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ON THREE DIMENSIONAL PARA-SASAKIAN MANIFOLDS

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Abstract. In this article, some special curvature conditions created by the W_0^*-curvature tensor with the Riemann, Ricci, projective, concircular, quasi-conformall curvature tensors in 3-dimensional para-Sasakian manifold are discussed. In addition, the behavior of the para-Sasakian manifold is investigated by considering the flatness, semi-symmetry, pseudo-symmetry and Ricci pseudo-symmetry properties of the W_0^*-curvature tensor.

1. Introduction

In 1976, Sato introduced almost paracontact manifolds [**16**]. Later, Kaneyuki and Kozai described the almost paracontact structure on the (2n+1)-dimensional pseudo-Riemann manifold [**8**]. Para-Sasakian manifolds were emerged as a very important subclass of almost paracontact manifolds. In 1977, Adati and Matsumoto defined para-Sasakian and special para-Sasakian manifolds [**1**]. In later times, Zamkovoy defined para-Sasakian manifolds as a normal paracontact manifold [**21**]. Zamkovoy obtained the necessary and sufficient condition for a paracontact metric manifold to be a para-Sasakian manifold [**21**]. Para-Sasakian manifolds satisfying the $R(X,Y).R=0$ condition have been studied by De and Tarafdar [**6**]. Para-Sasakian manfolds satisfying the $R(X,Y).P=0$ and $R(X,Y).S=0$ conditions are discussed by De and Pathak [**5**]. Para-Sasakian manifolds satisfying the $C(X,Y).S=0$ condition and also weyl-pseudo symmetric manifolds were studied by Özgür [**14**]. Again, para-Sasakian manifolds have been studied by many authors such as Adati and Miyazawa [**2**], Deshmuhk and Ahmed [**7**], De et al [**4**],

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Sharfuddin, Deshmuhk, Husain [**17**], Matsumoto, Ianus and Mihai [**10**], Özgür and Tripathi [**15**], and others [**18**], [**5**], [**9**].

Obtaining the characterization of a manifold with the help of curvature tensors is an important problem. This problem has been handled by many authorson many different manifolds and many important properties of these manifolds have been obtained ([**1**], [**11**], [**12**], [**13**], [**20**]).

A $(1,3)$-type W_0^*-curvature tensor was defined by M. Tripathi and P. Gunam [**19**], with

$$W_0^*(X,Y)Z = R(X,Y)Z + \frac{1}{2}[S(Y,Z)X - g(X,Z)QY]. \tag{1}$$

In this study, motivated by the above studies, some special curvature conditions obtained by Riemann, Ricci, projective, concircular and quasi-conformal curvature tensors on the W_0^*-curvature tensor are discussed. Various important properties of the para-Sasakian manifold have been characterized under these curvature conditions. Again, it has been shown that if the para-Sasakian manifold is W_0^*-flat, it is an $\eta-$Einstein manifold. In the last part of the study, the concepts of W_0^* pseudo-symmetry and W_0^* Ricci pseudo-symmetry are introduced and the properties of the para-Sasakian manifold are obtained according to these symmetrical states.

2. Preliminaries

Let M be the $n-$dimensional differentiable manifold. The quadrilateral (ϕ, ξ, η, g) satisfying the conditions

$$\phi\xi = 0, \eta(\xi) = 1, g(\xi, X) = \eta(X)$$

$$\phi^2 X = X - \eta(X)\xi$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$$

is called paracontact Riemann structure, where ϕ is the $(1,1)-$type tensor field, ξ is a vector field, η is a $1-$form and g is Riemann metric on M. In addition, if the paracontact Riemann structure (ϕ, ξ, η, g) satisfies the conditions

$$d\eta = 0,$$

$$\nabla_X \xi = \phi X,$$

$$(\nabla_X \phi)Y = -g(X,Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi,$$

the M manifold is called a para-Sasakian manifold. If the η which is $1-$form of the M para-Sasakian manifold satisfies the condition

$$(\nabla_X \eta)Y = -g(X,Y) + \eta(X)\eta(Y),$$

the M manifold is called a special para-Sasakian manifold.

An $n-$dimensional para-Sasakian manifold M provides the following relations for each $X, Y, Z \in \chi(M)$ vector field.

$$S(X,\xi) = -(n-1)\eta(X), \tag{2}$$

$$Q\xi = -(n-1)\xi, \tag{3}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X. \tag{4}$$

In the three-dimensional Riemannian manifold, the Riemannian curvature tensor is defined as

$$\begin{aligned} R(X,Y)Z = {} & g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X \\ & -S(X,Z)Y - \tfrac{r}{2}\left[g(Y,Z)X - g(X,Z)Y\right], \end{aligned} \tag{5}$$

where Q is the Ricci operator that is $g(QX,Y) = S(X,Y)$ and r is the scalar curvature of the manifold. If $Z = \xi$ is taken in (5), and (2) and (4) are used, we obtain

$$QX = \frac{1}{2}\left[(r+2)X - (r+6)\eta(X)\xi\right], \tag{6}$$

and

$$S(X,Y) = \frac{1}{2}\left[(r+2)g(X,Y) - (r+6)\eta(X)\eta(Y)\right]. \tag{7}$$

If we use the expression (7) in equation (5), the curvature tensor of the three-dimensional para-Sasakian manifold is obtained as

$$\begin{aligned} R(X,Y)Z = {} & \tfrac{r+4}{2}\left[g(Y,Z)X - g(X,Z)Y\right] \\ & -\tfrac{r+6}{4}\left[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \right. \\ & \left. +\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\right]. \end{aligned} \tag{8}$$

If $X = \xi, Y = \xi, Z = \xi$ are taken respectively in (8), we get the following relations.

$$R(\xi,Y)Z = -g(Y,Z)\xi + \eta(Z)Y, \tag{9}$$

$$R(X,\xi)Z = g(X,Z)\xi - \eta(Z)X, \tag{10}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{11}$$

$$\eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X). \tag{12}$$

Definition 2.1. Let M be a three dimensional Riemann manifold. The projective curvature tensor is defined as

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2}\left[S(Y,Z)X - S(X,Z)Y\right]. \tag{13}$$

If we choose $X=\xi, Y=\xi, Z=\xi$ respectively in equation (13), we get

$$P(\xi, Y) Z=\frac{r+6}{4}[-g(Y, Z) \xi+\eta(Y) \eta(Z) \xi], \tag{14}$$

$$P(X, \xi) Z=\frac{r+6}{4}[g(X, Z) \xi-\eta(X) \eta(Z) \xi], \tag{15}$$

$$P(X, Y) \xi=0. \tag{16}$$

Definition 2.2. Let M be a three dimensional Riemann manifold. The concircular curvature tensor is defined as

$$\tilde{Z}(X, Y) Z=R(X, Y) Z-\frac{r}{6}[g(Y, Z) X-g(X, Z) Y]. \tag{17}$$

If we choose $X=\xi, Y=\xi, Z=\xi$ respectively in equation (17), we get

$$\tilde{Z}(\xi, Y) Z=\frac{r+6}{6}[-g(Y, Z) \xi+\eta(Z) Y], \tag{18}$$

$$\tilde{Z}(X, \xi) Z=\frac{r+6}{6}[g(X, Z) \xi-\eta(Z) X], \tag{19}$$

$$\tilde{Z}(X, Y) \xi=\frac{r+6}{6}[\eta(X) Y-\eta(Y) X]. \tag{20}$$

Definition 2.3. Let M be a three dimensional Riemann manifold. The quasi-conformall curvature tensor is defined as

$$\begin{aligned} \tilde{C}(X, Y) Z= & a R(X, Y) Z+b[S(Y, Z) X-S(X, Z) Y \\ & +g(Y, Z) Q X-g(X, Z) Q Y]-\tfrac{r}{3}\left(\tfrac{a}{2}+2 b\right) \\ & [g(Y, Z) X-g(X, Z) Y]. \end{aligned} \tag{21}$$

If we choose $X=\xi, Y=\xi, Z=\xi$ respectively in equation (21), we get

$$\tilde{C}(\xi, Y) Z=\frac{(a+b)(r+6)}{6}[-g(Y, Z) \xi+\eta(Z) Y], \tag{22}$$

$$\tilde{C}(X, \xi) Z=\frac{(a+b)(r+6)}{6}[g(X, Z) \xi-\eta(Z) X], \tag{23}$$

$$\tilde{C}(X, Y) \xi=\frac{(a+b)(r+6)}{6}[\eta(X) Y-\eta(Y) X]. \tag{24}$$

It can also be easily shown that

$$Q \phi Y=\frac{r+2}{2} \phi Y. \tag{25}$$

Finally, let's write the formulas for the W_0^*-curvature tensor that we will use frequently in the next section. If we choose $X=\xi, Y=\xi, Z=\xi$ respectively in equation (1), we get

$$W_0^*(\xi, Y) Z=\frac{r-2}{4}[g(Y, Z) \xi-\eta(Z) Y], \tag{26}$$

$$W_0^*(X,\xi)Z = 2\left[g(X,Z)\xi - \eta(Z)X\right], \tag{27}$$

$$W_0^*(X,Y)\xi = -\frac{r-2}{4}\eta(X)Y - 2\eta(Y)X + \frac{r+6}{4}\eta(X)\eta(Y)\xi. \tag{28}$$

3. Para-Sasakian manifolds on W_0^*-curvature tensor

Let M be a three-dimensional para-Sasakian manifold. Let's first examine the W_0^*-flatness of the M manifold.

Theorem 3.1. *Let M be a three-dimensional para-Sasakian manifold. If the manifold M is W_0^*-flat, then M is an $\eta-$Einstein manifold.*

Proof. Let M be a W_0^*-flat. So it is clear from (1) that

$$R(X,Y)Z = \frac{1}{2}\left[-S(Y,Z)X + g(X,Z)QY\right]. \tag{29}$$

If we choose $X=\xi$ in expression (29), we get

$$R(\xi,Y)Z = \frac{1}{2}\left[-S(Y,Z)\xi + g(\xi,Z)QY\right],$$

ve if we use the expressions (7) and (9) here, then we obtain

$$\frac{r-2}{4}g(Y,Z)\xi + \eta(Z)Y - \frac{r+6}{4}\eta(Y)\eta(Z)\xi = \frac{1}{2}\eta(Z)QY. \tag{30}$$

If we take the inner product of the $X\in\chi(M)$ vector of both sides of equation (30) and choose $Z=\xi$, we get

$$S(X,Y) = 2g(X,Y) - 4\eta(X)\eta(Y).$$

This completes the proof. □

Let us now examine some special curvature conditions for the M manifold.

Theorem 3.2. *Let M be a three-dimensional para-Sasakian manifold. If $W_0^*(X,Y).R = 0$, then M is either a real space form with a constant section curvature or the scalar curvature of M is $r=2$.*

Proof. Let's assume that $(W_0^*(X,Y).R)(Z,W,U)=0$ for every $X,Y,Z,W,U\in\chi(M)$. So we can write

$$\begin{aligned} &W_0^*(X,Y)R(Z,W)U - R(W_0^*(X,Y)Z,W)U \\ &-R(Z,W_0^*(X,Y)W)U - R(Z,W)W_0^*(X,Y)U = 0. \end{aligned} \tag{31}$$

If we choose $X=\xi$ in the equation (31) and make use of the equation (26), we get

$$\begin{aligned} &\tfrac{r-2}{4}\{g(Y,R(Z,W)U)\xi - \eta(R(Z,W)U)Y - g(Y,Z)R(\xi,W)U \\ &+\eta(Z)R(Y,W)U - g(Y,W)R(Z,\xi)U + \eta(W)R(Z,Y)U \\ &-g(Y,U)R(Z,W)\xi + \eta(U)R(Z,W)Y\} = 0. \end{aligned} \tag{32}$$

If we use the equations $(9)\,,(10)\,,(11)$ in equation (32), we obtain

$$
\begin{aligned}
&\tfrac{r-2}{4}\left\{g\left(Y,R\left(Z,W\right)U\right)\xi-\eta\left(R\left(Z,W\right)U\right)Y+g\left(Y,Z\right)g\left(W,U\right)\xi\right.\\
&-g\left(Y,Z\right)\eta\left(U\right)W+\eta\left(Z\right)R\left(Y,W\right)U-g\left(Y,W\right)g\left(Z,U\right)\xi\\
&+g\left(Y,W\right)\eta\left(U\right)Z+\eta\left(W\right)R\left(Z,Y\right)U-g\left(Y,U\right)\eta\left(Z\right)W\\
&\left.+g\left(Y,U\right)\eta\left(W\right)Z+\eta\left(U\right)R\left(Z,W\right)Y\right\}=0.
\end{aligned}
\tag{33}
$$

If we choose $Z=\xi$ in the equation (33) and we make use of the (9) equation in (33) and make the necessary adjustments, we get

$$\frac{r-2}{4}\left[R\left(Y,W\right)U-\left(g\left(Y,U\right)W-g\left(W,U\right)Y\right)\right]=0.$$

It is clear from the last equation that M is either a real space form with a constant section curvature or the scalar curvature of M is $r=2$. □

Theorem 3.3. *Let M be a three-dimensional para-Sasakian manifold. If $W_0^*\left(X,Y\right).S=0$, then M is either an Einstein manifold or the scalar curvature of M is $r=2$.*

Proof. Let's assume that $\left(W_0^*\left(X,Y\right).S\right)\left(U,W\right)=0$ for every $X,Y,W,U\in\chi\left(M\right)$. So we can write

$$S\left(W_0^*\left(X,Y\right)U,W\right)+S\left(U,W_0^*\left(X,Y\right)W\right)=0. \tag{34}$$

If we choose $X=\xi$ in the equation (34) and make use of the equation (26), we get

$$
\begin{aligned}
&\tfrac{r-2}{4}\left\{-2g\left(Y,U\right)\eta\left(W\right)-\eta\left(U\right)S\left(Y,W\right)\right.\\
&\left.-2g\left(Y,W\right)\eta\left(U\right)-\eta\left(W\right)S\left(U,Y\right)\right\}=0.
\end{aligned}
$$

If we choose $Z=\xi$ in the last equation, we obtain

$$-\frac{r-2}{4}\left[S\left(Y,W\right)+2g\left(Y,W\right)\right]=0.$$

This completes the proof. □

Theorem 3.4. *Let M be a three-dimensional para-Sasakian manifold. If $W_0^*\left(X,Y\right).\tilde{Z}=0$, then M is either a real space form with a constant section curvature or the scalar curvature of M is $r=2$.*

Proof. Let's assume that $\left(W_0^*\left(X,Y\right).\tilde{Z}\right)\left(U,W,Z\right)=0$ for every $X,Y,Z,W,U\in\chi\left(M\right)$. So we can write

$$
\begin{aligned}
&W_0^*\left(X,Y\right)\tilde{Z}\left(U,W\right)Z-\tilde{Z}\left(W_0^*\left(X,Y\right)U,W\right)Z\\
&-\tilde{Z}\left(U,W_0^*\left(X,Y\right)W\right)Z-\tilde{Z}\left(U,W\right)W_0^*\left(X,Y\right)Z=0.
\end{aligned}
\tag{35}
$$

If we choose $X=\xi$ in the equation (35) and make use of the equation (26), we get

$$\begin{aligned}&\tfrac{r-2}{4}\Big\{g\left(Y,\tilde{Z}\left(U,W\right)Z\right)\xi-\eta\left(\tilde{Z}\left(U,W\right)Z\right)Y-g\left(Y,U\right)\tilde{Z}\left(\xi,W\right)Z\\&+\eta\left(U\right)\tilde{Z}\left(Y,W\right)Z-g\left(Y,W\right)\tilde{Z}\left(U,\xi\right)Z+\eta\left(W\right)\tilde{Z}\left(U,Y\right)Z\\&-g\left(Y,Z\right)\tilde{Z}\left(U,W\right)\xi+\eta\left(Z\right)\tilde{Z}\left(U,W\right)Y\Big\}=0.\end{aligned}\tag{36}$$

If we use the equations (18), (19), (20) in equation (36), we obtain

$$\begin{aligned}&\tfrac{r-2}{4}\Big\{g\left(Y,\tilde{Z}\left(U,W\right)Z\right)\xi-\eta\left(\tilde{Z}\left(U,W\right)Z\right)Y+\tfrac{r+6}{6}g\left(Y,U\right)g\left(W,Z\right)\xi\\&-\tfrac{r+6}{6}g\left(Y,U\right)\eta\left(Z\right)W+\eta\left(U\right)\tilde{Z}\left(Y,W\right)Z-\tfrac{r+6}{6}g\left(Y,W\right)g\left(Z,U\right)\xi\\&+\tfrac{r+6}{6}g\left(Y,W\right)\eta\left(Z\right)U+\eta\left(W\right)\tilde{Z}\left(U,Y\right)Z-\tfrac{r+6}{6}g\left(Y,Z\right)\eta\left(U\right)W\\&+\tfrac{r+6}{6}g\left(Y,Z\right)\eta\left(W\right)U+\eta\left(Z\right)\tilde{Z}\left(U,W\right)Y\Big\}=0.\end{aligned}\tag{37}$$

If we choose $U=\xi$ in the equation (37) and we make use of the (18) equation in (37) and make the necessary adjustments, we get

$$\frac{r-2}{4}\left[\frac{r+6}{6}g\left(W,Z\right)Y+\tilde{Z}\left(Y,W\right)Z-\frac{r+6}{6}g\left(Y,Z\right)W\right]=0.\tag{38}$$

If we substitute the expression (17) in the equation (38), we get

$$\frac{r-2}{4}\left[R\left(Y,W\right)Z-\left(g\left(Y,Z\right)W-g\left(W,Z\right)Y\right)\right]=0.$$

It is clear from the last equation that M is either a real space form with a constant section curvature or the scalar curvature of M is $r=2$. □

Theorem 3.5. *Let M be a three-dimensional para-Sasakian manifold. If $W_0^*\left(X,Y\right).P=0$, then M is either an $\eta-$Einstein manifold or the scalar curvature of M is $r=2$.*

Proof. Let's assume that $\left(W_0^*\left(X,Y\right).P\right)\left(U,W,Z\right)=0$ for every $X,Y,Z,W,U\in\chi\left(M\right)$. So we can write

$$\begin{aligned}&W_0^*\left(X,Y\right)P\left(U,W\right)Z-P\left(W_0^*\left(X,Y\right)U,W\right)Z\\&-P\left(U,W_0^*\left(X,Y\right)W\right)Z-P\left(U,W\right)W_0^*\left(X,Y\right)Z=0.\end{aligned}\tag{39}$$

If we choose $X=\xi$ in the equation (39) and make use of the equation (26), we get

$$\begin{aligned}&\tfrac{r-2}{4}\left\{g\left(Y,P\left(U,W\right)Z\right)\xi-\eta\left(P\left(U,W\right)Z\right)Y-g\left(Y,U\right)P\left(\xi,W\right)Z\right.\\&+\eta\left(U\right)P\left(Y,W\right)Z-g\left(Y,W\right)P\left(U,\xi\right)Z+\eta\left(W\right)P\left(U,Y\right)Z\\&\left.-g\left(Y,Z\right)P\left(U,W\right)\xi+\eta\left(Z\right)P\left(U,W\right)Y\right\}=0.\end{aligned}\tag{40}$$

If we use the equations $(13), (14), (15), (16)$ in equation (40), we obtain

$$
\begin{aligned}
&\tfrac{r-2}{4}\left\{g\left(Y,R\left(U,W\right)Z\right)\xi-\tfrac{1}{2}S\left(W,Z\right)g\left(Y,U\right)\xi+\tfrac{1}{2}S\left(U,Z\right)g\left(Y,W\right)\xi\right.\\
&-g\left(U,Z\right)\eta\left(W\right)Y+g\left(W,Z\right)\eta\left(U\right)Y+\tfrac{1}{2}S\left(W,Z\right)\eta\left(U\right)Y\\
&-\tfrac{1}{2}S\left(U,Z\right)\eta\left(W\right)Y+\tfrac{r+6}{4}g\left(Y,U\right)g\left(W,Z\right)\xi-\tfrac{r+6}{4}g\left(Y,U\right)\eta\left(W\right)\eta\left(Z\right)\xi\\
&+\eta\left(U\right)P\left(Y,W\right)Z-\tfrac{r+6}{4}g\left(Y,W\right)g\left(U,Z\right)\xi+\tfrac{r+6}{4}g\left(Y,W\right)\eta\left(U\right)\eta\left(Z\right)\xi\\
&\left.+\eta\left(W\right)P\left(U,Y\right)Z+\eta\left(Z\right)P\left(U,W\right)Y\right\}=0.
\end{aligned}
\tag{41}
$$

If we choose $U=\xi$ in the equation (41) and we make the necessary adjustments, we get

$$
\begin{aligned}
&\tfrac{r-2}{4}\left\{-g\left(W,Z\right)\eta\left(Y\right)\xi-\tfrac{1}{2}S\left(W,Z\right)\eta\left(Y\right)\xi+g\left(W,Z\right)Y\right.\\
&+\tfrac{r+6}{4}g\left(W,Z\right)\eta\left(Y\right)\xi+R\left(Y,W\right)Z+\tfrac{1}{2}S\left(Y,Z\right)W\\
&-\tfrac{r+6}{4}g\left(Y,W\right)\eta\left(Z\right)\xi-\tfrac{r+6}{4}g\left(Y,Z\right)\eta\left(W\right)\xi\\
&\left.+\tfrac{r+6}{4}\eta\left(Y\right)\eta\left(W\right)\eta\left(Z\right)\xi\right\}.
\end{aligned}
\tag{42}
$$

In equation (42), we take the inner product of both sides by $\xi\in\chi\left(M\right)$ and choose $Y=\xi$, we get

$$\frac{r-2}{8}\left\{-S\left(W,Z\right)+\left[\left(r+2\right)g\left(W,Z\right)-\left(r+6\right)\eta\left(W\right)\eta\left(Z\right)\right]\right\}.$$

It is clear from the last equation that M is either an Einstein manifold or the scalar curvature of M is $r=2$. □

THEOREM 3.6. *Let M be a three-dimensional para-Sasakian manifold. If $W_0^*\left(X,Y\right).\tilde{C}=0$, then M is either a real space form with a constant section curvature or the scalar curvature of M is $r=2$.*

PROOF. Let's assume that $\left(W_0^*\left(X,Y\right).\tilde{C}\right)\left(Z,U,W\right)=0$ for every $X,Y,Z,W,U\in\chi\left(M\right)$. So we can write

$$
\begin{aligned}
&W_0^*\left(X,Y\right)\tilde{C}\left(Z,U\right)W-\tilde{C}\left(W_0^*\left(X,Y\right)Z,U\right)W\\
&-\tilde{C}\left(Z,W_0^*\left(X,Y\right)U\right)W-\tilde{C}\left(Z,U\right)W_0^*\left(X,Y\right)W=0.
\end{aligned}
\tag{43}
$$

If we choose $X=\xi$ in the equation (43) and make use of the equation (26), we get

$$
\begin{aligned}
&\tfrac{r-2}{4}\left\{g\left(Y,\tilde{C}\left(Z,U\right)W\right)\xi-\eta\left(\tilde{C}\left(Z,U\right)W\right)Y-g\left(Y,Z\right)\tilde{C}\left(\xi,U\right)W\right.\\
&+\eta\left(Z\right)\tilde{C}\left(Y,U\right)W-g\left(Y,U\right)\tilde{C}\left(Z,\xi\right)W+\eta\left(U\right)\tilde{C}\left(Z,Y\right)W\\
&\left.-g\left(Y,W\right)\tilde{C}\left(Z,U\right)\xi+\eta\left(W\right)\tilde{C}\left(Z,U\right)Y\right\}=0.
\end{aligned}
\tag{44}
$$

If we use the equations $(22),(23),(24)$ in equation (44), we obtain

$$\begin{aligned}
&\tfrac{r-2}{4}\Big\{g\left(Y,\tilde{C}(Z,U)W\right)\xi-\eta\left(\tilde{C}(Z,U)W\right)Y+\eta(Z)\tilde{C}(Y,U)W\\
&+\tfrac{(a+b)(r+6)}{6}g(Y,Z)g(U,W)\xi-\tfrac{(a+b)(r+6)}{6}g(Y,Z)\eta(W)U\\
&-\tfrac{(a+b)(r+6)}{6}g(Y,U)g(Z,W)\xi+\tfrac{(a+b)(r+6)}{6}g(Y,U)\eta(W)Z\\
&+\eta(U)\tilde{C}(Z,Y)W-\tfrac{(a+b)(r+6)}{6}g(Y,W)\eta(Z)U\\
&+\tfrac{(a+b)(r+6)}{6}g(Y,W)\eta(U)Z+\eta(W)\tilde{C}(Z,U)Y\Big\}=0.
\end{aligned}\tag{45}$$

If we choose $Z=\xi$ in the equation (45) and we make the necessary adjustments, we get

$$\begin{aligned}
&\tfrac{r-2}{4}\Big\{\tfrac{(a+b)(r+6)}{6}g(U,W)Y+\tilde{C}(Y,U)W\\
&-\tfrac{(a+b)(r+6)}{6}g(Y,W)U\Big\}=0.
\end{aligned}\tag{46}$$

If we substitute ϕY for Y and ϕU for U in the equation (46) and also if we use the equation (21) and (25), we obtain

$$\begin{aligned}
&\tfrac{r-2}{4}\Big\{aR(\phi Y,\phi U)W-\left[a+\tfrac{b(r+6)}{2}\right]\\
&[g(\phi Y,W)\phi U-g(\phi U,W)\phi Y]\Big\}=0.
\end{aligned}$$

It is clear from the last equation that M is either a real space form with a constant section curvature or the scalar curvature of M is $r=2$. □

4. W_0^* pseudo-symmetric and W_0^* Ricci pseudo-symmetric para-Sasakian manifolds

Now let us examine the concepts of W_0^* pseudo-symmetry and W_0^* Ricci pseudo-symmetry for the three-dimensional para-Sasakian M manifold.

Definition 4.1. Let M be a three dimensional para-Sasakian manifold, R be the Riemann curvature tensor of M. If the pair $R.W_0^*$ and $Q(g,W_0^*)$ are linearly dependent, i.e., if a λ_1 function can be found on the set $M_1=\{x\in M|\, g(x)\neq W_0^*(x)\}$ such that

$$R.W_0^*=\lambda_1 Q(g,W_0^*)$$

the M manifold is called a W_0^* pseudo-symmetric manifold.

Theorem 4.1. *Let M be a three-dimensional para-Sasakian manifold. If M is a W_0^* pseudo-symmetric manifold, then M is an $\eta-$Einstein manifold.*

Proof. Let's assume that

$$(R(X,Y).W_0^*)(Z,W,U)=\lambda_1 Q(g,W_0^*)(Z,W,U;X,Y)$$

for every $X, Y, Z, W, U \in \chi(M)$. So, we can write

$$
\begin{aligned}
&R(X,Y)W_0^*(Z,W)U - W_0^*(R(X,Y)Z)U \\
&-W_0^*(Z,R(X,Y)W)U - W_0^*(Z,W)R(X,Y)U \\
&= -\lambda_1\{W_0^*((X\wedge_g Y)Z,W) + W_0^*(Z,(X\wedge_g Y)W)U \\
&+W_0^*(Z,W)[(X\wedge_g Y)U]\}.
\end{aligned}
\tag{47}
$$

If we choose $X = \xi$ in the equation (47) and make use of the equations (9), (26), (27), (28), we obtain

$$
\begin{aligned}
&-g(Y,W_0^*(Z,W)U)\xi + \eta(W_0^*(Z,W)U)Y \\
&+\tfrac{r-2}{4}g(Y,Z)g(W,U)\xi - \eta(Z)W_0^*(Y,W)U \\
&-\tfrac{r-2}{4}g(Y,Z)\eta(U)W + 2g(Y,W)g(Z,U)\xi \\
&-2g(Y,W)\eta(U)Z - \eta(W)W_0^*(Z,Y)U \\
&-\tfrac{r-2}{4}g(Y,U)\eta(Z)W - 2g(Y,U)\eta(W)Z \\
&+\tfrac{r+6}{4}g(Y,U)\eta(Z)\eta(W)\xi - \eta(U)W_0^*(Z,W)Y \\
&= -\lambda_1\left\{-\tfrac{r-2}{4}g(Y,Z)g(W,U)\xi - \tfrac{r-2}{4}g(Y,Z)\eta(U)W\right. \\
&-\eta(Z)W_0^*(Y,W)U + 2g(Y,W)g(Z,U)\xi \\
&-2g(Y,W)\eta(U)Z - \eta(W)W_0^*(Z,Y)U \\
&-\tfrac{r-2}{4}g(Y,U)\eta(Z)W - 2g(Y,U)\eta(W)Z\xi \\
&\left.+\tfrac{r+6}{4}g(Y,U)\eta(W)\eta(Z) - \eta(U)W_0^*(Z,W)Y\right\}.
\end{aligned}
\tag{48}
$$

If we choose $Z = \xi$ in equation (48), from (26), we get

$$
\begin{aligned}
&\tfrac{r-2}{4}g(W,U)Y - W_0^*(Y,W)U - \tfrac{r-2}{4}g(Y,U)W \\
&= -\lambda_1\left\{\tfrac{r-2}{4}g(W,U)\eta(Y)\xi - W_0^*(Y,W)U\right. \\
&+\tfrac{r-2}{4}\eta(U)\eta(W)Y - \tfrac{r-2}{4}g(Y,U)W \\
&\left.-\tfrac{r-2}{4}g(W,Y)\eta(U)\xi\right\}.
\end{aligned}
$$

If we substitute (1) in the last equation and then choose $U = \xi$, we have

$$\begin{aligned} &\tfrac{r-2}{4}\eta(W)Y - \eta(Y)W + 2\eta(W)Y \\ &+\tfrac{1}{2}\eta(Y)QW - \tfrac{r-2}{4}\eta(Y)W = \\ &-\lambda_1\left\{\tfrac{r-2}{4}\eta(W)\eta(Y)\xi - \eta(Y)W\right. \\ &+2\eta(W)Y + \tfrac{1}{2}\eta(Y)QW - \tfrac{r-2}{4}\eta(W)Y \\ &\left.-\tfrac{r-2}{4}\eta(Y)W - \tfrac{r-2}{4}g(Y,W)\xi\right\}. \end{aligned} \tag{49}$$

If we choose $Y = \xi$ in (49) and then we take inner product of both sides of the equation by $U \in \chi(M)$, we obtain

$$S(W,U) = \frac{1}{2}\left[(r+2)g(W,U) - (r+6)\eta(W)\eta(U)\right].$$

This completes the proof. □

Corollary 4.1. *Let M be a three-dimensional para-Sasakian manifold. If M is a W_0^* semi-symmetric manifold, then M is $\eta-$Einstein manifold.*

Definition 4.2. Let M be a three dimensional para-Sasakian manifold, R be the Riemann curvature tensor of M and S be the Ricci curvature tensor. If the pair $R.W_0^*$ and $Q(S, W_0^*)$ are linearly dependent, that is, if a λ_2 function can be found on the set $M_2 = \{x \in M|\, S(x) \neq W_0^*(x)\}$ such that

$$R.W_0^* = \lambda_2 Q(S, W_0^*)$$

the M manifold is called a W_0^* Ricci pseudo-symmetric manifold.

Theorem 4.2. *Let M be a three-dimensional para-Sasakian manifold. If M is a W_0^* Ricci pseudo-symmetric manifold, then M is an $\eta-$Einstein manifold.*

Proof. The proof can be easily done similarly to the proof above. □

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