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ON SEQUENTIAL HENSTOCK STIELTJES INTEGRAL FOR $L^p[0,1]$-INTERVAL VALUED FUNCTIONS

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Abstract. In this paper, we introduce the concept of Sequential Henstock Stieltjes integral for $L^p[0,1]$-interval valued functions and prove some of its properties.

1. Introduction and preliminaries

Ralph Henstock and Jaroslav Kursweil, sometimes in the late 1950s, independently introduced a generalised Riemann-type integral popularly known as the Henstock integral. It is well known that the Henstock integral is a kind of integral which comprises the Improper Riemann, Newton, Riemann and is a lot stronger and easier than the Werner, Lebesgue and the Feynmann integrals in handling difficult integration problems (see[1-12]). The Henstock integral has been proved to be equivalent to the Perron and Denjoy integral which recovers a continuous function from its derivative. In 2000, Wu and Gong [11] introduced the notion of the Henstock (H) integrals of interval valued functions and Fuzzy number-valued functions and obtain a number of properties. Two years earlier, Lim et al.[8] gave the idea of the Henstock-Stieltjes integrals of real-valued functions which is more general than the Henstock(H) integral and obtained its properties.
Yoon[12] established the interval valued Henstock-Stieltjes integral on time scales and investigated some properties of the integrals. In 2016, Paxton[10] introduced the notion of the Sequential Henstock(SH) integrals of real-valued functions as generalizations of the Henstock(H) integral and obtainedsome of its properties. Hamid and Elmuiz presented the idea of the Henstock-Stieltjes integrals of interval-valued

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functions and Fuzzy number-valued functions and obtain a number of properties. Cao [3] gave a generalization of the definition of the Henstock integral for Banach space-valued function, and then established some of its properties. Macalalag and Paluga [9] studied the Henstock-type integral for l_p-valued functions with $0 < p < 1$ and obtained its basic properties. It is well known that the class of $L^p[0,1]$-valued functions with $0 < p < 1$ is a Banach Space with the norm denoted by $\|.\|_{L^p}$.
In this paper, we introduce the Sequential Henstock Stieltjes(SHS)integral of $L^p[0,1]$-interval valued functions and discuss some of its properties.

Let $\mathbb{R}$ denote the set of real numbers, $F(X)$, an interval valued function, F^-, the left endpoint, F^+, right endpoint, $\{\delta_n(x)\}_{n=1}^{\infty}$, set of gauge functions, P_n, set of partitions of subintervals of a compact interval $[a,b]$, X, non empty interval in $\mathbb{R}$ and $\ll$ as much more smaller.

A gauge on $[a,b]$ is a positive real-valued function $\delta : [a,b] \to \mathbb{R}^+$. This gauge is δ-fine if $[u_{i-1}, u_i] \subset [t_i - \delta(t_i), t_i + \delta(t_i)]$ while a sequence of tagged partition P_n of [a,b] is a finite collection of ordered pairs $P_n = \{(u_{(i-1)_n}\, u_{i_n}), t_{i_n}\}_{i=1}^{m_n}$ where $[u_{i-1}, u_i] \in [a,b]$, $u_{(i-1)_n} \leqslant t_{i_n} \leqslant u_{i_n}$ and $a = u_0 < u_{i_1} < \ldots < u_{m_n} = b$.

Definition 1.1. ([12]) A function $f : [a,b] \to \mathbb{R}$ is Henstock integrable ($H[a,b]$) to α on $[a,b]$ if for any $\varepsilon > 0$ there exists a function $\delta(x) > 0$ such that for any $\delta(x)$-fine tagged partitions $P = \{(u_{i-1}, u_i), t_i\}_{i=1}^{n}$ we have

$$|\sum_{i=1}^{n} f(t_i)[u_i - u_{(i-1)}] - \alpha| < \varepsilon,$$

where $\alpha = (H)\int_{[a,b]} f(x)dx$ and $f \in H[a,b]$.

Definition 1.2. ([12]) Let $g : [a,b] \to \mathbb{R}$ be a non decreasing function. A real-valued function $f : [a,b] \to \mathbb{R}$ is Henstock-Stieltjes ($HS[a,b]$) integrable to $\alpha \in \mathbb{R}$ with respect to g on $[a,b]$ if for any $\varepsilon > 0$ there exists a function $\delta(x) > 0$ such that for each $\delta(x)$-fine tagged partitions $P = \{(u_{i-1}\, u_i), t_i\}_{i=1}^{n}$ we have

$$|\sum_{i=1}^{n} f(t_i)[g(u_i) - g(u_{(i-1)})] - \alpha| < \varepsilon,$$

where $\alpha = (H)\int_{[a,b]} f dg$ and $f \in HS[a,b]$.

Definition 1.3. ([12]) A function $f : [a,b] \to \mathbb{R}$ is Sequential Henstock integrable ($SH[a,b]$) to $\alpha \in \mathbb{R}$ on $[a,b]$ if for any $\varepsilon > 0$ there exists a sequence of gauge functions $\{\delta_n(x)\}_{n=1}^{\infty}$ such that for any $\delta_n(x) - fine$ tagged partitions $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}_{i=1}^{m_n}$, we have

$$|\sum_{i=1}^{m_n \in \mathbb{N}} f(t_{i_n})(u_{i_n} - u_{(i-1)_n}) - \alpha| < \varepsilon,$$

where $\alpha = (SH)\int_{[a,b]} f(x)dx$ and $f \in SH[a,b]$.

LEMMA 1.1. *[5]Let f, k be Sequential Henstock (SH)integrable functions on $[a, b]$, if $f \leqslant k$ is almost everywhere on $[a, b]$, then*

$$\int_a^b f \leqslant \int_a^b k.$$

DEFINITION 1.4. ([11,15]) Let $I_{\mathbb{R}} = \{I = [I^-, I^+]$: I is a closed bounded interval on the real line $\mathbb{R}\}$. For $X, Y \in I_{\mathbb{R}}$, we define
i. $X \leqslant Y$ if and only if $Y^- \leqslant X^-$ and $X^+ \leqslant Y^+$,
ii. $X + Y = Z$ if and only if $Z^- = X^- + Y^-$ and $Z^+ = X^+ + Y^+$,
iii. $X.Y = \{x.y : x \in X, y \in Y\}$, where

$$(X.Y)^- = \min\{X^-.Y^-, X^-.Y^+, X^+.Y^-, X^+.Y^+\}$$

Then $d(X, Y) = \max(|X^- - Y^-|, |X^+ - Y^+|)$ is the metric distance between intervals X and Y.

DEFINITION 1.5. ([5]) An interval valued function $F : [a, b] \to I_{\mathbb{R}}$ is Henstock integrable ($IH[a, b]$) to $I_0 \in I_{\mathbb{R}}$ on $[a, b]$ if for every $\varepsilon > 0$ on $[a, b]$ there exists a gauge function $\delta(x) > 0$ such that

$$|\sum_{i=1}^{n\in\mathbb{N}} F(t_i)(u_i - u_{i-1}) - I_o| < \varepsilon,$$

whenever $P = \{(u_{i-1}, u_i), t_i\}_{i=1}^n$ of $[a, b]$ is a $\delta(x)-fine$ Henstock partition of $[a, b]$. Where $\alpha = (IH) \int_{[a,b]} F = \alpha$ and $F \in IH[a, b]$.

DEFINITION 1.6. ([5]) Let $g : [a, b] \to \mathbb{R}$ be a non decreasing function. An interval-valued function $F : [a, b] \to I_{\mathbb{R}}$ is Henstock-Stieltjes(IHS) integrable with respect to g on $[a, b]$ to $\alpha \in \mathbb{R}$ if for every $\varepsilon > 0$ there exists a function $\delta(x) > 0$ on $[a, b]$ such that

$$|\sum_{i=1}^{n} F(t_i)[g(u_i) - g(u_{(i-1)})] - \alpha| < \varepsilon,$$

whenever $P = \{(u_{i-1}, u_i), t_i\}_{i=1}^n$ of $[a, b]$ is a $\delta(x)-fine$ Henstock partition of $[a, b]$. Then $(IHS) \int_{[a,b]} F = \alpha$ and $F \in IHS[a, b]$.

DEFINITION 1.7. ([5]) An interval valued function $F : [a, b] \to L^p$ is Henstock integrable(l_p-$IH[a, b]$) to $I_0 \in L^p[0, 1]$ on $[a, b]$ if for every $\varepsilon > 0$ there exists a positive gauge function $\delta(x) > 0$ on $[a, b]$ such that for every $\delta(x) - fine$ tagged partitions $P = \{(u_{i-1}, u_i), t_i\}_{i=1}^n$, we have

$$\|\sum_{i=1}^{n\in\mathbb{N}} F(t_i)(u_i - u_{i-1}) - I_o\|_{L^p} < \varepsilon$$

We say that I_0 is the Henstock integral of F on $[a, b]$ with $(L^p[0, 1]$-$IH) \int_{[a,b]} F = I_0$ and $F \in L^p[0, 1]$-$IH[a, b]$.

Now, we will define the Sequential Henstock Stieltjes integral of $L^p[0, 1]$-interval valued function and then discuss some of the properties of the integral.

DEFINITION 1.8. Let $g : [a, b] \to \mathbb{R}$ be a non decreasing function. An interval valued function $F : [a, b] \to L^p$ is Sequential Henstock integrable($L^p[0, 1]$-$ISH[a, b]$) to $I_0 \in L^p[0, 1]$ with respect to g on $[a, b]$ if for any $\varepsilon > 0$ there exists a sequence of positive gauge functions $\{\delta_n(x)\}_{n=1}^{\infty}$ such that for every $\delta_n(x) - fine$ tagged partitions $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}_{i=1}^{m_n}$, we have

$$\| \sum_{i=1}^{m_n \in \mathbb{N}} F(t_{i_n})(g(u_{i_n}) - g(u_{(i-1)_n})) - l_o\|_{L^p} < \varepsilon.$$

We say that $L^p[0, 1]$ is the Sequential Henstock integral of F on $[a, b]$ with $(L^p[0, 1]\text{-}ISH) \int_{[a,b]} F = \alpha$ and $F \in L^p[0, 1]$-$ISH[a, b]$.

REMARK 1.1. ([12]) If g is an identity function, we have a definition for $L^p[0, 1]$-interval Sequential Henstock integral.

2. Main results

In this section, we present some of the basic properties of the $L^p[0, 1]$-interval valued Sequential Henstock integrals.

THEOREM 2.1. *If $F \in L^p[0, 1]$-$ISHS[a, b]$, then there exists a unique integral value.*

PROOF. Suppose the integral values are not unique. Let $\alpha_1 = (L^p[0, 1]\text{-}ISHS) \int_{[a,b]} F$ and $\alpha_2 = (L^p[0, 1]\text{-}ISHS) \int_{[a,b]} F$ with $\alpha_i \neq \alpha_2$. Let $\varepsilon > 0$ then there exists a $\{\delta_n^1(x)\}_{n=1}^{\infty}$ and $\{\delta_n^2(x)\}_{n=1}^{\infty}$ such that for each $\delta_n^1(x)$-fine tagged partitions P_n^1 of $[a, b]$ and for each $\delta_n^2(x)$-fine tagged partitions P_n^2 of $[a, b]$, we have

$$\| \sum_{i=1}^{m_n \in \mathbb{N}} F(t_{i_n})(g(u_{i_n}) - g(u_{(i-1)_n})) - \alpha_1\|_{L^p} < \frac{\varepsilon}{2},$$

and

$$\| \sum_{i=1}^{m_n \in \mathbb{N}} F(t_{i_n})(g(u_{i_n}) - g(u_{(i-1)_n})) - \alpha_2\|_{L^p} < \frac{\varepsilon}{2}.$$

respectively.
Define a positive gauge function $\delta_n(x)$ on $[a, b]$ by $\delta_n(x) = \min\{\delta_n^1(x), \delta_n^2(x)\}$. Let P_n be any $\delta_n(x)$-fine tagged partition of $[a, b]$ and let $\varepsilon = \dfrac{\|\alpha_1 - \alpha_2\|_p}{2^{\frac{1}{p}}}$. Then we

have

$$\begin{aligned}\|\alpha_1-\alpha_2\|_{L^p} &= \|\sum_{i=1}^{m_n\in\mathbb{N}} F(t_{i_n})(g(u_{i_n})-g(u_{(i-1)_n}))-\alpha_1\\ &+\sum_{i=1}^{m_n\in\mathbb{N}} F(t_{i_n})(g(u_{i_n})-g(u_{(i-1)_n}))-\alpha_2\|_{L^p}\\ &\leqslant \|\sum_{i=1}^{m_n\in\mathbb{N}} F(t_{i_n})(g(u_{i_n})-g(u_{(i-1)_n}))-\alpha_1\|_{L^p}\\ &+\|\sum_{i=1}^{m_n\in\mathbb{N}} F(t_{i_n})(g(u_{i_n})-g(u_{(i-1)_n}))-\alpha_2\|_{L^p}\\ &< 2^{\frac{1}{p}}(\frac{\varepsilon}{2}+\frac{\varepsilon}{2})=2^{\frac{1}{p}}\varepsilon=\|\alpha_1-\alpha_2\|_{L^p},\end{aligned}$$

This is a contradiction. Thus $\alpha_1=\alpha_2$. □

THEOREM 2.2. *An interval valued function $F\in L^p[0,1]$-$ISHS[a,b]$ if and only if $F^-,F^+\in L^p[0,1]$-$SHS[a,b]$ and*

$$(L^p[0,1]\text{-}ISHS)\int_{[a,b]}F=[(l_p\text{-}SHS)\int_{[a,b]}F^-,(L^p[0,1]\text{-}SHS)\int_{[a,b]}F^+]$$

PROOF. Let $F\in L^p[0,1]$-$ISHS[a,b]$, from Definition 1.9 there is a unique interval number $I_o=[I_0^-,I_0^+]$ in the property, then for any $\varepsilon>0$, there exists a $\{\delta_n(x)\}_{n=1}^{\infty}$, $n\geqslant\mu$ on $[a,b]\in\mathbb{R}$ such that for any $\delta_n(x)$-fine tagged partition P_n, we have

$$\|\sum_{i=1}^{m_n\in\mathbb{N}} F(t_{i_n})(g(u_{i_n})-g(u_{(i-1)_n}))-I_0\|_{L^p}<\varepsilon.$$

Observe that

$$\|\sum_{i=1}^{m_n\in\mathbb{N}} F(t_{i_n})(g(u_{i_n})-g(u_{(i-1)_n}))-I_0\|_{L^p}=\max(I_1,I_2),$$

where $I_1=\|\sum_{i=1}^{m_n\in\mathbb{N}}F^-(t_{i_n})(g(u_{i_n})-g(u_{(i-1)_n}))-I_0^-\|_{L^p}$, and $I_2=\|\sum_{i=1}^{m_n\in\mathbb{N}}F^+(t_{i_n})(g(u_{i_n})-g(u_{(i-1)_n}))-I_0^+\|_{L^p}$.

Since $u_{i_n}-u_{(i-1)_n}\geqslant 0$ for $1\leqslant i_n\leqslant m_n$, then it follows that

$$\|\sum_{i=1}^{m_n\in\mathbb{N}} F^-(t_{i_n})(g(u_{i_n})-g(u_{(i-1)_n}))-I_0^-\|_{L^p}<\varepsilon,$$

and

$$\|\sum_{i=1}^{m_n\in\mathbb{N}} F^+(t_{i_n})(g(u_{i_n})-g(u_{(i-1)_n}))-I_0^+)\|_{L^p}<\varepsilon,$$

for every $\delta_n(x)$-tagged partition $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}_{i=1}^{m_n}$. Thus, by Definition 2.8, we obtain $F^+, F^- \in L^p[0,1]$-$SHS[a,b]$ and

$$I_o^- = (L^p[0,1]\text{-}SHS)\int_{[a,b]} F^-(x)dg$$

and

$$I_o^+ = (L^p[0,1]\text{-}SHS)\int_{[a,b]} F^+(x)dg.$$

Conversely, Let $F^- \in L^p[0,1]$-$SHS_{[a,b]}$. Then there exists a unique $\beta_1 \in \mathbb{R}$ with the property, let $\varepsilon > 0$ be given, then there exists a $\{\delta_n^1(x)\}_{n=1}^\infty$, such that for any $\delta_n^1(x)$-fine tagged partitions P_n^1 we have

$$\|\sum_{i=1}^{m_n\in\mathbb{N}} F^-(t_{i_n})(g(u_{i_n}) - g(u_{(i-1)_n})) - \beta_1\|_{L^p} < \varepsilon.$$

Similarly,
Let $F^+ \in L^p[0,1]$-$SHS[a,b]$. Then there exists a unique $\beta_2 \in \mathbb{R}$ with the property, let $\varepsilon > 0$ be given, then there exists a $\{\delta_n^2(x)\}_{n=1}^\infty$, such that for any $\delta_n^2(x)$-fine tagged partitions P_n^2 we have

$$\|\sum_{i=1}^{m_n\in\mathbb{N}} F^+(t_{i_n})(g(u_{i_n}) - g(u_{(i-1)_n})) - \beta_2)\|_{L^p} < \varepsilon.$$

Let $\beta = [\beta_1, \beta_2]$. If $F^- \leqslant F^+$, then $\beta_1 \leqslant \beta_2$. We define $\delta_n(x) = \min(\delta_n^1(x), \delta_n^2(x))$, then for any $\delta_n(x) - fine$ tagged partitions P_n we have

$$\|\sum_{i=1}^{m_n\in\mathbb{N}} F(t_{i_n})(g(u_{i_n}) - g(u_{(i-1)_n})) - \beta\|_{L^p} < \varepsilon.$$

Hence, $F : [a,b] \to L^p$ is Sequential Henstock Stieltjes integrable on $[a,b]$. □

THEOREM 2.3. *Let $F, K \in L^p[0,1]$-$ISHS[a,b]$ with $F = [F^-, F^+]$ and $K = [K^-, K^+]$ and $\gamma, \xi \in \mathbb{R}$. Then $\gamma F, \xi K \in L^p[0,1]$-$ISHS[a,b]$ and*

$$L^p[0,1]\text{-}ISHS)\int_{[a,b]}(\gamma F + \xi K)dg$$
$$= \gamma(L^p[0,1]\text{-}ISHS)\int_{[a,b]} Fdg + \xi(L^p[0,1]\text{-}ISHS)\int_{[a,b]} Kdg$$

PROOF. (i) If $F, K \in L^p[0,1]$-$ISHS[a,b]$, then $[F^-, F^+], K = [K^-, K^+] \in L^p[0,1]$-$SHS[a,b]$ by Theorem 3.2. Hence, $\gamma F^- + \xi K^-, \gamma F^- + \xi K^+, \gamma F^+ + \xi K^-, \gamma F^+ + \xi K^+ \in L^p[0,1]$-$SHS[a,b]$.
1) If $\gamma > 0$ and $\xi > 0$, then

$$(L^p[0,1]\text{-}SHS)\int_{[a,b]}(\gamma F + \xi K)^- dg$$

$$
\begin{aligned}
&= (L^p[0,1]\text{-}SHS)\int_{[a,b]}(\gamma F^- + \xi K^-)dg \\
&= \gamma(L^p[0,1]\text{-}SHS)\int_{[a,b]}F^- dg + \xi(L^p[0,1]\text{-}SHS)\int_{[a,b]}K^- dg \\
&= \gamma((L^p[0,1]\text{-}ISHS)\int_{[a,b]}Fdg)^- + \xi((L^p[0,1]\text{-}ISHS)\int_{[a,b]}Kdg)^- \\
&= (\gamma(L^p[0,1]\text{-}ISHS)\int_{[a,b]}Fdg + \xi(L^p[0,1]\text{-}ISHS)\int_{[a,b]}Kdg)^-.
\end{aligned}
$$

2) If $\gamma < 0$ and $\xi > 0$, then

$$
(L^p[0,1]\text{-}SHS)\int_{[a,b]}(\gamma F + \xi K)^- dg
$$

$$
\begin{aligned}
&= (L^p[0,1]\text{-}SHS)\int_{[a,b]}(\gamma F^+ + \xi K^+)dg \\
&= \gamma(L^p[0,1]\text{-}SHS)\int_{[a,b]}F^+ dg + \xi(L^p[0,1]\text{-}SHS)\int_{[a,b]}K^+ dg \\
&= \gamma((L^p[0,1]\text{-}ISHS)\int_{[a,b]}Fdg)^+ + \xi((L^p[0,1]\text{-}ISHS)\int_{[a,b]}Kdg)^+ \\
&= (\gamma(L^p[0,1]\text{-}ISHS)\int_{[a,b]}Fdg + \xi(L^p[0,1]\text{-}ISHS)\int_{[a,b]}Kdg)^-.
\end{aligned}
$$

3) If $\gamma > 0$ and $\xi < 0$ (or $\gamma < 0$ and $\xi > 0$), then

$$
(L^p[0,1]\text{-}ISHS)\int_{[a,b]}(\gamma F + \xi K)^- dg
$$

$$
\begin{aligned}
&= (L^p[0,1]\text{-}SHS)\int_{[a,b]}(\gamma F^- + \xi K^+)dg \\
&= \gamma(L^p[0,1]\text{-}SHS)\int_{[a,b]}F^- dg + \xi(L^p[0,1]\text{-}SHS)\int_{[a,b]}K^+ dg \\
&= \gamma((L^p[0,1]\text{-}ISHS)\int_{[a,b]}Fdg)^- + \xi((L^p[0,1]\text{-}ISHS)\int_{[a,b]}Kdg)^+ \\
&= (\gamma(L^p[0,1]\text{-}ISHS)\int_{[a,b]}Fdg + \xi(L^p[0,1]\text{-}ISHS)\int_{[a,b]}Kdg)^-.
\end{aligned}
$$

Similarly, for four cases above, we have

$$
\begin{aligned}
&(L^p[0,1]\text{-}ISHS)\int_{[a,b]}(\gamma F + \xi K)^+ dg \\
&= (\gamma(L^p[0,1]\text{-}ISHS)\int_{[a,b]}Fdg + \xi(L^p[0,1]\text{-}ISHS)\int_{[a,b]}Kdg)^+
\end{aligned}
$$

Hence, by Theorem 2.2, $\gamma F, \xi K \in L^p[0,1]$-$ISHS[a,b]$ and

$$(L^p[0,1]\text{-}ISHS)\int_{[a,b]}(\gamma F+\xi K)dg$$
$$=\gamma(L^p[0,1]\text{-}ISHS)\int_{[a,b]}Fdg+\xi(L^p[0,1]\text{-}ISHS)\int_{[a,b]}Kdg.$$

□

THEOREM 2.4. *Let $F, K \in L^p[0,1]$-$ISHS[a,b]$ and $F(x) \leqslant K(x)$ nearly everywhere on $[a,b]$, then*

$$(L^p[0,1]\text{-}ISHS)\int_{[a,b]}F(x)dg \leqslant (L^p[0,1]\text{-}ISHS)\int_{[a,b]}Kdg$$

PROOF. If $F(x) \leqslant K(x)$ nearly everywhere on $[a,b]$ and $F, K \in L^p[0,1]$-$ISHS[a,b]$, then $F^-, F^+, K^-, K^+ \in L^p[0,1]$-$SH[a,b]$ and $F^- \leqslant F^+, K^- \leqslant K^+$ nearly everywhere on $[a,b]$. By Lemma 2.5

$$(L^p[0,1]\text{-}SHS)\int_{[a,b]}F^-(x)dg \leqslant (L^p[0,1]\text{-}SHS)\int_{[a,b]}K^-dg$$

and

$$(L^p[0,1]\text{-}ISHS)\int_{[a,b]}F^+(x)dg \leqslant (L^p[0,1]\text{-}ISHS)\int_{[a,b]}K^+dg.$$

Hence by Theorem 2.2, we have

$$(L^p[0,1]\text{-}ISHS)\int_{[a,b]}F(x)dg \leqslant (L^p[0,1]\text{-}ISHS)\int_{[a,b]}Kdg.$$

□

THEOREM 2.5. *Let $k \in \mathbb{R}$.*
1. If $F \in L^p[0,1]$-$ISHS[a,b]$, then $kF \in L^p[0,1]$-$ISHS[a,b]$. Moreover,

$$\int_a^b kF = k\int_a^b Fdg.$$

2. If $F \in L^p[0,1]$-$ISHS[a,b]$ and $G \in L^p[0,1]$-$ISHS[c,b]$, then $(F+G) \in L^p[0,1]$-$ISHS[a,b]$. Moreover

$$\int_a^b (F+G) = \int_a^b Fdg + \int_a^b Gdg.$$

PROOF. (1) Suppose $F \in L^p[0,1]$-$ISHS[a,b]$. The case $k=0$ is obvious. Suppose $k \neq 0$ and $F \in L^p[0,1]$-$ISHS[a,b]$, there exists a sequence of positive functions$\{\delta_n(x)\}_{n=1}^{\infty}$ on $[a,b]$ such that

$$\| \sum_{i=1}^{m_n\in\mathbb{N}} F(t_{i_n})(g(u_{i_n})-g(u_{(i-1)_n})) - \int_a^b F\|_{L^p} < \frac{\varepsilon}{|k|_{L^p}}$$

whenever P_n is $\delta_n(x) - fine$ tagged partitions of $[a,b]$. Then, exists a sequence of positive functions $\{\delta_n^2(x)\}_{n=1}^{\infty}$ on $[a,c]$ such that

$$\begin{aligned}
&\|\sum_{i=1}^{m_n\in\mathbb{N}} kF(t_{i_n})(g(u_{i_n}) - g(u_{(i-1)_n})) - k\int_a^b Fdg\|_{L^p} \\
= \ & \|k\sum_{i=1}^{m_n\in\mathbb{N}} F(t_{i_n})(g(u_{i_n}) - g(u_{(i-1)_n})) - k\int_a^b Fdg\|_{L^p} \\
< \ & |k|_{L^p}\frac{\varepsilon}{|k|_{L^p}} \\
= \ & \varepsilon.
\end{aligned}$$

(2) Let $\varepsilon > 0$ Suppose $\int_a^b Fdg = \alpha_1$ and $\int_a^b Gdg = \alpha_2$. Then there exists a sequence of positive functions$\{\delta_n^1(x)\}_{n=1}^{\infty}$ on $[a,b]$ such that

$$\|\sum_{i=1}^{m_n\in\mathbb{N}} F(t_{i_n})(g(u_{i_n}) - g(u_{(i-1)_n})) - \alpha_1\|_{L^p} < \frac{\varepsilon}{2(2^{\frac{1}{p}})}$$

whenever P_n^1 is $\delta_n^1(x) - fine$ tagged partitions of $[a,b]$. Also, there exists a sequence of positive functions $\{\delta_n^2(x)\}_{n=1}^{\infty}$ on $[a,b]$ such that

$$\|\sum_{i=1}^{m_n\in\mathbb{N}} G(t_{i_n})(g(u_{i_n}) - g(u_{(i-1)_n})) - \alpha_2\|_{L^p} < \frac{\varepsilon}{2(2^{\frac{1}{p}})}$$

whenever P_n^2 is $\delta_n^2(x) - fine$ tagged partitions of $[a,b]$.
Define a positive gauge function $\delta_n(x)$ on $[a,b]$ by $\delta_n(x) = \min\{\delta_n^1(x), \delta_n^2(x)\}$. Let P_n be any $\delta_n(x)$-fine tagged partition of $[a,b]$. Then

$$\begin{aligned}
&\|\sum_{i=1}^{m_n\in\mathbb{N}} (F+G)(t_{i_n})(g(u_{i_n}) - g(u_{(i-1)_n})) - (\alpha_1+\alpha_2)\|_{L^p} = \\
&= (\|\sum_{i=1}^{m_n\in\mathbb{N}} F(t_{i_n})(g(u_{i_n}) - g(u_{(i-1)_n})) \\
&+ \sum_{i=1}^{m_n\in\mathbb{N}} G(t_{i_n})(g(u_{i_n}) - g(u_{(i-1)_n})) - (\alpha_1+\alpha_2)\|_{L^p}) \\
&\leqslant 2^{\frac{1}{p}}(\|\sum_{i=1}^{m_n\in\mathbb{N}} F(t_{i_n})(g(u_{i_n}) - g(u_{(i-1)_n})) - \alpha_1\|_{L^p}) \\
&+ 2^{\frac{1}{p}}(\|\sum_{i=1}^{m_n\in\mathbb{N}} G(t_{i_n})(g(u_{i_n}) - g(u_{(i-1)_n})) - \alpha_2\|_{L^p}) \\
&< 2^{\frac{1}{p}}(\frac{\varepsilon}{2(2\frac{1}{p})} + \frac{\varepsilon}{2(2\frac{1}{p})}) = \varepsilon.
\end{aligned}$$

□

References

1. A. Boccuto and A.Ṙ. Sambucini, The Henstock-Kurzweil Integral For Functions Defined on Unbounded Interval with Values in Banach Spaces. *Acta mathematica*, **7**(2004), 3–17.
2. S.Ṡ. Cao, The Henstock Integral For Banach Valued Functions. *SEA Bull. Math.,* 16, No. **1**, (1992), 35-40.
3. R. Gordon, The Integral of Lebesgues, Denjoy, Perron and Henstock. Graduate studies in Mathematics. *American Mathematical Society. Providence, RI,* **4**, 1994.
4. M. E Hamid and A. H, Elmuiz, On Henstock-Stieltjes Integrals of interval-Valued Functions and Fuzzy-Number-Valued Functions. *Journal of Applied Mathematics and Physics,* **4**. (2016), 779-786.
5. R. Henstock, The General Theory of Integration. *Oxford University Press, Oxford, UK,* 1991.
6. V. O. Iluebe and A. A. Mogbademu, Equivalence Of Henstock And Certain Sequential Henstock Integral. *Bangmond International Journal of Mathematical and Computational Science.,* **1**.(1 and 2), (2020), 9–16.
7. J. M. R. Macalalag and R. N. Paluga, On The Henstock-Stieltjes Integral For l_p Valued Functions, $0 < p < 1$. *Annal of Studies in Science and Humanities.* **1**, (2015), 25-34.
8. R. E. Moore, R. B. Kearfott, and J. C. Michael, Introduction to Interval Analysis. *Society for Industrial and Applied Mathematics*, 2009.
9. L. A. Paxton, Sequential Approach to the Henstock Integral. *Washington State University, arXiv:1609.05454v1* [maths.CA],(2016), 9-13.
10. C.X. Wu and Z.T. Gong, On Henstock Integrals of interval-Valued Functions and Fuzzy-Number-Valued Functions. *Fuzzy Set and Systems.* **115**, (2016), 377-391.
11. J.H. Yoon, On Henstock-Stieltjes Integrals of interval-Valued Functions On Time Scale. *Journal of the Chungcheong Mathematical Society*,**29**, (2016), 109-115.

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