_n is n [**15**]). Let $G = K_4$ then, choose all four point vertices is required to form hub set, that is $S = \{v_1, v_2, v_3, v_4\}$. If we remove all four point vertices then we get a disconnected graph with K_2 components. Thus, $|S| = 4$ and $m(G - S) = 2$. Therefore, $HI(K_4^{---}) = 6$. Let G be a star graph, then G^{---} is a disconnected graph with two components. Each point vertex is adjacent to nonincident line vertices and each line vertices are adjacent to nonadjacent line vertices in G^{---}. Choose all point vertices in G^{---} which forms a hub set. Thus, $S = \{v_1, v_2, v_3, ..., v_n\}$ that is $|S| = n$ and $m(G - S) = 1$. Therefore, $HI(G^{---}) = n + 1$. Let G be a complete graph, then each point vertex is adjacent to nonincident line vertices and each line vertices are adjacent to nonadjacent line vertices in G^{---}. Now choose all line vertices in G^{---} which forms a hub set. Thus, $S = \{e_1, e_2, e_3, ..., e_m\}$ that is $|S| = m$ and $m(G - S) = 1$. Therefore, $HI(G^{---}) = m + 1$. Otherwise, point vertices are adjacent to nonadjacent point vertices and adjacent to nonincident line vertices in G^{---}. Each line vertex is adjacent to nonincident point vertices and nonadjacent line vertices in G^{---}. Then, choose all $(n - 1)$ point vertices and $(m - 2)$ line vertices are required to form hub set in G^{---}. If we remove $(n - 1)$ point vertices and $(m - 2)$ line vertices in G^{---} then we get a completely disconnected graph. Therefore, hub set in G^{---} is $S = \{v_1, v_2, v_3, ...v_{n-1}, e_1, e_2, e_3, ..., e_{m-2}\}$ that is $|S| = n + m - 3$ and $m(G - S) = 1$. Hence, $HI(G^{---}) = n + m - 2$. □

Theorem 2.6. *Let G be any graph of order n and size m, then*

$$HI(G^{--+}) = \begin{cases} 2 & if\ G = P_2, \\ 4 & if\ G = P_3,\ C_3, \\ n+1 & star,\ if\ n \geqslant 4, \\ m+1 & complete\ graph,\ if\ n \geqslant 4, \\ n+m-3 & otherwise,\ if\ n \geqslant 5. \end{cases}$$

Proof. Let $G = P_2$, then $P_2^{--+} \cong P_3$. Therefore, hub-integrity of P_2^{--+} is 2 (since hub-integrity any path P_n is $n - 1$ if $n \geqslant 3$ [**15**]). Let $G = P_3$, then $P_3^{--+} \cong C_5$. Therefore, hub-integrity of P_3^{--+} is 4 (since hub-integrity any cycle C_n is $n - 1$ if $n = 4, 5$ [**15**]). Let $G = C_3$, then $C_3^{--+} \cong C_6$. Therefore, hub-integrity of C_3^{--+} is 4 (since hub-integrity any cycle C_n is $n - 2$ if $n \geqslant 6$ [**15**]). Let G be a star graph, then each point vertex is adjacent to nonadjacent point vertices and nonincident line vertices in G^{--+}. Each line vertices are adjacent to nondjacent line vertices in G^{--+}. Choose all point vertices in G^{--+}. Therefore, $|S| = n$ and $m(G - S) = 1$. Therefore, $HI(G^{--+}) = n + 1$. Let G be a complete graph, then choose all m line vertices in G^{--+} which forms a hub set. If we remove m line vertices in G^{--+} then we get a completely disconnected graph. Therefore, $|S| = m$ and $m(G - S) = 1$. Therefore, $HI(G^{--+}) = m + 1$. Otherwise, each point vertex is adjacent to nonadjacent point vertices and nonincident line vertices in G^{--+}. Each line vertices are adjacent to nondjacent line vertices in

G^{--+}. Choose all $(n-2)$ point vertices and $(m-2)$ line vertices in G^{--+} which forms a hub set. If we remove $(n-2)$ point vertices and $(m-2)$ line vertices in G^{--+} then we get a completely disconnected graph. Therefore, hub set in G^{--+} is $S=\{v_1, v_2, v_3, ...v_{n-2}, e_1, e_2, e_3, ..., e_{m-2}\}$ that is $|S| = n+m-4$ and $m(G-S) = 1$. Therefore, $HI(G^{--+}) = n+m-3$. □

Theorem 2.7. *Let G be any graph of order n and size m, then*

$$HI(G^{+-+}) = \begin{cases} 3 & if\ G = P_2,\ P_3, \\ 5 & if\ G = P_4, \\ 4 & if\ G = C_3, \\ 6 & if\ G = C_4, \\ \lfloor \frac{n}{2} \rfloor + m + 1 & if\ G\ is\ path,\ n \geqslant 5, \\ 3 & if\ G\ is\ star,\ n \geqslant 4, \\ \lceil \frac{n}{2} \rceil + m & if\ G\ is\ cycle,\ n \geqslant 5, \\ n+m-3 & otherwise,\ n \geqslant 4. \end{cases}$$

Proof. Let $G = P_2$, then P_2^{+-+} is a complete graph with three vertices. Therefore, hub-integrity of P_2^{+-+} is 3 (since hub-integrity of complete graph K_n is n [**15**]). Let $G = P_3$, then the central vertex of P_3 is adjacent to every other vertex in P_3^{+-+}. Therefore, central vertex of P_3 itself sufficient to forms a hub set in P_3^{+-+}. Thus, $|S| = 1$ and $m(G-S) = 2$. Therefore, $HI(P_3^{+-+}) = 3$. Let $G = P_4$. Then choose nonpendant vertices v_1 and v_2 and a line vertex e_1 which is incident with one of point vertex v_1 in P_4^{+-+} which forms a hub set. Thus, $|S| = 3$ and $m(G-S) = 2$. Therefore, $HI(P_4^{+-+}) = 5$. Let $G = C_3$, then each point vertices are adjacent to adjacent point vertices and incident line vertices in C_3^{+-+}. Choose all the three point vertices $S = \{v_1, v_2, v_3\}$ which forms a hub set in C_3^{+-+}. Thus, $|S| = 3$ and $m(G-S) = 1$. Therefore, $HI(C_3^{+-+}) = 4$. Similarly, let $G = C_4$, then choose all point vertices is required to forms hub set that is $S = \{v_1, v_2, v_3, v_4\}$. If we remove all point vertices then we get a disconnected graph with K_2 components. Thus, $|S| = 4$ and $m(G-S) = 2$. Therefore, $HI(C_4^{+-+}) = 6$. Let G be a path. Then each point vertex is adjacent to incident line vertices and adjacent point vertices in G^{+-+}. Each line vertex is adjacent to incident line vertices and nonadjacent line vertices in G^{+-+}. Choose all $\lfloor \frac{n}{2} \rfloor$ point vertices and m line vertices in G^{+-+} which forms a hub set. If we remove $\lfloor \frac{n}{2} \rfloor$ point vertices and m line vertices in G^{+-+} then we get a completely disconnected graph. Therefore, hub set in G^{+-+} is $S = \{v_1, v_2, v_3, ..., v_{\lfloor \frac{n}{2} \rfloor}, e_1, e_2, e_3, ..., e_m\}$ that is $|S| = \lfloor \frac{n}{2} \rfloor + m$ and $m(G-S) = 1$. Therefore, $HI(G^{+-+}) = \lfloor \frac{n}{2} \rfloor + m + 1$. Let G be a star. Then choose a point vertex v_i which has maximum degree in G^{+-+}. If we remove point vertices v_i in G^{+-+} then we get a completely disconnected graph with K_2 components. $|S| = 1$ and $m(G-S) = 2$. Therefore, $HI(G^{+-+}) = 3$. Similarly, let G be a cycle. Choose all $\lceil \frac{n}{2} \rceil$ point vertices and $(m-1)$ line vertices in G^{+-+} which forms a hub set. If we remove $\lceil \frac{n}{2} \rceil$ point vertices and $(m-1)$ line vertices in G^{+-+} then we get a completely disconnected graph. Therefore, hub set in G^{+-+} is $S = \{v_1, v_2, v_3, ..., v_{\lceil \frac{n}{2} \rceil}, e_1, e_2, e_3, ..., e_{m-1}\}$ that is $|S| = \lceil \frac{n}{2} \rceil + m - 1$ and $m(G-S) = 1$. Therefore, $HI(G^{+-+}) = \lceil \frac{n}{2} \rceil + m$. Otherwise, each point vertex is adjacent to adjacent point vertices and incident line vertices in G^{+-+}. Each line vertices are adjacent to nonadjacent line vertices in G^{+-+}. Choose all $(n-1)$ point vertices and $(m-3)$ line vertices in G^{+-+} which forms a hub set. If we remove $(n-1)$ point vertices and $(m-3)$ line vertices in G^{+-+} then we get a completely disconnected graph. Therefore, hub set in G^{+-+} is $S=\{v_1,v_2,v_3,...v_{n-1},e_1,e_2,e_3,...,e_{m-3}\}$ that is $|S|=n+m-3$ and $m(G-S)=1$. Therefore, $HI(G^{+-+})=n+m-3$. □

THEOREM 2.8. *Let G be any graph of order n and size m, then*

$$HI(G^{+--})=\begin{cases} 3 & if\ G=P_2, \\ 4 & if\ G=P_3,\ C_3, \\ \lfloor \frac{n}{2} \rfloor+m+1 & if\ G\ is\ path,\ n\geqslant 4, \\ n & if\ G\ is\ star\ graph,\ n\geqslant 4, \\ n+m-2 & otherwise. \end{cases}$$

PROOF. Let $G=P_2$, then P_2^{+--} is a disconnected graph with two components K_2 and K_1. Therefore, hub-integrity of P_2^{+--} is 3 (since hub-integrity of complete graph K_n is n [**15**]). Let $G=P_3$, then $P_3^{+--}\cong P_5$. Therefore, hub-integrity of P_3^{+--} is 4 (since hub-integrity any path P_n is $n-1$ if $n\geqslant 3$ [**15**]). Similarly, let $G=C_3$, then C_3^{+--} is complete graph K_3 with each point vertex is adjacent with pendant vertices. Choose all the three point vertices $S=\{v_1,v_2,v_3\}$ which forms a hub set. If we remove all point vertices in C_3^{+--} then we get totally disconnected graph. Thus, $|S|=3$ and $m(G-S)=1$. Therefore, $HI(C_3^{+--})=4$. Let G be a path, then each point vertex is adjacent to nonincident line vertices and adjacent point vertices in G^{+--}. Each line vertex is adjacent to nonincident point vertices and nonadjacent line vertices in G^{+--}. Choose all $\lfloor \frac{n}{2} \rfloor$ point vertices and m line vertices in G^{+--} which forms a hub set. If we remove $\lfloor \frac{n}{2} \rfloor$ point vertices and m line vertices in G^{+--} then we get a completely disconnected graph. Therefore, hub set in G^{+--} is $S=\{v_1,v_2,v_3,...,v_{\lfloor \frac{n}{2} \rfloor},e_1,e_2,e_3,...,e_m\}$ that is $|S|=\lfloor \frac{n}{2} \rfloor+m$ and $m(G-S)=1$. Therefore, $HI(G^{+--})=\lfloor \frac{n}{2} \rfloor+m+1$. Let G be a star. Then, choose $(n-1)$ point vertices in G^{+--}, which are pendant vertices in G. If we remove $n-1$ point vertices in G^{+--} then we get a completely disconnected graph. Therefore, hub set in G^{+--} is $S=\{v_1,v_2,v_3,...,v_{n-1}\}$ that is $|S|=n-1$ and $m(G-S)=1$. Therefore, $HI(G^{+--})=n$. Otherwise, each point vertex is adjacent to adjacent point vertices and nonincident line vertices in G^{+--}. Each line vertices are adjacent to nonadjacent line vertices in G^{+--}. Choose all $(n-1)$ point vertices and $(m-2)$ line vertices in G^{+--} which forms a hub set. If we remove $(n-1)$ point vertices and $(m-2)$ line vertices in G^{+--} then we get a completely disconnected graph. Therefore, hub set in G^{+--} is $S=\{v_1,v_2,v_3,...v_{n-1},e_1,e_2,e_3,...,e_{m-2}\}$ that is $|S|=n+m-3$ and $m(G-S)=1$. Therefore, $HI(G^{+--})=n+m-2$. □

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