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DOUBLE DOMINATION IN SHADOW GRAPHS

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Abstract. Let $G = (V, E)$ be a graph having vertex set $V(G)$. For $S \subseteq V(G)$, if every vertex is of $V(G)$ is dominated at least twice by the vertices of S, then the set S is a double dominating set of G. The double domination number, denoted by $\gamma_{dd}(G)$, is the minimum cardinality among all double dominating sets of G. In this work, we discuss the double domination of shadow graphs of some graphs such as cycle, path, star, complete bipartite and wheel graphs.

1. Introduction

Graph theory is one of the most evolving branches of modern mathematics and computer applications. The development of graph theory has been witnessed in the last 30 years, as graph theory can be applied to many problems such as discrete optimization problems, combinatorial problems, and classical algebraic problems. Graph domination has been an extensively researched branch of graph theory. Domination theory which has a very wide range of applications in many fields such as engineering, physical, social and biological sciences, linguistics, etc., has recently been the core of research activities in graph theory.

The domination set problem requires determining the domination number of a given graph. In addition, many facilities have natural applications in location problems. In such problems, the vertices of a graph correspond to locations and adjacency represents some concept of accessibility. The aim is to determine the places where the fire stations, bus stops, post offices or similar facilities will be established and which can be reached from other places with optimum ease. There are also domination set applications in coding theory and social networks.

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One of the reason why domination parameters is so popular is that it is suitable for the formation of new parameters that can be developed from simple definitions. Also, its close relationship with NP-complete other fundamental domination problems and other NP-complete problems has contributed to the growth of research activity in domination theory [**1, 2, 6, 11**].

A set $D \subseteq V(G)$ of vertices in a graph G is a dominating set of G if every vertex in $V(G)\backslash D$ is adjacent to some vertex in D and the domination number $\gamma(G)$ is the minimum order of a dominating set of G. Equivalently, D is a dominating set of G if for every vertex $v \in V$, $|N[v] \cap D| \geqslant 1$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set [**10**].

Let Sbe a subset of vertices in G. If every vertex in $V(G) - S$ has at least two neighbors in the set S and every vertex in S has a neighbor in the set S, then the set S is called a double dominating set, abbreviated DDS. This requires $|N[v] \cap S| \geqslant 2$ for each $v \in V(G)$.The minimal cardinality of a double dominating set of G is the double domination number $\gamma_{dd}(G)$. A $\gamma_{dd}(G)$-set is a G double dominating set with minimum cardinality. Harary and Haynes [**9**] presented double domination, which was later investigated in [**3, 4, 8, 13**]. The double domination number can be defined for any graph that does not contain isolated vertex. Let illustrates application of the double domination parameter by a prisoners-guard example. In this example the meaning of domination is that each prisoner can be seen by some guards. Here, double domination increases security by requiring each prisoner to be protected by two or more guards.

Let G be a graph having vertex set $V(G)$ and edge set $E(G)$. For two vertices u and v if there is an edge joining them, then they are adjacent (or neighbors). The distance $d_G(u, v)$ between u and v is the length of the shortest path joining them in G. The greatest distance between any pair of vertices of G is the diameter of G and denoted by $diam(G)$ [**14**].

The shadow graph of G, denoted by $D_2(G)$ is the graph constructed from G by taking two copies of G namely G itself and G' and by joining each vertex u in G to the neighbors of the corresponding vertex u' in G' [**7**].

The purpose of this research is to demonstrate what the double domination number in shadow network topologies is. The theoretical background and literature overview on the double domination number is presented in Section 2. Main results for the double domination number shadow networks are provided and discussed in Section 3.

2. Known results

THEOREM 2.1. [**5**] *If G is a graph without isolated peaks, then $2 \leqslant \gamma_{dd}(G) \leqslant n$.*

THEOREM 2.2. [**5**] *For any graph G without isolated vertices, $\gamma(G) \leqslant \gamma_{dd}(G) - 1$.*

THEOREM 2.3. [**3, 5, 9**]

a) *For $n \geqslant 2$, $\gamma_{dd}(P_n) = \left[\frac{2n+2}{3}\right]$*
b) *For $n \geqslant 3$, $\gamma_{dd}(C_n) = \left\lceil\frac{2n}{3}\right\rceil$*

c) *For* $m > 1$, $\gamma_{dd}(K_{1,m}) = m + 1$.

THEOREM 2.4. [**5**] *For a graph* G, $\gamma_{dd}(G) \leqslant \lfloor \frac{n}{2} \rfloor + \gamma(G) - 1$ *with* $\delta(G) \geqslant 2$*and* $n \neq 3, 5$.

OBSERVATION 2.1. [**4**] *Each* $DD - set$*of any graph contains all leaves and support vertices.*

3. Shadow graph $D_2(G)$

The double domination number of shadow graphs of several specific graphs, such as cycle, path, star, complete bipartite, and wheel graphs, is determined in this section. Throughout the paper, we will label vertices of $D_2(G)$ for $G \neq W_{1,n}, K_{r,s}$ as the vertices in the first copy of G by $1, 2, \ldots, n$ and the vertices in the second copy of G by $n + 1$, $n + 2, \ldots, 2n$ starting from the left.

THEOREM 3.1. *If* $D_2(P_n)$ *is a shadow graph of a path with* $n \geqslant 3$*, then*

$$\gamma_{dd}(D_2(P_n)) = \begin{cases} \left\lceil \frac{6n}{7} \right\rceil + 1, & \textit{if } n \equiv 1, 2 (\bmod 7) \\ \left\lceil \frac{6n}{7} \right\rceil, & \textit{otherwise.} \end{cases}$$

PROOF. We first establish the upper bound for $\gamma_{dd}(D_2(P_n))$. Let

$$S_1 = \{(7i+2), (7i+3) \quad | \quad 0 \leqslant i \leqslant \left\lceil \frac{n-1}{7} \right\rceil - 1\} \cup \{7i+6 \quad | \quad 0 \leqslant i \leqslant \left\lceil \frac{n-5}{7} \right\rceil - 1\}$$

and Let $S = S_1 \cup S_2$. Furthermore, $|S| = 3\left\lceil \frac{n-1}{7} \right\rceil + \left\lceil \frac{n-5}{7} \right\rceil + 2\left\lceil \frac{n-4}{7} \right\rceil$. If $n \equiv i (\bmod 7)$ for each $i \in \{0, 3, 6\}$, then $D = S$. If $n \equiv 1 (\bmod 7)$, then $D = S \cup \{(n - 1), (2n - 1)\}$. If $n \equiv 2 (\bmod 7)$, then the $\{(n + 1)\}$ vertex is in the set S. However, since there is $(n + 1) \notin V(D_2(P_n))$, this vertex should be removed from the set S. Thus, we get $D = (S-(n+1)) \cup \{(n-1)\}$. If $n \equiv 4(\bmod 7)$, then $D = S \cup \{(2n-1)\}$. If $n \equiv 5 (\bmod 7)$, then the $\{(2n + 1)\}$ vertex is in the set S. However, since there is $(2n+1) \notin V(D_2(P_n))$, this vertex should be removed from the set S. Thus, we get $D = \{S - ((2n + 1), (2n))\} \cup \{(n - 1), (2n - 1)\}$. In all cases of n based on mod 7 , the set D is a DD-set of $D_2(P_n)$. Thus, if $n \equiv 1, 2 (\bmod 7)$, then $|D| = \left\lceil \frac{6n}{7} \right\rceil + 1$ and for other cases $|D| = \left\lceil \frac{6n}{7} \right\rceil$. Hence, $\gamma_{dd}(D_2(P_n)) \leqslant \left\lceil \frac{6n}{7} \right\rceil + 1]$ if $n \equiv 1,\ 2$ (mod 7), and the otherwise, $\gamma_{dd}(D_2(P_n)) \leqslant \left\lceil \frac{6n}{7} \right\rceil$.
Let us prove the reverse inequality. Assume that $T = \{v_1, v_2, \ldots, v_i, \ldots, v_j, \ldots, v_t\}$ is a γ_{dd}-set of $D_2(P_n)$ with $v_1 < v_2 < \ldots < v_i < \ldots < v_m < v_{m+1} < \ldots < v_j < \ldots < v_t$, where v_i and v_j are any positive integers such that $1 \leqslant v_i \leqslant n$ for $i \in \{1, 2, \ldots, m\}$ and $n+1 \leqslant v_j \leqslant 2n$ for $j \in \{m+1, m+2, \ldots, t\}$. Let $f_x = v_{x+3} - v_x$ for $x \in \{1, 2, \ldots, t - 3\}$ with $x \neq m, (m - 1), (m - 2)$. We must prove $f_x \leqslant 7$ for each $x \in \{1, 2, \ldots, t - 3\}$ provided that $x \neq m, (m - 1), (m - 2)$. Let us suppose that $f_x \geqslant 8$ for every x. We claim that $f_x = 8$ for some $x \in \{1, 2, \ldots, t - 3\}$ with $x \neq m, (m - 1), (m - 2)$. In accordance with this claim, we construct the set

$$S_1' = \{2\} \cup \left\{ \bigcup_{i=0}^{\lceil \frac{n-5}{7} \rceil - 1} (7i+6) \right\} \cup \left\{ \bigcup_{i=0}^{\lceil \frac{n-8}{7} \rceil - 1} (7i+9), (7i+10) \right\}.$$

Let $T = S_1' \cup S_2'$. It is easy to see that the value of n in all its cases is the same as the set discussed above. So $T = D$ is obtained. So lets assume that there is $f_x \geqslant 9$ for at least one x. Let $f_1 = 9$ be in order not to disturb the generality.

$$A = \{2\} \cup \left\{ \bigcup_{i=0}^{\lceil \frac{n-5}{7} \rceil - 1} (7i+6) \right\} \cup \left\{ \bigcup_{i=0}^{\lceil \frac{n-9}{7} \rceil - 1} \{(7i+10), (7i+11)\} \right\}$$

$$B = \left\{ \bigcup_{i=0}^{\lceil \frac{n-1}{7} \rceil - 1} \{(n+7i+2), (n+7i+3)\} \right\}$$

$$\cup \left\{ \bigcup_{i=0}^{\lceil \frac{n-5}{7} \rceil - 1} \{(n+7i+5), (n+7i+6), (n+7i+7)\} \right\}.$$

This contradicts the upper bounds we have established on $\gamma_{dd}\left(D_2\left(P_n\right)\right)$ earlier. So, it should be $f_x \leqslant 8$. However, this condition is only possible when there is exactly one value of x, and we have proven that this value is the same as if there is $f_x \leqslant 7$ for every value of x. Therefore, $f_x \leqslant 7$ for all $x \in \{1, 2, \ldots, t-3\}$ with $x \neq m, (m-1), (m-2)$. Thus, it is clear that $\sum\limits_{x_1=1}^{m-3} f_{x_1} + \sum\limits_{x_2=m+1}^{t-3} f_{x_2} \leqslant 7(t-6)$. Since $v_1 = 2, v_2 = 3, v_3 = 6, v_{m+1} = n+2, v_{m+2} = n+5$ and $v_{m+3} = n+6$, we $\sum\limits_{x_1=1}^{m-3} f_{x_1} + \sum\limits_{x_2=m+1}^{t-3} f_{x_2} = v_m + v_{m-1} + v_{m-2} + v_t + v_{t-1} + v_{t-2} - (3n+24)$.

- $n \equiv 0(\text{mod}7)$
  $v_m = n-1, v_{m-1} = n-4, v_{m-2} = n-5, v_t = 2n-1, v_{t-1} = 2n-2$ and $v_t = 2n-5$. Thus, we get $6n - 42 = \sum\limits_{x_1=1}^{m-3} f_{x_1} + \sum\limits_{x_2=m+1}^{t-3} f_{x_2} \leqslant 7(t-6)$. This yields $|T| = t \geqslant \lceil \frac{6n}{7} \rceil$.

- $n \equiv 1\,(\bmod 7)$
  $v_m = n-1, v_{m-1} = n-2, v_{m-2} = n-5, v_t = 2n-1, v_{t-1} = 2n-2$ and $v_t = 2n-3$. Thus, we get $6n - 38 = \sum\limits_{x_1=1}^{m-3} f_{x_1} + \sum\limits_{x_2=m+1}^{t-3} f_{x_2} \leqslant 7(t-6)$. This yields $|T| = t \geqslant \lceil \frac{6n+4}{7} \rceil = \lceil \frac{6n}{7} \rceil + 1$.

- $n \equiv 2\,(\bmod 7)$
  $v_m = n, v_{m-1} = n-1, v_{m-2} = n-3, v_t = 2n, v_{t-1} = 2n-3$ and $v_t = 2n-4$. Thus, we get $6n - 35 = \sum\limits_{x_1=1}^{m-3} f_{x_1} + \sum\limits_{x_2=m+1}^{t-3} f_{x_2} \leqslant 7(t-6)$. This yields

$|T| = t \geqslant \left\lceil \frac{6n+7}{7} \right\rceil = \left\lceil \frac{6n}{7} \right\rceil + 1.$

- $n \equiv 3\,(\bmod 7)$
  $v_m = n, v_{m-1} = n-1, v_{m-2} = n-4, v_t = 2n-1, v_{t-1} = 2n-4$ and $v_t = 2n-5$. Thus, we get $6n - 39 = \sum\limits_{x_1=1}^{m-3} f_{x_1} + \sum\limits_{x_2=m+1}^{t-3} f_{x_2} \leqslant 7(t-6)$. This yields $|T| = t \geqslant \left\lceil \frac{6n+3}{7} \right\rceil = \left\lceil \frac{6n}{7} \right\rceil$.

- $n \equiv 4\,(\bmod 7)$
  $v_m = n-1, v_{m-1} = n-2, v_{m-2} = n-5, v_t = 2n-1, v_{t-1} = 2n-2$ and $v_t = 2n-5$. Thus, we get $6n - 40 = \sum\limits_{x_1=1}^{m-3} f_{x_1} + \sum\limits_{x_2=m+1}^{t-3} f_{x_2} \leqslant 7(t-6)$. This yields $|T| = t \geqslant \left\lceil \frac{6n+2}{7} \right\rceil = \left\lceil \frac{6n}{7} \right\rceil$.

- $n \equiv 5\,(\bmod 7)$
  $v_m = n-1, v_{m-1} = n-2, v_{m-2} = n-3, v_t = 2n-1, v_{t-1} = 2n-3$ and $v_t = 2n-6$. Thus, we get $6n - 40 = \sum\limits_{x_1=1}^{m-3} f_{x_1} + \sum\limits_{x_2=m+1}^{t-3} f_{x_2} \leqslant 7(t-6)$. This yields $|T| = t \geqslant \left\lceil \frac{6n+2}{7} \right\rceil = \left\lceil \frac{6n}{7} \right\rceil$.

- $n \equiv 6\,(\,bmod\,7)$
  $v_m = n, v_{m-1} = n-3, v_{m-2} = n-4, v_t = 2n, v_{t-1} = 2n-1$ and $v_t = 2n-4$. Thus, we get $6n - 36 = \sum\limits_{x_1=1}^{m-3} f_{x_1} + \sum\limits_{x_2=m+1}^{t-3} f_{x_2} \leqslant 7(t-6)$. This yields $|T| = t \geqslant \left\lceil \frac{6n+6}{7} \right\rceil = \left\lceil \frac{6n}{7} \right\rceil$.

The proof is completed by combining the lower and upper bounds for $\gamma_{dd}(D_2(P_n))$. □

For the cycle graph C_n, the value of $\gamma_{dd}\left(D_2\left(C_n\right)\right)$ is equal to the $\gamma_{dd}\left(D_2\left(P_n\right)\right)$ value of the path graph P_n. So the proof is similar to the proof of Theorem 3.1. Therefore, the proof has been removed from the article.

Corollary 3.1. *If $D_2\left(C_n\right)$ is a shadow graph of a cycle with $n \geqslant 3$, then $\gamma_{dd}\left(D_2\left(C_n\right)\right) = \gamma_{dd}\left(D_2\left(P_n\right)\right)$.*

Theorem 3.2. *If $G \cong S_{1,n}$ be a star graph with $(n+1)$vertices, then the double domination number of the star graph $\gamma_{dd}(D_2(G)) = 3$.*

Proof. Lets label the set of vertices of the $D_2(G)$ graph as the union of two sets $V\left(D_2\left(G\right)\right) = V \cup V'$, where $V = V\left(G\right) = \left\{v_1, v_2, ..., v_n\right\}$ and $V' = V'\left(G\right) = \left\{v_1', v_2', ..., v_n'\right\}$. Let v_1 be the central vertex of the graph G. Furthermore, let the set D be the $\gamma_{dd} - set$ of the graph $D_2\left(G\right)$. Since there are $N_{D_2(G)}\left(v_1\right) = V \cup \left(V' - \left\{v_1'\right\}\right) = V\left(D_2\left(G\right)\right) - \left\{v_1'\right\}$ and $N_{D_2(G)}\left(v_1'\right) = V' \cup \left(V - \left\{v_1\right\}\right) =$

$V(D_2(G)) - \{v_1\}$, $\deg_{D_2(G)}(v_1) = \deg_{D_2(G)}\left(v_1^{'}\right) = 2n-2$. This requires $v_1, v_1^{'} \in D$. So, every vertex in the $V(D_2(G)) - \left\{v_1, v_1^{'}\right\}$ set are double dominated by D.

Since the vertex v_1 is not adjacent to the vertex $v_1^{'}$, in order to give double domination to the vertices in D we need to add any vertex that is adjacent to both vertices to D. Without loss of generality, lets assume that this vertex is v_2. Hence, we have $D = \left\{v_1, v_1^{'}, v_2\right\}$.

In this case, with the set D, every vertex of the graph $D_2(G)$are double-dominated. As a result, the double domination number of the graph $D_2(G)$ is $\gamma_{dd}(D_2(G)) = |D| = 3$.

Hence, the proof of the theorem holds. □

THEOREM 3.3. *If $G \cong W_n$ be a wheel graph with n vertices, then the domination number of the wheel graph $\gamma(D_2(G)) = 2$.*

PROOF. Let the vertices of the graph G be $V(G) = \{v_1, v_2, ..., v_n\}$and the vertex v_1 the central vertex. Since the vertex v_1 is adjacent to all the vertices except the vertex $v_1^{'}$, which is its copy in $D_2(G)$, $\deg_{D_2(G)}(v_1) = 2(n-1)$. If the set S is the $\gamma_{dd} - set$ of the graph $D_2(G)$, then the vertex v_1 (or its copy vertex $v_1^{'}$) must be added to the S set.

Assume that $v_1 \in S$. In this case, every vertex of $D_2(G)$ are dominated with S except for the vertex $v_1^{'}$. In order to dominate the vertex $v_1^{'}$, we need to add any vertex in the set$N_{D_2(G)}\left[v_1^{'}\right]$ to S. Thus, we get $|S| = 2$. As a result, the domination number of the graph $D_2(G)$ is $\gamma(D_2(G)) = |S| = 2$.

Hence, the proof of the theorem holds. □

THEOREM 3.4. *If $G \cong W_n$ be a wheel graph with n vertices, then the double domination number of the wheel graph $\gamma_{dd}(D_2(G)) = 3$.*

PROOF. From Theorem 2.2 , it is known that $\gamma_{dd}(G) \geqslant \gamma(G) + 1$. Let the set of vertices of $D_2(G)$ be divided into two sets of $V(D_2(G)) = V(G) \cup V^{'}(G)$, where $V(G) = \{v_1, v_2, ..., v_n\}$ and $V^{'}(G) = \left\{v_1^{'}, v_2^{'}, ..., v_n^{'}\right\}$. The vertex $v_1$is the central vertex of the first copy of the graph $D_2(G)$, while the vertex $v_1^{'}$ is the central vertex of the second copy of the graph.

From Theorem 3.3 and Theorem 2.2, we get $\gamma_{dd}(D_2(G)) \geqslant 3$. To prove the inverse of the inequality, lets assume $S = \left\{v_1, v_1^{'}, v_2\right\}$. Thus, every vertex of the graph $D_2(G)$ are double dominated by the set S. In this case, the set S is a $\gamma_{dd} - set$ of $D_2(G)$. Thus, we get $\gamma_{dd}(D_2(G)) \leqslant 3$. As a result, the double domination number of the graph $D_2(G)$, $\gamma_{dd}(D_2(G)) = 3$, is obtained from the lower and upper limits.

Hence, the proof of the theorem holds. □

THEOREM 3.5. *If $G \cong K_{m,n}$ be a bipartite complete graph with $(m+n)-$vertices, then the domination number of the bipartite complete graph $\gamma(D_2(G)) = 2$.*

Proof. It is known that $\gamma(G) = 2$ [**12**]. Let the $\gamma - set$ giving this value be$S$. It is easy to see that the set $S$is also a $\gamma_{dd} - set$ of the graph $D_2(G)$. Thus, we have $\gamma(D_2(G)) = 2$.

Hence, the proof of the theorem holds. □

Theorem 3.6. *If* $G \cong K_{m,n}$ *be a bipartite complete graph with* $(m+n)-$*vertices, then the double domination number of the bipartite complete graph* $\gamma_{dd}(D_2(G)) = 4$.

Proof. Let the set of vertices of G be divided into two sets of $V(G) = V_1 \cup V_2$, where $V_1 = \{v_1, v_2, ..., v_m\}$ and $V_2 = \{u_1, u_2, ..., u_n\}$. In this case, the set of vertices of the graph $D_2(G)$ is $V(D_2(G)) = V_1 \cup V_2 \cup V_1^{'} \cup V_2^{'}$. Let the set D be the $\gamma_{dd} - set$ of the graph $D_2(G)$.

From Theorem 3.5 and Theorem 2.2 we get $\gamma_{dd}(D_2(G)) \geqslant 3$. Assume that $\gamma_{dd}(D_2(G)) = 3$. In order to be able to double dominated each vertex in the set V_1, there must be two vertices in D. Both of these vertices can be in V_2 or $V_2^{'}$, or one at V_2 and one at $V_2^{'}$. Thus, double dominate of every vertex in V_1 and $V_1^{'}$ is provided by the set D. One vertex is not enough to double dominated the remaining vertices in V_2 and $V_2^{'}$. This requires $\gamma_{dd}(D_2(G)) = |D| \geqslant 4$ for the graph $D_2(G)$.

Now, lets prove the inverse of the inequality. Let the set S be the $\gamma_{dd} - set$ of the graph $D_2(G)$. Assume that $S = \{v_i, v_j, u_k, u_t\}$, where $v_i, v_j \in V_1$ and $u_k, u_t \in V_2$. Thus, double domination of every vertex in the graph $D_2(G)$is provided by S. Thus, we get $\gamma_{dd}(D_2(G)) = |S| \leqslant 4$. As a result, the double domination number of the graph $D_2(G)$, $\gamma_{dd}(D_2(G)) = 4$, is obtained from the lower and upper limits. Hence, the proof of the theorem holds. □

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