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ON TRIANGULAR PELL AND PELL-LUCAS NUMBERS

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Abstract. In this paper, we define triangular Pell and triangular Pell-Lucas numbers. We carry with little differences an elegant result given in the literature on the sum of any two consecutive triangular numbers from triangular numbers to triangular Pell numbers and Pell-Lucas numbers. Also, we present some interesting identities satisfied by the triangular Pell and triangular Pell-Lucas numbers which are connected with Pell and Pell-Lucas numbers. Furthermore, we give some important properties of triangular Pell and triangular Pell-Lucas numbers.

1. Introduction

Binomial coefficients play an important role in many areas of mathematics. The binomial coefficients are defined by

$$\begin{pmatrix} n \\ k \end{pmatrix} = \begin{cases} \frac{n!}{k!(n-k)!}; & n \geqslant k \\ 0; & n < k \end{cases}$$

for n and k non-negative integers. The theories of binomial coefficients in the classical algebra can be found in, e.g. [**2**], [**4**] and [**10**].

Triangular numbers are defined by $T_n = \begin{pmatrix} n+1 \\ 2 \end{pmatrix}$. There is an interesting relationship between triangular numbers and Pascal's triangle. The study of triangular numbers, their properties and their identities from the algebraic, analytic and geometric point of view has a long history (see, e.g., the monographs [**1**], [**5**], [**8**]).

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The Fibonomial coefficients known as interesting generalizations of binomial coefficients are defined by the relation for $n \geqslant m \geqslant 1$

$$\left[\begin{array}{c} n \\ m \end{array}\right]_F = \frac{F_1 F_2 \cdots F_n}{(F_1 F_2 \cdots F_{n-m})(F_1 F_2 \cdots F_m)}.$$

There has been a lot of interest in the Fibonomial coefficients and some profound results were established (for more details see [**6**], [**7**], [**9**] ,[**11**]).

The Pell numbers P_n are defined by $P_0 = 0, P_1 = 1$ and, for $n \geqslant 2$,

$$P_n = 2P_{n-1} + P_{n-2}.$$

The Pell-Lucas numbers Q_n are defined by the same recurrence, with the initial conditions $Q_0 = Q_1 = 1$. The first six Pell numbers are $0, 1, 2, 5, 12, 29$, and 70; and the first six Pell-Lucas numbers are $1, 1, 3, 7, 17, 41$, and 99. The solutions of Pell's equation $x^2 - 2y^2 = (-1)^n$ are (Q_n, P_n). These numbers have many interesting number-theoretic and combinatorial properties. For a full introduction to Pell and Pell-Lucas numbers, see [**5**].

The focus of this paper is the study of triangular Pell and triangular Pell-Lucas numbers. We first define the Pellnomial and Pell-Lucanomial coefficients and then we define triangular Pell and triangular Pell-Lucas numbers according to these coefficients. We present some interesting identities satisfied by the triangular Pell and triangular Pell-Lucas numbers which are connected with Pell and Pell-Lucas numbers. Furthermore, we give some important properties of triangular Pell and triangular Pell-Lucas numbers.

2. Main results

In this section, we will address triangular Pell and triangular Pell-Lucas numbers from a number theory science perspective, where a special emphasis is put on integer sequences issues.

We define the Pellnomial and Pell-Lucanomial coefficients for $1 \leqslant k \leqslant m$ by

$$\left(\begin{array}{c} m \\ k \end{array}\right)_P = \frac{P_m P_{m-1} \cdots P_{m-k+1}}{P_1 \cdots P_k} \tag{2.1}$$

and

$$\left(\begin{array}{c} m \\ k \end{array}\right)_Q = \frac{Q_m Q_{m-1} \cdots Q_{m-k+1}}{Q_1 \cdots Q_k}, \tag{2.2}$$

respectively.

The Pell and Pell-Lucas analogs of $n!$ are defined by $P_m! = \prod_{k=1}^{n} P_k$ and $Q_m! = \prod_{k=1}^{n} Q_k$, respectively. Now, we can give the Pell analog in (2.1) and Pell-Lucas analog in (2.2) of the binomial coefficients by the following relations, respectively, for $n \geqslant m \geqslant 1$

$$\left(\begin{array}{c} m \\ k \end{array}\right)_P = \frac{P_m!}{P_{m-k}! P_k!}$$

and

$$\begin{pmatrix} m \\ k \end{pmatrix}_Q = \frac{Q_m!}{Q_{m-k}!Q_k!}$$

with

$$\begin{pmatrix} m \\ 0 \end{pmatrix}_P = \begin{pmatrix} m \\ m \end{pmatrix}_P = \begin{pmatrix} m \\ 0 \end{pmatrix}_Q = \begin{pmatrix} m \\ m \end{pmatrix}_Q = 1$$

where P_n and Q_n are the nth Pell and Pell-Lucas numbers, respectively.

For the sake of simplicity, we will denote the nth Pell and Pell-Lucas numbers by $[n]$ and $\langle n \rangle$, respectively. The Pell numbers are defined by $[0] = 0, [1] = 1$ and, for $n \geqslant 2$,

$$[n] = 2\,[n-1] + [n-2]\,. \tag{2.3}$$

For example, we have

$$[2] = 2, \quad [3] = 5, \quad [4] = 12, \quad [5] = 29.$$

The Pell-Lucas numbers are defined by

$$\langle n \rangle = 2\langle n-1 \rangle + \langle n-2 \rangle, \quad n \geqslant 2$$

together with the initial conditions $\langle 0 \rangle = \langle 1 \rangle = 1$. Here is a list of the Pell-Lucas numbers

$$\langle 2 \rangle = 3, \quad \langle 3 \rangle = 7, \quad \langle 4 \rangle = 17, \quad \langle 5 \rangle = 41.$$

We are now in a position to define corresponding triangular Pell and triangular Pell-Lucas numbers. For the Pellnomial and Pell-Lucasnomial coefficients, it is natural to define the triangular Pell and triangular Pell-Lucas numbers by

$$T_{[n]} = \begin{pmatrix} n+1 \\ 2 \end{pmatrix}_P = \frac{P_{n+1}!}{P_{n-1}!P_2!} = \frac{P_{n+1}P_n}{P_2!} = \frac{[n+1]\,[n]}{2}, \quad n \geqslant 1 \tag{2.4}$$

and

$$T_{\langle n \rangle} = \begin{pmatrix} n+1 \\ 2 \end{pmatrix}_Q = \frac{Q_{n+1}!}{Q_{n-1}!Q_2!} = \frac{Q_{n+1}Q_n}{Q_2!} = \frac{\langle n+1 \rangle \langle n \rangle}{6}, \quad n \geqslant 1,$$

respectively.

It is clear from the Definition 2.3 and 2.4 that $T_{[n]} = \begin{pmatrix} n+1 \\ 2 \end{pmatrix}_P$ is an integer for all $n \geqslant 2$.

In [**5**], it is given an elegant result on the sum of any two consecutive triangular numbers:

$$T_n + T_{n-1} = n^2.$$

In the following first two propositions, we carry with little differences the above result from triangular numbers to triangular Pell numbers and Pell-Lucas numbers.

Proposition 2.1. *The difference of any two consecutive triangular Pell numbers is a square:*

$$T_{[n]} - T_{[n-1]} = [n]^2 \quad n \geqslant 3. \tag{2.5}$$

Proof. For $n \geqslant 3$, by formula (2.4) and definition (2.3), we get

$$\begin{aligned} T_{[n]} - T_{[n-1]} &= \left(\begin{array}{c} n+1 \\ 2 \end{array} \right)_P - \left(\begin{array}{c} n \\ 2 \end{array} \right)_P \\ &= \frac{[n+1]\,[n]}{2} - \frac{[n]\,[n-1]}{2} \\ &= \frac{[n]\,([n+1]-[n-1])}{2} \\ &= [n]^2 . \end{aligned}$$

This completes the proof of Proposition 2.1. □

Proposition 2.2. *For the difference of any two consecutive triangular Pell-Lucas numbers, we have*

$$T_{\langle n\rangle} - T_{\langle n-1\rangle} = \frac{\langle n\rangle^2}{3} \quad n \geqslant 1. \tag{2.6}$$

Proof. The proof is similar to the proof of Proposition 2.1 since

$$\langle n\rangle = 2\langle n-1\rangle + \langle n-2\rangle.$$

□

Next, we present some fundamental identities satisfied by the triangular Pell and triangular Pell-Lucas numbers.

Proposition 2.3. *For the triangular Pell and Pell-Lucas numbers, we have*

$$36T_{\langle n\rangle}^2 + 16T_{[n]}^2 = [2n+1]^2 .$$

Proof. It is easily seen from the definition triangular Pell and Pell-Lucas numbers that

$$\begin{aligned} 36T_{\langle n\rangle}^2 + 16T_{[n]}^2 &= \left(6T_{\langle n\rangle}\right)^2 + \left(4T_{[n]}\right)^2 \\ &= \left(6\frac{\langle n+1\rangle\,\langle n\rangle}{6}\right)^2 + \left(4\frac{[n+1]\,[n]}{2}\right)^2 \\ &= \left(\langle n+1\rangle\,\langle n\rangle\right)^2 + \left(2\,[n+1]\,[n]\right)^2 . \qquad (2.7) \end{aligned}$$

Combining formula $(\langle n+1\rangle\,\langle n\rangle)^2 + (2\,[n+1]\,[n])^2 = [2n+1]^2$ and (2.7) we get the desired result. □

Proposition 2.4. *For the triangular Pell and Pell-Lucas numbers, we have*

$$T_{\langle n\rangle} + T_{[n-1]} - T_{[n]} - T_{\langle n-1\rangle} = \frac{1}{3}\left((-1)^n - [n]^2\right).$$

Proof. We have from (2.5) and (2.6)

$$T_{\langle n\rangle} - T_{\langle n-1\rangle} - T_{[n]} + T_{[n-1]} = \frac{\langle n\rangle^2}{3} - [n]^2 .$$

Therefore, this relation can be rewritten as

$$T_{\langle n\rangle} - T_{\langle n-1\rangle} - T_{[n]} + T_{[n-1]} = \frac{1}{3}\left(\langle n\rangle^2 - 2\,[n]^2 - [n]^2\right). \tag{2.8}$$

Since every solution of $x^2 - 2y^2 = (-1)^n$ is $(\langle n\rangle, [n])$ and $(1,1)$ is its fundamental solution, it follows that

$$\langle n\rangle^2 - 2\,[n]^2 = (-1)^n. \tag{2.9}$$

It follows from (2.8) and (2.9) that

$$T_{\langle n\rangle} - T_{\langle n-1\rangle} - T_{[n]} + T_{[n-1]} = \frac{1}{3}\left((-1)^n - [n]^2\right)$$

thus we get the desired result. □

The following theorem establishes a criterion for two consecutive triangular numbers to have the same parity (oddness or evenness).

THEOREM 2.1 ([5, Theorem 5.1]). *The triangular numbers t_n and t_{n+1} have the same parity if and only if n is odd.*

We now establish a criterion for two consecutive triangular Pell numbers to have the same parity by the following theorems.

THEOREM 2.2. *The triangular Pell numbers $T_{[n]}$ and $T_{[n+1]}$ have the same parity if and only if n is odd.*

PROOF. Suppose $T_{[n]} \equiv T_{[n+1]}$ $(mod 2)$. Therefore, by Proposition 2.1, we have that $T_{[n+1]} - T_{[n]} = [n+1]^2$ is even. So $[n+1]$ is even. Thus, $n+1$ is even. Hence n is odd.
Conversely, let n be odd. Then, $n+1$ is even. Moreover, we have that $[n+1]$ and$[n+1]^2$ are even. By Proposition 2.1, we get $T_{[n+1]} - T_{[n]}$ is even. Consequently, we get the desired result: $T_{[n]} \equiv T_{[n+1]}(mod 2)$. Therefore, the proof is completed. □

THEOREM 2.3. *For any positive integer $n \geqslant 3$, we have*

$$T_{[n+1]} \equiv T_{[n-1]} + 1 (mod 2).$$

PROOF. From (2.5), we get for $n \geqslant 3$

$$T_{[n+1]} - T_{[n]} = [n+1]^2 \tag{2.10}$$

and

$$T_{[n]} - T_{[n-1]} = [n]^2. \tag{2.11}$$

The fact that

$$T_{[n+1]} - T_{[n-1]} = [n+1]^2 + [n]^2$$

directly follows from (2.10) and (2.11). Since $[n+1]^2 + [n]^2 = [2n+1]$ and $[2n+1]$ is odd, we thus get

$$T_{[n+1]} \equiv T_{[n-1]} + 1 (mod 2)$$

which concludes the proof. □

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