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E-SUPER ARITHMETIC GRACEFUL LABELLING OF SOME STANDARD CLASSES OF CUBIC GRAPHS RELATED TO CYCLES

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Abstract. We introduced a new concept called E-super arithmetic graceful labelling of graphs. A (p, q)-graph G is said to be ***E-super arithmetic graceful*** if there exists a bijection f from $V(G) \cup E(G)$ to $\{1, 2, ..., p+q\}$ such that $f(E(G)) = \{1, 2, ..., q\}$, $f(V(G)) = \{q+1, q+2, ..., q+p\}$ and the induced mapping f^* given by $f^*(uv) = f(u) + f(v) - f(uv)$ for $uv \in E(G)$ has the range $\{p+q+1, p+q+2, ..., p+2q\}$.

In this paper we prove that the complete graph,flower snarks and its related graphs, the cubic graphs $F^{(3)}(C_n)$, generalised Petersen graphs $P(n, 2)$, the Petersen graph which has chromatic number 3, Desargues graph and Heawood graph are E-super arithmetic graceful.

1. Introduction

Rosa [**9**] in 1967, called a function f a β - valuation of a graph G with q edges if f is an injection from the vertices of G to the set $\{0, 1, ..., q\}$ such that when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. Golomb[**3**] subsequently called such labelling graceful. Acharya and Hedge [**1**] have defined $(k, d)-$ arithmetic graphs. Let G be a graph with q edges and let k and d be positive integers.A labelling f of G is said to be $(k, d)-$ arithmetic if the vertex labels are distinct nonnegative integers and the edge labels induced by $f(x) + f(y)$ for each edge xy are $k, k+d, k+2d, ..., k+(q-1)d$. The case where $k = 1$ and $d = 1$ was called additively graceful by Hedge [**4**].

Joseph A. Gallian [**2**] surveyed numerous graph labelling methods.

2010 *Mathematics Subject Classification.* Primary 05C78.

Key words and phrases. E-super, complete graph, flower snarks, $F^{(3)}(C_n)$, generalised Petersen graphs, the Petersen graph with chromatic number 3, Desargues graph, Heawood graph.

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V. Ramachandran and C.Sekar [**8**] introduced $(1, N)$-arithmetic labelling.

In 1970 Kotzig and Rosa [**5**] defined a magic valuation of a graph $G(V, E)$ as a bijection f from $V \cup E$ to $\{1, 2, ..., |V \cup E|\}$ such that for all edges xy, $f(x) + f(y) + f(xy)$ is constant. Ringel and Llado in 1996 called this labelling edge - magic. If the vertex labels are 1 to $|V|$, it is called Super edge - magic total labelling.

MacDougall, Slamin, Miller and Wallis [**6**] introduced the notion of a vertex-magic total labelling in 1999. For a graph $G(V, E)$ an injective mapping f from $V \cup E$ to the set $\{1, 2, .., |V| + |E|\}$ is a vertex - magic total labeling if there is a constant k, called the magic constant such that for every vertex v, $f(u) + \Sigma f(vu) = k$ where the sum is taken over all vertices u adjacent to v.

A vertex magic total labelling of $G(V, E)$ is said to be E-super if $f(E(G)) = \{1, 2, 3, ...|E(G)|\}$.

A labelling of $G(V, E)$ is said to be E-super if $f(E(G)) = \{1, 2, 3, ...|E(G)|\}$.

Marimuthu and Balakrishnan [**7**] defined a graph $G(V, E)$ to be edge magic graceful if there exists a bijection f from $V(G) \cup E(G)$ to $\{1, 2, ..., p + q\}$ such that $|f(u) + f(v) - f(uv)|$ is a constant for all edges uv of G.
We introduced a new concept called E-super arithmetic graceful labelling of graphs [**10**]. We define a graph $G(p, q)$ to be ***E-super arithmetic graceful*** if there exists a bijection f from $V(G) \cup E(G)$ to $\{1, 2, ..., p + q\}$ such that $f(E(G)) = \{1, 2, ..., q\}$, $f(V(G)) = \{q + 1, q + 2, ..., q + p\}$ and the induced mapping f^* given by $f^*(uv) = f(u) + f(v) - f(uv)$ for $uv \in E(G)$ has the range $\{p + q + 1, p + q + 2, ..., p + 2q\}$.
In the field of graph theory, the flower snarks form an infinite family of snarks introduced by Rufao Issac in 1975. As snarks, the flower snarks are connected bridgeless cubic graphs with chromatic index equal to 4. The flower snarks are nonplanar and non-hamiltonian

In this paper we prove that the complete graphs, flower snarks and its related graphs, the cubic graphs $F^{(3)}(C_n)$, generalised Petersen graphs $P(n, 2)$, the Petersen graph which has chromatic number 3, Desargues graph and the Heawood graphs are E-super arithmetic graceful.

2. Preliminaries

Definition 2.1. For $n = 3$, and even $n \geqslant 4$, the graph related to flower snark, denoted by denoted by F_n is a cubical graph with vertex set

$$V(F_n) = \{a_i, i = 0, 1, 2, \ldots, n-1\} \cup \{b_i,\ i = 0, 1, \ldots, n-1\} \cup \{c_i,\ i = 0, 1, 2, \ldots, 2n-1\}$$

and edge set

$$E(F_n) = \left\{a_i a_{i+1(\mod n)},\ 0 \leqslant i \leqslant n-1\right\} \cup \{a_i b_i, 0 \leqslant i \leqslant n-1\} \cup \{b_i c_i,\ 0 \leqslant i \leqslant n-1\} \cup \{b_i c_{n+i},\ 0 \leqslant i \leqslant n-1\} \cup \left\{c_i c_{i+1(\mod 2n)}, 0 \leqslant i \leqslant 2n-1\right\}$$

F_n has $4n$ vertices and $6n$ edges. For odd $n \geqslant 5$ similar graphs are called flower snarks.

F_5 :

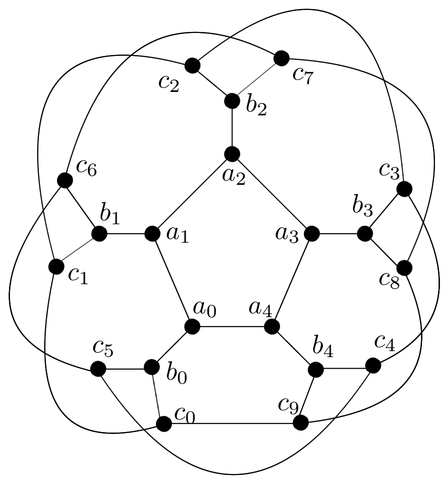

Fig 2.1

Definition 2.2. $F^{(3)}(C_n)$ for $n \geqslant 3$ denotes a cubic graph with vertex set $V = \{a_i,\ 0 \leqslant i \leqslant n-1\} \cup \{b_i,\ 0 \leqslant i \leqslant n-1\} \cup \{c_i,\ 0 \leqslant i \leqslant 2n-1\}$, and edge set

$$\begin{aligned} E = &\left\{a_i a_{i+1(\mod n)},\ 0 \leqslant i \leqslant n-1\right\} \cup \{a_i b_i,\ 0 \leqslant i \leqslant n-1\} \\ &\cup \{b_i c_i, 0 \leqslant i \leqslant n-1\} \cup \{b_i c_{n+i}, 0 \leqslant i \leqslant n-1\} \\ &\cup \left\{c_i c_{i+1(\mod n)}, 0 \leqslant i \leqslant n-1\right\} \cup \{c_{n+i} c_{n+i+1}, 0 \leqslant i \leqslant n-2\} \cup \{c_{2n-1} c_n\} \end{aligned}$$

$F^{(3)}(C_n)$ has $4n$ vertices and $6n$ edges.
$F^{(3)}(C_4)$:

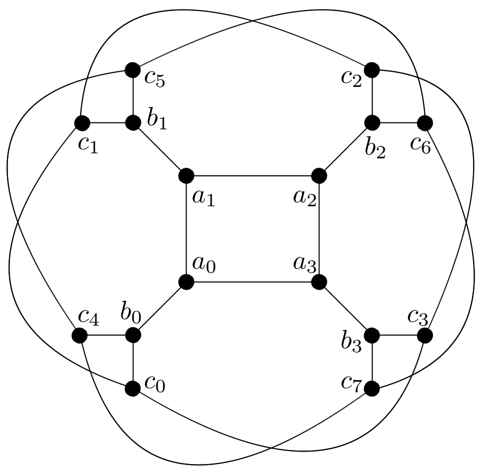

Fig 2.2

DEFINITION 2.3. The *generalized Peterson graph* $P(n,k)$, $n \geqslant 5$, $k \geqslant 2$ is the graph with vertex set $\{u_1, u_2, \dots u_n\} \cup \{v_1, v_2, \dots v_n\}$ and edge set

$$\{u_i u_{i+1} \mid i = 1, 2, \dots, n \quad where \quad u_{n+1} = u_1\} \cup \{u_i v_i \mid i = 1, 2, \dots, n\}$$
$$\cup \{v_i v_{i+k} \mid i = 1, 2, \dots, n \quad where \quad v_{n+j} = v_j\}$$

The usual *Petersen graph* is $P(5,2)$.
$P(n,2)$ for all $n \geqslant 5$ is a cubic graph related to cycle C_n.
Generalised Petersen graph $P(7,2)$:

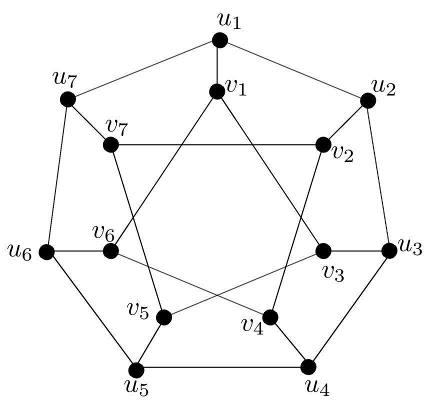

Fig 2.3

DEFINITION 2.4. Let G be the graph having vertices $u_0, u_1, u_2, \dots u_9$ and edge set $\{u_0u_1, u_0u_4, u_0u_7\} \cup \{u_iu_{i+1} \mid i = 1, 2, \dots 9 \text{ where } u_{10} = u_1\} \cup \{u_2u_6, u_3u_8, u_5u_9\}$. This graph G is a cubic graph called *Petersen graph which has chromatic number 3* which is given in the adjoined figure.

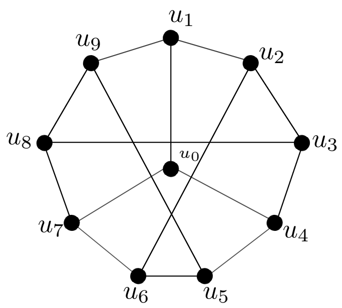

Fig 2.4

Definition 2.5. The Desargues graph is a distance transitive cubic graph with 20 vertices and 30 edges. The graph is as shown in the figure.

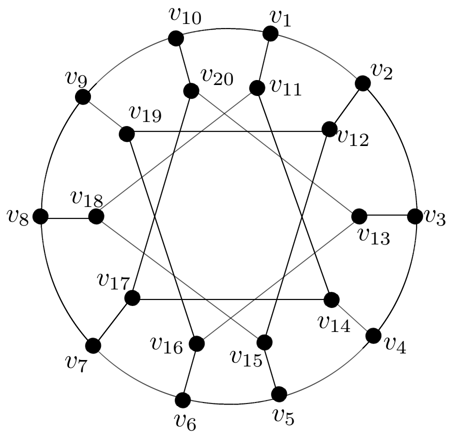

Fig 2.5

Definition 2.6. Heawood graph is a cubic graph with 14 vertices and 21 edges as given in the adjoined figure.

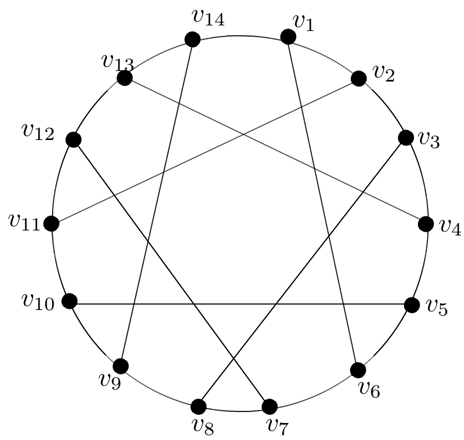

Fig 2.6

3. Main results

Theorem 3.1. *The graphs F_n for $n = 3$ and even $n \geqslant 4$ related to flower snarks and the flower snarks F_n for odd $n \geqslant 5$ are E-super arithmetic graceful.*

Proof. We give a common labelling for the graphs related to flower snarks and flower snarks.

Consider F_n, $n \geqslant 3$.
Let $\{a_i,\ \ i = 0, 1, 2, \ldots, n-1\} \cup \{b_i,\ \ i = 0, 1, \ldots, n-1\} \cup \{c_i,\ \ i = 0, 1, 2, \ldots, 2n-1\}$ be the vertices of F_n.
F_n has $4n$ vertices and $6n$ edges.
Define $f : V(F_n) \cup E(F_n) \longrightarrow \{1, 2, .., 10n\}$ as follows:
$f(a_i) = 6n + 1 + i, \quad for \quad i = 0, 1, ..., n-1$
$f(b_i) = 8n - i, \quad for \quad i = 0, 1, ..., n-1$
$f(c_i) = 9n + 1 + i, \quad for \quad i = 0, 1, ..., n-1$
$f(c_{n+i}) = 8n + 2 + i, \quad for \quad i = 0, 1, ..., n-2$
$f(c_{2n-1}) = 8n + 1.$
$f(a_i a_{i+1(\mod n)}) = n + 1 + i, \quad for \quad i = 0, 1, ..., n-1$
$f(a_i b_i) = 3n + 1 + i, \quad for \quad i = 0, 1, ..., n-1$
$f(b_i c_i) = 2n + 1 + i, \quad for \quad i = 0, 1, ..., n-1$
$f(b_i c_{n+i}) = 2 + i, \quad for \quad i = 0, 1, ..., n-2$
$f(b_{n-1} c_{2n-1}) = 1.$
$f(c_0 c_{2n-1}) = 5n + 1.$
$f(c_i c_{i+1}) = 5n + 2 + i, \quad for \quad i = 0, 1, 2, ...n-2$
$f(c_{n-1} c_n) = 4n + 2$
$f(c_{n+i} c_{n+i+1}) = 4n + 3 + i, \quad for \quad i = 0, 1, 2, ...n-3$
$f(c_{2n-2} c_{2n-1}) = 4n + 1.$
Clearly f is a bijection.
$f(E(F_n)) = \{1, 2, \ldots, 6n\}$.

$$\begin{aligned}\left\{f^*(a_i a_{i+1(\mod n)}) \mid 0 \leqslant i \leqslant n-1\right\} &= \{11n + 1 + i \mid 0 \leqslant i \leqslant n-1\}\\ &= \{11n+1, 11n+2, \ldots, 12n\}\end{aligned}$$

$\{f^*(a_i b_i) | 0 \leqslant i \leqslant n-1\} = \{11n - i \mid 0 \leqslant i \leqslant n-1\} = \{10n+1, 10n+2, \ldots, 11n\}$

$\{f^*(b_i c_i) \mid 0 \leqslant i \leqslant n-1\} = \{15n + i \mid 0 \leqslant i \leqslant n-1\} = \{14n+1, 14n+2, \ldots, 15n\}$
$\{f^*(b_i c_{n+i}) \mid 0 \leqslant i \leqslant n-2\} = \{16n - i \mid 0 \leqslant i \leqslant n-2\} = \{15n+2, 15n+3, \ldots, 16n\}$
$f^*(b_{n-1} c_{2n-1}) = 15n + 1$

$$\begin{aligned}\{f^*(c_i c_{i+1}) \mid 0 \leqslant i \leqslant n-2\} &= \{13n + 1 + i \mid 0 \leqslant i \leqslant n-2\}\\ &= \{13n+1, 13n+2, \ldots, 14n-1\}\end{aligned}$$

$f^*(c_{n-1} c_n) = 14n$

$$\begin{aligned}\{f^*(c_{n+i} c_{n+i+1}) \mid 0 \leqslant i \leqslant n-3\} &= \{12n + 2 + i \mid 0 \leqslant i \leqslant n-3\}\\ &= \{12n+2, 12n+3, \ldots, 13n-1\}\end{aligned}$$

$f^*(c_{2n-2} c_{2n-1}) = 13n$
$f^*(c_0 c_{2n-1}) = 12n + 1$
Therefore $f^*(E(F_n)) = \{10n+1, 10n+2, \ldots, 16n\}$.
Thus F_n is E-super arithmetic graceful for all $n \geqslant 3$. □

EXAMPLE 3.1. E-super arithmetic graceful labelling of flower snark F_7.

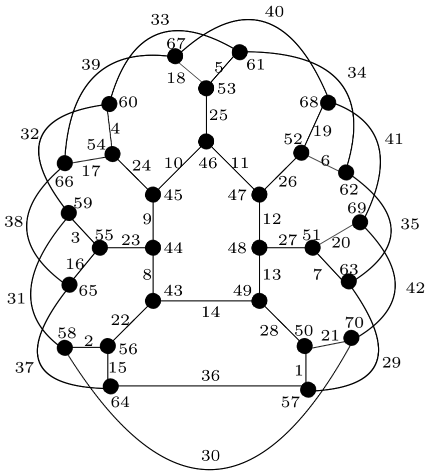

Fig 3.1

EXAMPLE 3.2. E-super arithmetic graceful labelling of the related graph F_4

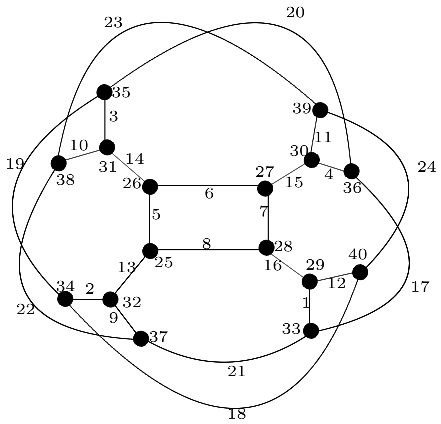

Fig 3.2

THEOREM 3.2. *$F^{(3)}(C_n)$ is E-super arithmetic graceful for all $n \geqslant 3$.*

PROOF. : Let $\{a_i, 0 \leqslant i \leqslant n-1\} \cup \{b_i, 0 \leqslant i \leqslant n-1\} \cup \{c_i, 0 \leqslant i \leqslant 2n-1\}$ be the vertices of $F^{(3)}(C_n)$.
$F^{(3)}(C_n)$ has $4n$ vertices and $6n$ edges.
Define $f : V \cup E \longrightarrow \{1, 2, 3, ..., 10n\}$ as follows:
$f(a_i) = 8n - i, \qquad i = 0, 1, ..., n-1$

$f(b_i) = 6n + 1 + i, \qquad i = 0, 1, ..., n - 1$
$f(c_i) = 9n + 1 + i, \qquad i = 0, 1, ..., n - 1$
$f(c_{n+i}) = 8n + 1 + i, \qquad i = 0, 1, ..., n - 1$
$f(a_i a_{i+1}) = 5n - 1 - i, \qquad i = 0, 1, 2, .., n - 2$
$f(a_{n-1} a_0) = 5n.$
$f(a_i b_i) = n + 1 + i, \qquad i = 0, 1, ..., n - 1.$
$f(b_i c_i) = 2n + 1 + i, \qquad i = 0, 1, ..., n - 1.$
$f(b_i c_{n+i}) = 1 + i, \qquad i = 0, 1, ..., n - 1.$
$f(c_i c_{i+1(\mod n)}) = 3n + 1 + i \qquad i = 0, 1, ..., n - 1.$
$f(c_{n+i} c_{n+i+1}) = 5n + 2 + i \qquad i = 0, 1, ..., n - 2.$
$f(c_{2n-1} c_n) = 5n + 1$
Clearly f is a bijection.
$f(E(F^{(3)}(C_n))) = \{1, 2, \ldots, 6n\}$
$\{f^*(a_i a_{i+1}) \mid 0 \leqslant i \leqslant n-2\} = \{11n-i \mid 0 \leqslant i \leqslant n-2\} = \{10n+2, 10n+3, \ldots, 11n\}$
$f^*(a_{n-1} a_0 = 10n + 1)$
$\{f^*(a_i b_i) \mid 0 \leqslant i \leqslant n - 1\} = \{13n - i \mid 0 \leqslant i \leqslant n - 1\} = \{12n + 1, 12n + 2, \ldots, 13n\}$
$\{f^*(b_i c_i) | 0 \leqslant i \leqslant n-1\} = \{14n+1+i \mid 0 \leqslant i \leqslant n-1\} = \{14n+1, 14n+2, \ldots, 15n\}$
$\{f^*(c_i c_{i+1(\mod n)}) \mid 0 \leqslant i \leqslant n - 1\} = \{15n + 1, 15n + 2, \ldots, 16n\}$

$$\begin{aligned}\{f^*(c_{n+i} c_{n+i+1}) \mid 0 \leqslant i \leqslant n - 2\} &= \{11n + 1 + i \mid 0 \leqslant i \leqslant n - 2\} \\ &= \{11n + 1, 11n + 2, \ldots, 12n - 1\}\end{aligned}$$

$f^*(c_{2n-1} c_n) = 12n$
Therefore $f^*(E(F^{(3)}(C_n))) = \{10n + 1, 10n + 2, ..., 16n\}$.
Thus $F^{(3)}(C_n)$ is E-super arithmetic graceful. □

EXAMPLE 3.3. E-super arithmetic graceful labelling of $F^{(3)}(C_5)$.

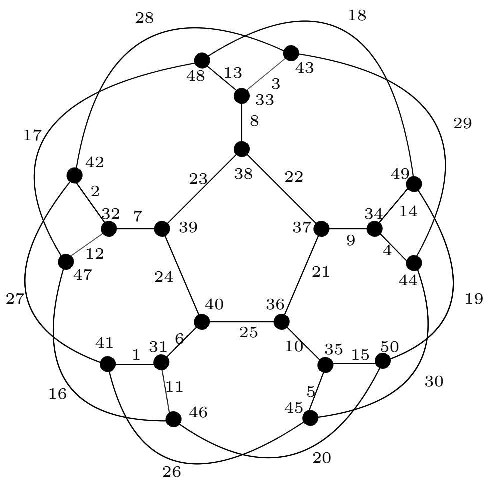

Fig 3.3

THEOREM 3.3. *$P(n,2)$ for $n \geqslant 5$ is E-super arithmetic graceful.*

PROOF. The generalized Petersen graph $P(n,2)$ for $n \geqslant 5$ has $2n$ vertices and $3n$ edges.
Define $f : V \cup E \longrightarrow \{1,2,\ldots,5n\}$ as follows:
$f(u_i) = 3n + i, \quad i = 1,2,\ldots,n$
$f(v_i) = 4n + i, \quad i = 1,2,\ldots,n$
$f(u_iu_{i+1}) = n + i, \quad i = 1,2,\ldots,n$ where $u_{n+1} = u_1$
$f(u_iv_i) = i, \quad i = 1,2,\ldots,n$
$f(v_iv_{i+2}) = 2n + i, \quad i = 1,2,\ldots,n$ where $v_{n+1} = v_1$ and $v_{n+2} = v_2$
Clearly f is a bijection.
$f(E(P(n,2))) = \{1,2,\ldots,3n\}$
$f^*(E(P(n,2))) = \{5n+1, 5n+2, \ldots 8n\}$

$$\begin{aligned}\{f^*(u_iu_{i+1}) \mid i = 1,2,\ldots,n-1\} &= \{5n+1+i \mid i = 1,2,\ldots,n-1\}\\ &= \{5n+2, 5n+3,\ldots,6n\}\end{aligned}$$

$f^*(u_nu_1) = 5n + 1$
$\{f^*(u_iv_i) \mid i = 1,2,\ldots,n\} = \{7n + i \mid i = 1,2,\ldots,n\} = \{7n+1, 7n+2,\ldots,8n\}$

$$\begin{aligned}\{f^*(v_iv_{i+2}) \mid ; i = 1,2,\ldots,n-2\} &= \{6n+i+2 \mid i = 1,2,\ldots,n-2\}\\ &= \{6n+3, 6n+4,\ldots,7n\}\end{aligned}$$

$f^*(v_{n-1}v_1) = 6n + 1$
$f^*(v_nv_2) = 6n + 2$
Therefore $P(n,2)$ for $n \geqslant 5$ is E-super arithmetic graceful for all $n \geqslant 5$.

□

EXAMPLE 3.4. E-super arithmetic graceful labelling of $P(7,2)$.

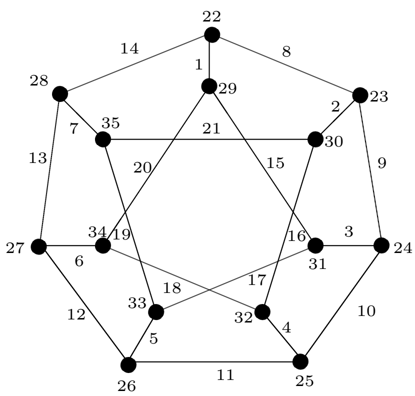

Fig 3.4

A particular labelling:

E-super arithmetic graceful labelling of the Petersen graph which has chromatic number 3 is given below:

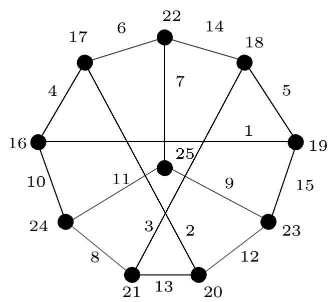

Fig 3.5

THEOREM 3.4. *The Desargues graph is E-super arithmetic graceful.*

PROOF. Let G be the Desargues graph with 20 vertices and 30 edges. Let $V(G) = \{v_1, v_2, \ldots, v_{20}\}$. The edge set

$$E(G) = \{v_i v_{i+1} \mid 1 \leqslant i \leqslant 9\} \cup \{v_1 v_{10}\} \cup \{v_i v_{i+10} \mid 1 \leqslant i \leqslant 10\} \cup \{v_i v_{i+3} \mid 11 \leqslant i \leqslant 17\} \cup \{v_i v_{i+7} \mid 11 \leqslant i \leqslant 13\}$$

Define $f : V(G) \cup E(G) \longrightarrow \{1, 2, \ldots, 50\}$ as given below.
$f(v_i) = 30 + i, \quad i = 1, 2, \ldots, 20$
$f(v_i v_{i+1}) = 10 + i, \quad i = 1, 2, \ldots, 9$
$f(v_1 v_{10}) = 20$
$f(v_i v_{i+10}) = i, \quad i = 1, 2, \ldots, 10$
$f(v_i v_{i+3}) = 10 + i, \quad 11 \leqslant i \leqslant 17$
$f(v_i v_{i+7}) = 17 + i, \quad 11 \leqslant i \leqslant 13$
Clearly $f(E(G)) = \{1, 2, \ldots, 30\}$
$f(V(G)) = \{31, 32, \ldots, 50\}$
$f^*(E(G)) = \{51, 52, \ldots, 80\}$.
Therefore G is E-super arithmetic graceful. □

EXAMPLE 3.5. E-super arithmetic graceful labelling of Desargues Graph.

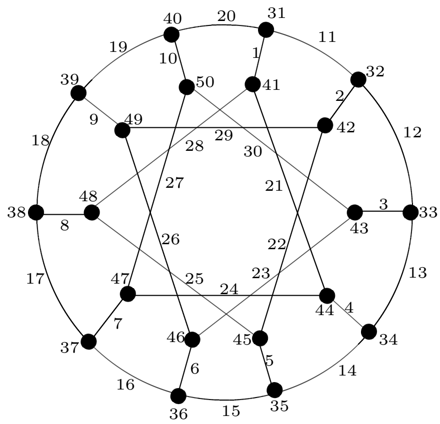

Fig 3.6

THEOREM 3.5. *Heawood graph is E-super arithmetic graceful.*

PROOF. Let G be the Heawood graph with 14 vertices and 21 edges. Let $V(G) = \{v_1, v_2, \ldots, v_{14}\}$. The edge set

$$E(G) = \{v_i v_{i+1} \mid i = 1, 2, \ldots, 13\} \cup \{v_1 v_{14}\} \cup \{v_i v_{i+5} \mid i = 1, 3, , 5, 7, 9\} \cup \{v_2 v_{11}, v_4 v_{13}\}$$

Define $f : V(G) \cup E(G) \to \{1, 2, \ldots, 35\}$ as follows:

$f(v_i) = 22 + i, \quad i = 1, 2, \ldots, 13$
$f(v_{14}) = 22$
$f(v_i v_{i+1}) = 2 + i, \quad i = 1, 2, \ldots, 12$
$f(v_{13} v_{14}) = 1$
$f(v_1 v_{14}) = 2$
$f(v_i v_{i+5}) = 14 + i, \quad i = 1, 3, 5, 7$
$f(v_9 v_{14}) = 16$
$f(v_2 v_{11}) = 18$
$f(v_4 v_{13}) = 20$
Clearly $f(E(G)) = \{1, 2, \ldots, 21\}$ and $f(V(G)) = \{22, 23, \ldots, 35\}$.
$f^*(E(G)) = \{36, 37, \ldots, 56\}$
Therefore G is E-super arithmetic graceful. □

EXAMPLE 3.6. E-super arithmetic graceful labelling of Heawood graph.

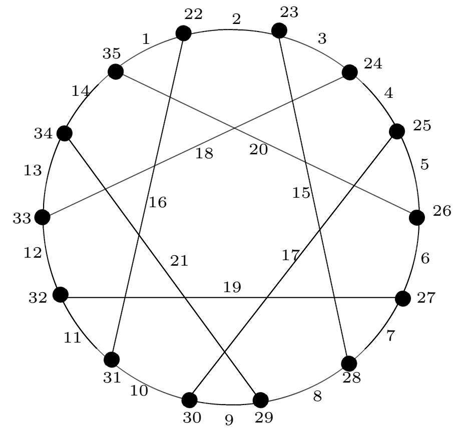

Fig 3.7

REMARK 3.1. The complete graph K_4, a cubic graph related to cycle is also E-super arithmetic graceful. It is shown as a particular case in the following generalised result.

THEOREM 3.6. *Complete graphs K_n, $n \geqslant 3$ are E-super arithmetic graceful.*

PROOF. K_n has n vertices and $\frac{n(n-1)}{2}$ edges.

Case:(i) $n = 3$.
Let u_1, u_2, u_3 be the vertices of K_3.
E-super arithmetic graceful labelling of K_3 is given below.

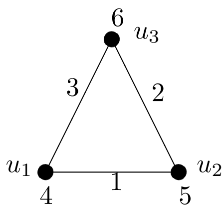

Fig 3.8

Clearly $f^*(E(K_3)) = \{7, 8, 9\}$

Case:(ii) $n = 4$.
Let u_1, u_2, u_3, u_4 be the vertices of K_4.
E-super arithmetic graceful labelling of K_4 is given below.

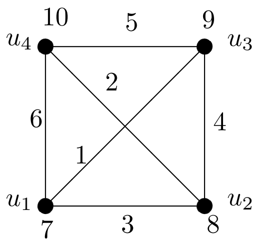

Fig 3.9

Clearly $f^*(E(K_4)) = \{11, 12, 13, 14, 15, 16\}$

Case:(iii) $n \geqslant 5$, n is odd.
Let $u_1, u_2, ..., u_n$ be the vertices of K_n.
Define $f : V(K_n) \cup E(K_n) \longrightarrow \left\{1, 2, \dots, \frac{n(n+1)}{2}\right\}$ as follows:
$f(u_i) = \frac{n(n-1)}{2} + i, \quad for \quad i = 1, 2, ..., n$
$f(u_i u_{i+1}) = \frac{n(n-3)}{2} + i, \quad for \quad i = 1, 2, ..., n$
where $u_{n+1} = u_1$
For $2 \leqslant k \leqslant \frac{n-1}{2}$,
define $f(u_i u_{i+k}) = (k-2)n + i, \quad for \quad i = 1, 2, ..., n$
where $u_{n+k} = u_k$ for all k.
Clearly $f(E(K_n)) = \left\{1, 2, \dots, \frac{n(n-1)}{2}\right\}$

$$\{f^*(u_i u_{i+1}) \mid 1 \leqslant i \leqslant n-1\} = \left\{\frac{n(n+1)}{2} + i + 1 \mid 1 \leqslant i \leqslant n-1\right\}$$
$$= \left\{\frac{n(n+1)}{2} + 2, \dots, \frac{n(n+3)}{2}\right\}$$

$f^*(u_n u_1) = \frac{n(n+1)}{2} + 1$

$$\left\{f^*(u_i u_{i+k}) \mid 1 \leqslant i \leqslant n, \quad 2 \leqslant k \leqslant \frac{n-1}{2}\right\} =$$
$$= \left\{(n-k)(n-1) + 2n + i \mid 1 \leqslant i \leqslant n,\ 2 \leqslant k \leqslant \frac{n-1}{2}\right\} = \left\{\frac{n(n+3)}{2} + 1, \dots, n^2\right\}.$$

Therefore $f^*(E(K_n)) = \left\{\frac{n(n+1)}{2} + 1, \frac{n(n+1)}{2} + 2, ..., n^2\right\}$
Thus K_n is E-super arithmetic graceful

Case:(iv) $n \geqslant 6$, n is even.
Let $u_1, u_2, ..., u_n$ be the vertices of K_n.
Define $f : V(K_n \cup E(K_n)) \longrightarrow \left\{1, 2, \dots, \frac{n(n+1)}{2}\right\}$ as follows:

$f(u_i) = \frac{n(n-1)}{2} + i, \quad for \quad i = 1, 2, ..., n$
$f(u_i u_{i+1}) = \frac{n(n-3)}{2} + i, \quad for \quad i = 1, 2, ..., n$
where $u_{n+1} = u_1$
For $2 \leqslant k \leqslant \frac{n}{2} - 1$,
define $f(u_i u_{i+k}) = (k-2)n + i$, for $i = 1, 2, ..., n$
where $u_{n+k} = u_k$ for all k.
Define $f(u_i u_{i+\frac{n}{2}}) = \frac{n(n-4)}{2} + i, \quad i = 1, 2, ..., \frac{n}{2}$
As in the above case, $f^*(E(K_n)) = \left\{\frac{n(n+1)}{2} + 1, \frac{n(n+1)}{2} + 2, ..., n^2\right\}$
Thus K_n is E-super arithmetic graceful. □

EXAMPLE 3.7. E-super arithmetic graceful labelling of K_7.

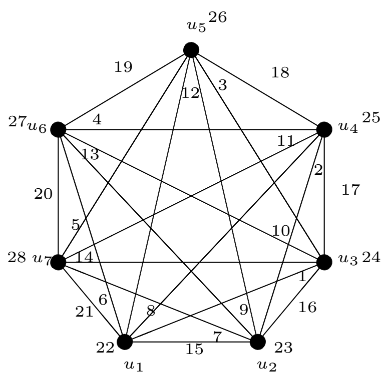

Fig 3.10

EXAMPLE 3.8. E-super arithmetic gaceful labelling of K_6.

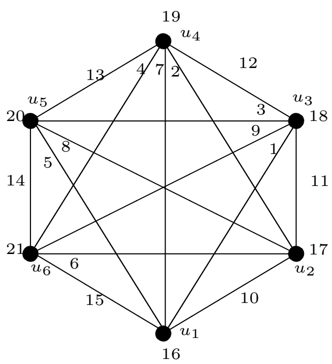

Fig 3.11

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