BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Bull. Int. Math. Virtual Inst., **12**(3)(2022), 409-415
DOI: 10.7251/BIMVI2203409M

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

ON $\mu-$FILTERS OF ALMOST DISTRIBUTIVE LATTICES

Y. Monikarchana

Abstract. In this paper, we prove some important properties on the set $Spec^{\mu}_{F}(L)$ of all prime $\mu-$filters of an Almost Distributive Lattice(ADL) topologically. We established a set of equivalent conditions for $Spec^{\mu}_{F}(L)$ to become a Hausdorff space.

1. Introduction

After Booles axiomatization of two valued propositional calculus as a Boolean algebra, a number of generalizations both ring theoretically and lattice theoretically have come into being. The concept of an Almost Distributive Lattice (ADL) was introduced by Swamy and Rao [**7**] as a common abstraction of many existing ring theoretic generalizations of a Boolean algebra on one hand and the class of distributive lattices on the other. In that paper, the concept of an ideal in an ADL was introduced analogous to that in a distributive lattice and it was observed that the set $PI(L)$ of all principal ideals of L forms a distributive lattice. This enables us to extend many existing concepts from the class of distributive lattices to the class of ADLs. In lattices, ideals plays a crucial role. Many algebraists have studied ideals extensively. Because of the lattice theoretic duality principle, the filters(dual ideals) have not assumed much importance in lattice theory. In [**6**], Rao and Badawy derived important results on prime $\mu-$filters topologically. After that in [**5**], Rafi and Ravi kumar Bandaru introduced the concept of $\mu-$filters in an ADL and studied their properties. In this paper, we derived some important properties of the space$Spec^{\mu}_{F}(L)$ of all prime $\mu-$filters of an ADL topologically. Given a set of equivalent conditions for the space $Spec^{\mu}_{F}(L)$ to become a T_1-space.

2010 *Mathematics Subject Classification.* Primary 06D75.
Key words and phrases. Almost Distributive Lattice(ADL), $\mu-$filrer, Prime filter, Compact, Haudorff space.
Communicated by Dusko Bogdanic.

We given a necessary and sufficient condition for $Spec_F^{\mu}(L)$ to become a Hausdorff space.

2. Preliminaries

In this section, we recall certain definitions and important results, those will be required in the text of the paper.

DEFINITION 2.1. [**7**] An Almost Distributive Lattice with zero or simply ADL is an algebra $(L, \vee, \wedge, 0)$ of type $(2, 2, 0)$ satisfying:
1. $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$
2. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
3. $(x \vee y) \wedge y = y$
4. $(x \vee y) \wedge x = x$
5. $x \vee (x \wedge y) = x$
6. $0 \wedge x = 0$
7. $x \vee 0 = x$, for all $x, y, z \in L$.

EXAMPLE 2.1. Every non-empty set X can be regarded as an ADL as follows. Let $x_0 \in X$. Define the binary operations $\vee, \wedge$ on X by

$$x \vee y = \begin{cases} x & \text{if } x \neq x_0 \\ y & \text{if } x = x_0 \end{cases} \qquad x \wedge y = \begin{cases} y & \text{if } x \neq x_0 \\ x_0 & \text{if } x = x_0. \end{cases}$$

Then $(X, \vee, \wedge, x_0)$ is an ADL (where x_0 is the zero) and is called a discrete ADL.

If $(L, \vee, \wedge, 0)$ is an ADL, for any $a, b \in L$, define $a \leqslant b$ if and only if $a = a \wedge b$ (or equivalently, $a \vee b = b$), then $\leqslant$ is a partial ordering on L.

THEOREM 2.1. [**7**] *If $(L, \vee, \wedge, 0)$ is an ADL, for any $a, b, c \in L$, we have the following:*

(1). $a \vee b = a \Leftrightarrow a \wedge b = b$
(2). $a \vee b = b \Leftrightarrow a \wedge b = a$
(3). $\wedge$ *is associative in* L
(4). $a \wedge b \wedge c = b \wedge a \wedge c$
(5). $(a \vee b) \wedge c = (b \vee a) \wedge c$
(6). $a \wedge b = 0 \Leftrightarrow b \wedge a = 0$
(7). $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
(8). $a \wedge (a \vee b) = a$, $(a \wedge b) \vee b = b$ *and* $a \vee (b \wedge a) = a$
(9). $a \leqslant a \vee b$ *and* $a \wedge b \leqslant b$
(10). $a \wedge a = a$ *and* $a \vee a = a$
(11). $0 \vee a = a$ *and* $a \wedge 0 = 0$
(12). *If* $a \leqslant c$, $b \leqslant c$ *then* $a \wedge b = b \wedge a$ *and* $a \vee b = b \vee a$
(13). $a \vee b = (a \vee b) \vee a$.

It can be observed that an ADL L satisfies almost all the properties of a distributive lattice except the right distributivity of $\vee$ over $\wedge$, commutativity of $\vee$, commutativity of $\wedge$. Any one of these properties make an ADL L a distributive lattice. That is

THEOREM 2.2. [**7**] *Let $(L, \vee, \wedge, 0)$ be an ADL with 0. Then the following are equivalent:*

1). $(L, \vee, \wedge, 0)$ *is a distributive lattice*
2). $a \vee b = b \vee a$*, for all* $a, b \in L$
3). $a \wedge b = b \wedge a$*, for all* $a, b \in L$
4). $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$*, for all* $a, b, c \in L$.

As usual, an element $m \in L$ is called maximal if it is a maximal element in the partially ordered set $(L, \leqslant)$. That is, for any $a \in L$, $m \leqslant a \Rightarrow m = a$.

THEOREM 2.3. [**7**] *Let L be an ADL and $m \in L$. Then the following are equivalent:*

1). m *is maximal with respect to* $\leqslant$
2). $m \vee a = m$*, for all* $a \in L$
3). $m \wedge a = a$*, for all* $a \in L$
4). $a \vee m$ *is maximal, for all* $a \in L$.

As in distributive lattices [**1, 2**], a non-empty subset I of an ADL L is called an ideal of L if $a \vee b \in I$ and $a \wedge x \in I$ for any $a, b \in I$ and $x \in L$. Also, a non-empty subset F of L is said to be a filter of L if $a \wedge b \in F$ and $x \vee a \in F$ for $a, b \in F$ and $x \in L$.

The set $I(L)$ of all ideals of L is a bounded distributive lattice with least element $\{0\}$ and greatest element L under set inclusion in which, for any $I, J \in I(L)$, $I \cap J$ is the infimum of I and J while the supremum is given by $I \vee J := \{a \vee b \mid a \in I, b \in J\}$. A proper ideal P of L is called a prime ideal if, for any $x, y \in L$, $x \wedge y \in P \Rightarrow x \in P$ or $y \in P$. A proper ideal M of L is said to be maximal if it is not properly contained in any proper ideal of L. It can be observed that every maximal ideal of L is a prime ideal. Every proper ideal of L is contained in a maximal ideal. A proper filter G of L is called a prime filter of L if, for any $x, y \in L$, $x \vee y \in G \Rightarrow x \in G$ or $y \in G$. For any subset S of L the smallest ideal containing S is given by $(S] := \{(\bigvee_{i=1}^{n} s_i) \wedge x \mid s_i \in S, x \in L \text{ and } n \in N\}$. If $S = \{s\}$, we write $(s]$ instead of $(S]$. Similarly, for any $S \subseteq L$, $[S) := \{x \vee (\bigwedge_{i=1}^{n} s_i) \mid s_i \in S, x \in L \text{ and } n \in N\}$ is the smallest filter containing S. If $S = \{s\}$, we write $[s)$ instead of $[S)$. The set $F(L)$ of all filters of L forms a bounded distributive lattice, where $F \cap G$ is the infimum and $F \vee G = \{a \wedge b \mid a \in F,\ b \in G\}$ is the supremum in $F(L)$.

THEOREM 2.4. [**7**] *For any x, y in L the following are equivalent:*

1). $(x] \subseteq (y]$
2). $y \wedge x = x$
3). $y \vee x = y$
4). $[y) \subseteq [x)$.

For any $x, y \in L$, it can be verified that $(x] \vee (y] = (x \vee y]$ and $(x] \wedge (y] = (x \wedge y]$. Hence the set $PI(L)$ of all principal ideals of L is a sublattice of the distributive lattice $I(L)$ of ideals of L.

THEOREM 2.5. [**3**] *Let I be an ideal and F a filter of L such that $I \cap F = \emptyset$. Then there exists a prime ideal P such that $I \subseteq P$ and $P \cap F = \emptyset$.*

For any subset S of an ADL L with maximal elements, define $S^+ = \{x \in L \mid s \vee x$ is a maximal element, for all $s \in S\}$. Here we say that S^+ is a dual annihilator of S. For $S = \{x\}$, then we denote simply $(x)^+$ for $(\{x\})^+$. It is clear that $L^+ = \mathcal{M}_{max.elt}$, where $\mathcal{M}_{max.elt}$ is the set of all maximal elements of an ADL L, for any maximal element m of an ADL L, we have $m^+ = L$ and it is easy to verify that S^+ is a filter of an ADL L.

DEFINITION 2.2. [**5**] A filter F of an ADL L is said to be a $\mu-$filter if for any $x, y \in L$, $x^+ = y^+$ and $x \in F$, then $y \in F$. A $\mu-$filter P of an ADL L is said to be prime if for any $x, y \in$, $x \vee y \in P$ implies either $x \in P$ or $y \in P$.

THEOREM 2.6. [**5**] *Let F be a μ filter and I an ideal of a ADL L with 0 such that $F \cap I = \emptyset$. Then there exists a prime $\mu-$filter P such that $F \subseteq P$ and $P \cap I = \emptyset$.*

COROLLARY 2.1. [**5**] *Let F be a $\mu-$filter of an ADL L and $x \notin F$. Then there exists a prime $\mu-$filter P of L such that $F \subseteq P$ and $x \notin P$.*

COROLLARY 2.2. [**5**] *For any $\mu-$filter F of an ADL L, we have*

$$F = \bigcap \{P | P \text{ is a prime } \mu-\text{filter of } L \text{ and } F \subseteq P\}$$

COROLLARY 2.3. [**5**] *The intersection of all prime $\mu-$filters of an ADL is equal to the set of all maximal elements of L.*

3. On $\mu-$filters of ADLs

In this section we discuss some topological concepts on the collection of prime $\mu-$filters of a ADL. A necessary and sufficient condition is derived for the space of all prime $\mu-$filters of a ADL to become a Hausdorff space.

Let $Spec_F^{\mu}(L)$ be the set of all prime $\mu-$filters of a ADL L. For any $X \subseteq L$, let $h(X) = \{N \in Spec_F^{\mu}(L) | X \nsubseteq N\}$ and for any $x \in L, h(x) = h(\{x\})$. For any two subsets X and Y of L, it is obvious that $X \subseteq Y$ implies $h(X) \subseteq h(Y)$. The following observations can be verified directly.

LEMMA 3.1. *Let L be an ADL with maximal elements. Then for any $a, b \in L$, the following conditions holds.*

(1) $\bigcup_{a \in L} h(a) = Spec_F^{\mu}(L)$

(2) $h(a) \cap h(b) = h(a \vee b)$

(3) $h(a) \cup h(b) = h(a \wedge b)$

(4) $h(a) = \emptyset \Leftrightarrow a$ *is a maximal element of L*

(5) $h(a) = Spec_F^{\mu}(L)$ *if and only if $(a)^+$ is the set of all maximal elements of L.*

From above lemma it can be easily observed that the collection $\{h(a)/a \in L\}$ forms a base for a topology on $Spec_F^{\mu}(L)$ which is called a hull kernel topology.

PROPOSITION 3.1. *Let L be an ADL. Then for any $a \in L$, we have the following*
(1) $h(a) = h((a)^{++})$
(2) $h((a)^{++}) \subseteq Spec_F^{\mu}(L) \setminus h((a)^{+})$

PROOF. (1). Let $N \in h(a)$. Then $a \notin N$. That implies $(a)^{++} \nsubseteq N$. Therefore $N \in h((a)^{++})$ and hence $h(a) \subseteq h((a)^{++})$. Let $N \in h((a)^{++})$. Then $(a)^{++} \nsubseteq N$. Since N is a $\mu-$filter, we get that $a \notin N$. Therefore $N \in h(a)$ and hence $h((a)^{++}) \subseteq h(a)$. Thus $h(a) = h((a)^{++})$.

(2). Let $N \in h((a)^{++})$. Then $(a)^{++} \nsubseteq N$. Since N is a $\mu-$filter, we get that $a \notin N$. That implies $(a)^{+} \subseteq N$. Therefore $N \in Spec_F^{\mu}(L) \setminus h((a)^{+})$ and hence $h((a)^{++}) \subseteq Spec_F^{\mu}(L) \setminus h((a)^{+})$ □

THEOREM 3.1. *Let F be any filter of an ADL L. Then $h(F) = h(\overleftarrow{\mu}\mu(F))$.*

PROOF. Since $F \subseteq \overleftarrow{\mu}\mu(F)$, we have that $h(F) \subseteq h(\overleftarrow{\mu}\mu(F))$. Let $N \in Spec_F^{\mu}(L) \cap h(\overleftarrow{\mu}\mu(F))$. Then $\overleftarrow{\mu}\mu(F) \nsubseteq N$. That implies there exists an element $a \in \overleftarrow{\mu}\mu(F)$ such that $a \notin N$. Since $a \in \overleftarrow{\mu}\mu(F)$, we get that $(a)^{++} \in \mu(F)$. Then there exists an element $x \in F$ such that $(a)^{++} = (x)^{++}$. Suppose $N \notin h(F)$. Then $x \in F \subseteq N$. Since N is a $\mu-$filter, we get that $(x)^{++} \subseteq N$. Hence $a \in (a)^{++} \subseteq N$, which is a contradiction. Thus $N \in h(F)$. Therefore $h(\overleftarrow{\mu}\mu(F)) \subseteq h(F)$. □

In the following theorem, the compact open set of $Spec_F^{\mu}(L)$ are characterized.

THEOREM 3.2. *Let L be an ADL. Then the set of all compact open sets of $Spec_F^{\mu}(L)$ is the base $\{h(a)/a \in L\}$.*

PROOF. Let $\{h(a_i)\}_{i\in\Delta}$ be any basic open cover for$h(a)$. Let F be a filter generated by $\{a_i | i \in \Delta\}$.

Suppose $a \notin \overleftarrow{\mu}\mu(F)$. Then there exists a prime μ-filter N such that $\overleftarrow{\mu}\mu(F) \subseteq N$ and $a \notin N$. Thene $P \in h(a) \subseteq \bigcup_{i\in\Delta} h(a_i)$. That implies there exists $i \in \Delta$ such that $a_i \notin N$, which is a contradiction to $F \subseteq \overleftarrow{\mu}\mu(F) \subseteq N$. Therefore $a \in \overleftarrow{\mu}\mu(F)$.

Since $a \in \overleftarrow{\mu}\mu(F)$, there exists an element $x \in F$ such that $a \in (x)^{++}$. Since F is a filter generated by $\{a_i | i \in \Delta\}$, there exist $a_1, a_2, \cdots, a_n \in \{x_i | i \in \Delta\}$ such that $x = a_1 \wedge a_2 \wedge \cdots \wedge a_n$. Therefore $a \in (x)^{++} = (a_1 \wedge a_2 \wedge \cdots \wedge a_n)^{++}$.

Let $N \in h(a)$. Then $a \notin N$. Suppose $N \notin \bigcup_{i=1}^{n} h(a_i)$. Then $a_i \in N$ for all $i = 1, 2, \cdots, n$. Hence $a_1 \wedge a_2 \wedge \cdots \wedge a_n \in N$. Since N is a $\mu-$filter, we get $a \in (a_1 \wedge a_2 \wedge \cdots \wedge a_n)^{++} \subseteq N$, which is a contradiction to that $a \notin N$. Therefore $N \in \bigcup_{i=1}^{n} h(a_i)$ and hence $h(a) \subseteq \bigcup_{i=1}^{n} h(a_i)$, which is a finite subcover for $h(a)$. Thus

$h(a)$ is compact in $Spec^{\mu}_{F}(L)$.

Now we prove that every compact open subset of $Spec^{\mu}_{F}(L)$ is of the form $h(a)$ for some $a \in L$. Let C be a compact open subset of $Spec^{\mu}_{F}(L)$. Since C is open, there exists $A \subseteq L$ such that $C = \bigcup_{x \in A} h(x)$. Since C is compact, there exists $x_1, x_2, \cdots, x_n \in A$ such that

$$C = \bigcup_{i=1}^{n} h(x_i) = h(\bigwedge_{i=1}^{n} x_i)$$

Therefore $C = h(a)$ for some $a \in L$. □

THEOREM 3.3. *Let L be an ADL with maximal elements. Then $Spec^{\mu}_{F}(L)$ is compact if and only if L has an element of the form $(a)^+$ is the set of all maximal elements of L.*

COROLLARY 3.1. *If L is an ADL with maximal elements, then $Spec^{\mu}_{F}(L)$ is compact.*

It is already observed that every minimal prime filter of a ADL is a prime μ-filter. In general, the converse is not true. However, in the following theorem, some equivalent conditions are derived for every prime μ-filter of a ADL to become a minimal prime filter.

THEOREM 3.4. *Let L be an ADL. Then the following conditions are equivalent*
(1) $Spec^{\mu}_{F}(L)$ is a T_1-space.
(2) every prime $\mu-$filter is maximal
(3) every prime $\mu-$filter is minimal.

PROOF. (1) $\Rightarrow$ (2) : Assume that $Spec^{\mu}_{F}(L)$ is a T_1-space. Let N be a prime $\mu-$filter. Now we prove that N is maximal. Suppose M be any proper $\mu-$filter with $N \subset M$. Since $Spec^{\mu}_{F}(L)$ is a T_1-space, there exists open sets $h(a)$ and $h(b)$ such that $N \in h(a) \setminus h(b)$ and $M \in h(b) \setminus h(a)$. Since $N \notin h(b)$, we get that $b \in N \subset M$. That implies $M \notin h(b)$, which is a contradiction. Therefore N is a maximal $\mu-$filter.
(2) $\Rightarrow$ (3) : Clear
(3) $\Rightarrow$ (1) : Assume that every prime $\mu-$filter is minimal. Let N and M be two distinct elements of $Spec^{\mu}_{F}(L)$. Since N and M are minimal, it is clear that $N \nsubseteq M$ and $M \nsubseteq N$. Then there exists $a, b \in L$ such that $a \in N \setminus M$ and $b \in M \setminus N$. That implies $N \in h(b) \setminus h(a)$ and $M \in h(a) \setminus h(b)$. Therefore $Spec^{\mu}_{F}(L)$ is a T_1-space. □

THEOREM 3.5. *Let L be a ADL such that each $(a)^+$ is a direct factor of L. Then the following conditions are equivalent.*
(1) $Spec^{\mu}_{F}(L)$ is a Hausdorff space.
(2) For any two distinct prime μ-filters N, M in L, there exists $x, y \in L$ such that $(x)^+ \subseteq N$ and $(y)^+ \subseteq M$ and there does not exist any $P \in Spec^{\mu}_{F}(L)$ such that $x \vee y \notin P$.

PROOF. (1) $\Rightarrow$ (2) : Assume that $Spec_F^{\mu}(L)$ is a Hausdorff space. Let N, M be two distinct prime $\mu-$filters of L. Since$Spec_F^{\mu}(L)$ is Hausdorff, there exists two open sets $h(x)$ and $h(y)$ such that $N \in h(x)$ and $M \in h(y)$ and $h(x) \cap h(y) = \emptyset$. Since $N \in h(x) = h((x)^{++})$, we get that $(x)^{++} \nsubseteq N$. Choose an element $a \in (x)^{++}$ and $a \notin N$. Hence we get $(x)^+ \subseteq (a)^+$ and $(a)^+ \subseteq N$. Hence $(x)^+ \subseteq N$. Similarly, we get that $(y)^+ \subseteq M$. Suppose there exists a prime $\mu-$filter P such that $x \vee y \notin P$. Then $P \in h(x \vee y) = h(x) \cap h(y)$, which is a contradiction to that $h(x) \cap h(y) = \emptyset$.

(2) $\Rightarrow$ (1) : Assume the condition (2). Let N, M be two distinct elements of $Spec_F^{\mu}(L)$. By our assumption, there exists $x, y \in L$ such that $(x)^+ \subseteq N$ and $(y)^+ \in M$. Hence by the assumption, we get $(x)^+$ and $(y)^+$ are direct factors of L. That implies $(x)^+ \vee (x)^{++} = L$. Suppose $x \in N$. Since N is a $\mu-$filter, we get that $(x)^{++} \subseteq N$. Hence $L = (x)^+ \vee (x)^{++} \subseteq N$, which is a contradiction. Hence $x \notin N$. Similarly, we get that $y \notin M$. Therefore $N \in h(x)$ and $M \in h(y)$. Suppose $h(x) \cap h(y) \neq \emptyset$. Then there exists a prime $\mu-$filter P such that $P \in h(x) \cap h(y) = h(x \vee y)$. Hence $x \vee y \notin P$, which is a contradiction. Therefore $Spec_F^{\mu}(L)$ is a Hausdorff space. □

References

1. G.Birkhoff. Lattice Theory. *Amer. Math. Soc. Colloq. Publ. XXV, Providence* (1967), U.S.A.
2. G. Gratzer. General Lattice Theory. *Academic Press, New York, Sanfransisco* (1978).
3. G.C. Rao. Almost Distributive Lattices. Doctoral Thesis (1980), Dept. of Mathematics, Andhra University, Visakhapatnam.
4. G.C. Rao and S. Ravi Kumar. Minimal prime ideals in an ADL. *Int. J. Contemp. Sciences*, **4** (2009), 475-484.
5. N. Rafi and Ravi Kumar Bandaru. $\mu-$filters of Almost Distributive Lattices, *Chamchuri Journal of Mathematics*, **10**(2018), 53-65.
6. M. Sambasiva Rao and Badawy Abd El-Mohsen. $\mu-$filters of Distributive Lattices, *Southeast Asian Bulletin of Mathematics*, **4(2)**(2016), 251-264.
7. U.M. Swamy and G.C. Rao. Almost Distributive Lattices. *J. Aust. Math. Soc. (Series A)*, **31** (1981), 77-91.

Receibed by editors 4.5.2022; Revised version 22.8.2022; Available online 10.9.2022.

Y. Monikarchana, Department of Mathematics, Sree Vidyanikethan Engineering College, A.Rangampet, Tirupati, Andhra Pradesh, India-517 102.

Email address: `285archana@gmail.com`