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BI-IDEALS OF Γ-NEARNESS SEMIRINGS

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Abstract. As a generalization of the concept of a quasi-ideal, bi-ideal is defined. In this article, we define bi-ideals in gamma semirings on weak nearness approximation spaces and explain some of the properties and definitions. By these properties, further studies of nearness semirings will be more progressive.

1. Introduction

Nobusawa [**10**] and Barnes [**1**] gave the definition of Γ-rings and studied some properties and related definitions in different concepts. As a generalization of a ring , the concept of a semiring was introduced and studied in detail by Vandiver [**26**]. Afterwards, Rao introduced Γ-semirings in [**7**],[**8**] as a generalization of Γ-ring and semiring. Ideals in semirings were studied by Iseki in [**4**]. Lajos and Szasz introduced theory of bi ideals in rings and semirings [**9**]. Dutta and Sardar in [**2**] gave properties of prime and semiprime ideals in Γ-semirings. Also, Jagatap and Pawar discussed quasi ideals and bi-ideals in Γ-semirings [**5, 6**].

In 2002, Peters gave the definition of near sets that is a generalization of rough sets [**22**]. In the near set theory, Peters defined an indiscernibility relation by using the features of the objects to investigate the nearness of the objects [**23**]. In addition to this, he generalized approach theory of the nearness of non-empty sets resembling each other [**24, 25**].

İnan and Öztürk gave the definition of nearness groups and nearness ring by using nearness sets in 2012 [**3, 17**]. After that, Öztürk et al. defined some other nearness algebraic structures such as nearness semirings [**15**], prime ideals in nearness semirings [**21**], Γ-nearness semiring in the sense of Barnes theory [**16**] and

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Nobusawa Γ-nearness semirings in [**20**] and Γ-nearness hemiring [**19**] and Nobusawa Γ-nearness hemirings in [**14**]. Besides this, Tekin defined quasi nearness ideals and bi-nearness ideals in semirings [**13, 12**] and also quasi-ideals of Γ-nearness semirings [**11**].

In this article, bi-ideals of Γ nearness semirings are defined and some of the concepts and definitions on weak nearness approximation spaces are explained. Then, we study some basic properties of bi-ideals of a Γ nearness semirings.

2. Preliminaries

An object description is specified by means of a tuple of function values $\Phi(x)$ deal with an object $x \in X$. $B \subseteq \mathcal{F}$ is a set of probe functions and these functions stand for features of sample objects $X \subseteq \mathcal{O}$. Let $\varphi_i \in B$, that is $\varphi_i : \mathcal{O} \to \mathbb{R}$. The functions showing object features supply a basis for $\Phi : \mathcal{O} \to \mathbb{R}^L, \Phi(x) = (\varphi_1(x), \varphi_2(x), ..., \varphi_L(x))$ a vector consisting of measurements deal with each functional value $\varphi_i(x)$, where the description length $\mid \Phi \mid = L$ ([**23**]).

The selection of functions $\varphi_i \in B$ is very fundamental by using to determine sample objects. $X \subseteq \mathcal{O}$ are near each other if and only if the sample objects have similar characterization. Each φ shows a descriptive pattern of an object. Hence, $\triangle_{\varphi_i}$ means $\triangle_{\varphi_i} = \mid \varphi_i(x') - \varphi_i(x) \mid$, where $x', x \in \mathcal{O}$. The difference φ means to a description of the indiscernibility relation " $\sim_B$ " defined by Peters in [**23**]. B_r is probe functions in B for $r \leqslant |B|$.

Definition 2.1. [**23**] Indiscernibility relation on $\mathcal{O}$ is found

$$\sim_B = \{(x, x') \in \mathcal{O} \times \mathcal{O} \mid \triangle_{\varphi_i} = 0 \forall \varphi_i \in B \, B \subseteq \mathcal{F}\}$$

where description length $i \leqslant \mid \Phi \mid$. $\sim_{B_r}$ is also indiscernibility relation determined by utilizing B_r.

Near equivalence class is defined as

$$[x]_{B_r} = \{x' \in \mathcal{O} \mid x \sim_{B_r} x'\}.$$

Quotient set is given as

$$\mathcal{O} / \sim_{B_r} = \{[x]_{B_r} \mid x \in \mathcal{O}\} = \xi_{\mathcal{O}, B_r}$$

From here, set of partitions are written as $N_r(B) = \{\xi_{\mathcal{O}, B_r} \mid B_r \subseteq B\}$. Upper approximation set $N_r(B)^* X = \bigcup_{[x]_{B_r} \cap X \neq \varnothing} [x]_{B_r}$ can be attained with the help of near equivalence classes [**23**].

Definition 2.2. [**18**] Let $\mathcal{O}$ be a set of sample objects, $\mathcal{F}$ a set of the probe functions, $\sim_{B_r}$ an indiscernibility relation, and N_r a collection of partitions. Then, $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B))$ is called a weak nearness approximation space.

Theorem 2.1. [**18**] *Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B))$ be a weak nearness approximation space and $X, Y \subset \mathcal{O}$. Then the following statements hold:*

i) $X \subseteq N_r(B)^* X$,

ii) $N_r(B)^* (X \cup Y) = N_r(B)^* X \cup N_r(B)^* Y$,

iii) $X \subseteq Y$ *implies* $N_r(B)^* X \subseteq N_r(B)^* Y$,
iv) $N_r(B)^* (X \cap Y) \subseteq N_r(B)^* X \cap N_r(B)^* Y$.

Definition 2.3. [**16**] Let $S = \{x, y, z, \ldots\} \subseteq \mathcal{O}$, and $\Gamma = \{\alpha, \beta, \gamma, \ldots\} \subseteq \acute{\mathcal{O}}$ where $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B))$ and $(\acute{\mathcal{O}}, \mathcal{F}, \sim_{B_r}, N_r(B))$ are two different weak near approximation spaces. S is called a Γ-semiring on $\mathcal{O} - \acute{\mathcal{O}}$ denote by $(S, +, \cdot)$ if the following properties are hold:
$NGSR_1$) $(S, +)$ is an abelian monoid on $\mathcal{O}$ with identity element 0_S,
$NGSR_2$) $(\Gamma, +)$ is an abelian monoid on $\acute{\mathcal{O}}$ with identity element 0_Γ,
$NGSR_3$) $(S, \cdot)$ is an Γ monoid on $\mathcal{O} - \acute{\mathcal{O}}$ with identity element 1_S,
$NGSR_4$) For all $x, y, z \in S$, and $\gamma, \beta \in \Gamma$ such that

$$i)\ x\gamma(y+z) = (x\gamma y) + (x\gamma z),$$
$$ii)\ x(\beta+\gamma)z = (x\beta z) + (x\gamma z),$$
$$iii)\ (x+y)\gamma z = x\gamma z + y\gamma z.$$

hold in $N_r(B)^* S$,
$NGSR_5$) For all $x \in S$, and $\gamma \in \Gamma$ such that

$$0_S\gamma x = 0_S = x\gamma 0_S$$

hold in $N_r(B)^* S$,
$NGSR_6$) $1_S \neq 0_S$.

Let S be a Γ-semiring on weak nearness approximation spaces $\mathcal{O} - \acute{\mathcal{O}}$. If $\mathcal{O} = \acute{\mathcal{O}}$, then S is a Γ-semiring on weak nearness approximation spaces $\mathcal{O}$.

Lemma 2.1. [**16**] *Let S be a Γ-nearness semiring. The following properties hold:*
i) If $X, Y \subseteq S$, then $(N_r(B)^* X) + (N_r(B)^* Y) \subseteq N_r(B)^* (X + Y)$,
ii) If $X, Y \subseteq S$, then $(N_r(B)^* X)\Gamma(N_r(B)^* Y) \subseteq N_r(B)^* (X\Gamma Y)$.

Definition 2.4. [**16**] Let S be a Γ-nearness semiring and A be a non-empty subsets of S. Then
i) A is called a sub Γ-semiring of S if $A + A \subseteq N_r(B)^* A$ and $A\Gamma A \subseteq N_r(B)^* A$.
ii) A is called an upper-near sub Γ-semiring of H if $(N_r(B)^* A) + (N_r(B)^* A) \subseteq N_r(B)^* A$ and $(N_r(B)^* A)\Gamma(N_r(B)^* A) \subseteq N_r(B)^* A$.

Definition 2.5. [**16**] Let S be a Γ-nearness semiring and A be a sub Γ-semigroup of S.
i) A is called a right (left) Γ-ideal of S if $A\Gamma S \subseteq N_r(B)^* A$ ($S\Gamma A \subseteq N_r(B)^* A$).
ii) A is called an upper-near right (left) Γ-ideal of S if $(N_r(B)^* A)\Gamma S \subseteq N_r(B)^* A$ ($S\Gamma(N_r(B)^* A) \subseteq N_r(B)^* A$).

Definition 2.6. [**16**] Let S be a Γ-nearness semiring. For nonempty subsets A, B of S, $A\Gamma B = \{\sum_{\text{finite}} a_i\alpha b_i : a_i \in A \quad \text{and} \quad b_i \in B, \alpha \in \Gamma\}$.

Definition 2.7. [**12**] Let S be a nearness semiring and A be a subsemigroup of S, where $A \subseteq S$.
(i)A is called bi-nearness ideal of S if $ASA \subseteq N_r(B)^* A$.

(ii)A is called a bi upper-near ideal of S if $(N_r(B)^* A)S(N_r(B)^* A) \subseteq N_r(B)^* A$.

LEMMA 2.2. [**13**] *Let S be a nearness semiring. If S is commutative, then each quasi-nearness ideal of S is two-sided ideal of S.*

DEFINITION 2.8. [**11**] Let S be a Γ-nearness semiring and Q be a Γ-subsemigorup of S, where $Q \neq S$.
i) Q is called quasi Γ-ideal of S if $Q\Gamma S \cap S\Gamma Q \subseteq N_r(B)^* Q$.
ii) Q is called a quasi upper-near Γ-ideal of S if $(N_r(B)^* Q)\Gamma S \cap S\Gamma(N_r(B)^* Q) \subseteq N_r(B)^* Q$.

LEMMA 2.3. [**11**] *Let S be a Γ-nearness semiring. If S is commutative, then each quasi Γ-ideal of S is two-sided Γ-ideal of S.*

3. Bi-ideals of gamma nearness semirings

DEFINITION 3.1. Let M be a nearness semiring and A be a subsemigroup of M, where $A \subseteq M$.
(i)A is called bi-nearness ideal of M if $A\Gamma M\Gamma A \subseteq N_r(B)^* A$.
(ii)A is called a bi upper-near ideal of M if $(N_r(B)^* A)\Gamma M\Gamma(N_r(B)^* A) \subseteq N_r(B)^* A$.

EXAMPLE 3.1. Let $\mathcal{O} = \{p, q, r, s, t, u, v, w, m, y, z\}$ be a set of sample objects where

$$p = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, q = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, r = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, s = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$
$$t = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, u = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, v = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, w = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$
$$m = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, z = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

for universal set $U = \{ [a_{ij}]_{2x3} \mid a_{ij} \in \mathbb{Z}_2\}$.
Furthermore, $\acute{\mathcal{O}} = \{\theta, \alpha, \beta, \gamma, \mu, \eta, \nu, \sigma\}$ be a set of sample objects where

$$\theta = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \alpha = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \beta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \gamma = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$
$$\mu = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \eta = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}, \nu = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \sigma = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$$

for universal set $\acute{U} = \{ [a_{ij}]_{3x2} \mid a_{ij} \in \mathbb{Z}_2\}, r = 1$, $B = \{\chi_1, \chi_2, \chi_3\} \subseteq \mathcal{F}$ be a set of probe functions and $M = \{t, m, y\} \subset \mathcal{O}, \Gamma = \{\mu, \nu\} \subset \acute{\mathcal{O}}$.

Values of the probe functions for M:

$$\chi_1 : \mathcal{O} \to V_1 = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5\},$$
$$\chi_2 : \mathcal{O} \to V_2 = \{\gamma_2, \gamma_3, \gamma_4, \gamma_5\},$$
$$\chi_3 : \mathcal{O} \to V_3 = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$$

are presented in the following table:

	p	q	r	s	t	u	v	w	m	y	z
χ_1	γ_1	γ_2	γ_3	γ_2	γ_4	γ_4	γ_5	γ_5	γ_1	γ_5	γ_4
χ_2	γ_2	γ_3	γ_3	γ_4	γ_2	γ_4	γ_2	γ_2	γ_5	γ_3	γ_3
χ_3	γ_1	γ_2	γ_1	γ_1	γ_3	γ_2	γ_4	γ_4	γ_3	γ_3	γ_2

Also, values of the probe functions for Γ:

$$\chi_1 : \mathcal{O} \to V_1 = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\},$$
$$\chi_2 : \mathcal{O} \to V_2 = \{\gamma_1, \gamma_2, \gamma_3\},$$
$$\chi_3 : \mathcal{O} \to V_3 = \{\gamma_2, \gamma_3, \gamma_4\}$$

are given in the following table:

	θ	α	β	γ	μ	η	ν	σ
χ_1	γ_1	γ_2	γ_3	γ_1	γ_4	γ_4	γ_1	γ_2
χ_2	γ_1	γ_2	γ_1	γ_2	γ_3	γ_3	γ_3	γ_1
χ_3	γ_2	γ_3	γ_3	γ_2	γ_2	γ_4	γ_4	γ_3

It can be found the near equivalence classes according to the indiscernibility relation $\sim_{B_r}$ of elements in $\mathcal{O}$ and $\acute{\mathcal{O}}$:

$$\begin{aligned}
[p]_{\chi_1} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_1(p) = \gamma_1\} = \{p, m\} \\
&= [m]_{\chi_1}, \\
[q]_{\chi_1} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_1(q) = \gamma_2\} = \{q, s\} \\
&= [s]_{\chi_1}, \\
[r]_{\chi_1} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_1(r) = \gamma_3\} = \{r\}, \\
[t]_{\chi_1} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_1(t) = \gamma_4\} = \{t, u, z\} \\
&= [u]_{\chi_1} = [z]_{\chi_1}, \\
[v]_{\chi_1} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_1(v) = \gamma_5\} = \{v, w, y\} \\
&= [w]_{\chi_1} = [y]_{\chi_1}.
\end{aligned}$$

Then, we get $\xi_{\chi_1} = \left\{[p]_{\chi_1}, [q]_{\chi_1}, [r]_{\chi_1}, [t]_{\chi_1}, [v]_{\chi_1}\right\}$.

$$
\begin{aligned}
[p]_{\chi_2} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_2(p) = \gamma_2\} = \{p, t, v, w, y\} \\
&= [t]_{\chi_2} = [v]_{\chi_2} = [w]_{\chi_2} = [y]_{\chi_2}, \\
[q]_{\chi_2} &= \{x \in \mathcal{O} \mid \chi_2(x) = \chi_2(q) = \gamma_3\} = \{q, r, z\} \\
&= [r]_{\chi_1} = [z]_{\chi_1}, \\
[s]_{\chi_2} &= \{x \in \mathcal{O} \mid \chi_2(x) = \chi_2(s) = \gamma_4\} = \{s, u\} \\
&= [u]_{\chi_2}, \\
[m]_{\chi_2} &= \{x \in \mathcal{O} \mid \chi_2(x) = \chi_2(m) = \gamma_5\} = \{m\}.
\end{aligned}
$$

Then, we get $\xi_{\chi_2} = \left\{[p]_{\chi_2}, [q]_{\chi_2}, [s]_{\chi_2}, [m]_{\chi_2}\right\}$.

$$
\begin{aligned}
[p]_{\chi_3} &= \{x \in \mathcal{O} \mid \chi_3(x) = \chi_3(p) = \gamma_1\} = \{p, r, s\} \\
&= [r]_{\chi_3} = [s]_{\chi_3}, \\
[q]_{\chi_3} &= \{x \in \mathcal{O} \mid \chi_3(x) = \chi_3(q) = \gamma_2\} = \{q, u, z\} \\
&= [u]_{\chi_3} = [z]_{\chi_3}, \\
[t]_{\chi_3} &= \{x \in \mathcal{O} \mid \chi_3(x) = \chi_3(t) = \gamma_3\} = \{t, m, y\} \\
&= [m]_{\chi_2} = [y]_{\chi_2}, \\
[v]_{\chi_3} &= \{x \in \mathcal{O} \mid \chi_3(x) = \chi_3(v) = \gamma_4\} = \{v, w\} \\
&= [w]_{\chi_2}.
\end{aligned}
$$

Then, we get $\xi_{\chi_3} = \left\{[p]_{\chi_3}, [q]_{\chi_3}, [t]_{\chi_3}, [v]_{\chi_3}\right\}$.

Therefore, for $r = 1$, a set of partitions of $\mathcal{O}$ is $N_1(B) = \{\xi_{\chi_1}, \xi_{\chi_2}, \xi_{\chi_3}\}$. Then, we can write

$$
\begin{aligned}
N_1(B)^* M &= \bigcup_{[x]_{\chi_i} \cap M \neq \varnothing} [x]_{\chi_i} \\
&= [p]_{\chi_1} \cup [t]_{\chi_1} \cup [v]_{\chi_1} \cup [p]_{\chi_2} \cup [m]_{\chi_2} \cup [t]_{\chi_3} \\
&= \{p, t, u, v, w, m, y, z\}.
\end{aligned}
$$

Considering the following table of operation:

$+$	t	m	y
t	p	y	m
m	y	p	t
y	m	t	p

In that case, $(M, +)$ is an abelian monoid on $\mathcal{O}$ with identity element $0_M = p$. Moreover,

$$\begin{aligned}
[\theta]_{\chi_1} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_1(\theta) = \gamma_1\} = \{\theta, \gamma, \nu\} \\
&= [\gamma]_{\chi_1} = [\nu]_{\chi_1}, \\
[\alpha]_{\chi_1} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_1(\alpha) = \gamma_2\} = \{\alpha, \sigma\} \\
&= [\sigma]_{\chi_1}, \\
[\beta]_{\chi_1} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_1(\beta) = \gamma_3\} = \{\beta\}, \\
[\mu]_{\chi_1} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_1(\mu) = \gamma_4\} = \{\mu, \eta\} \\
&= [\eta]_{\chi_1}.
\end{aligned}$$

Then, we get $\xi_{\chi_1} = \left\{[\theta]_{\chi_1}, [\alpha]_{\chi_1}, [\beta]_{\chi_1}, [\mu]_{\chi_1}\right\}$.

$$\begin{aligned}
[\theta]_{\chi_2} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_2(\theta) = \gamma_1\} = \{\theta, \beta, \sigma\} \\
&= [\beta]_{\chi_2} = [\sigma]_{\chi_2}, \\
[\alpha]_{\chi_2} &= \{x \in \mathcal{O} \mid \chi_2(x) = \chi_2(\alpha) = \gamma_2\} = \{\alpha, \gamma\} \\
&= [\gamma]_{\chi_1}, \\
[\mu]_{\chi_2} &= \{x \in \mathcal{O} \mid \chi_2(x) = \chi_2(\mu) = \gamma_3\} = \{\mu, \eta, \nu\} \\
&= [\eta]_{\chi_2} = [\nu]_{\chi_2}.
\end{aligned}$$

Then, we get $\xi_{\chi_2} = \left\{[\theta]_{\chi_2}, [\alpha]_{\chi_2}, [\mu]_{\chi_2}\right\}$.

$$\begin{aligned}
[\theta]_{\chi_3} &= \{x \in \mathcal{O} \mid \chi_3(x) = \chi_3(\theta) = \gamma_2\} = \{\theta, \gamma, \mu\} \\
&= [\gamma]_{\chi_3} = [\mu]_{\chi_3}, \\
[\alpha]_{\chi_3} &= \{x \in \mathcal{O} \mid \chi_3(x) = \chi_3(\alpha) = \gamma_3\} = \{\alpha, \beta, \sigma\} \\
&= [\beta]_{\chi_3} = [\sigma]_{\chi_3}, \\
[\eta]_{\chi_3} &= \{x \in \mathcal{O} \mid \chi_3(x) = \chi_3(\eta) = \gamma_4\} = \{\eta, \nu\} \\
&= [\nu]_{\chi_3}.
\end{aligned}$$

Then, we get $\xi_{\chi_3} = \left\{[\theta]_{\chi_3}, [\alpha]_{\chi_3}, [\eta]_{\chi_3}\right\}$.

Therefore, for $r = 1$, a set of partitions of $\acute{\mathcal{O}}$ is $N_1(B) = \{\xi_{\chi_1}, \xi_{\chi_2}, \xi_{\chi_3}\}$. Then, we can write

$$\begin{aligned}
N_1(B)^* \Gamma &= \bigcup_{[x]_{\chi_i} \cap \Gamma \neq \varnothing} [x]_{\chi_i} \\
&= [\theta]_{\chi_1} \cup [\mu]_{\chi_1} \cup [\mu]_{\chi_2} \cup [\theta]_{\chi_3} \cup [\eta]_{\chi_3} \\
&= \{\theta, \gamma, \mu, \eta, \nu\}.
\end{aligned}$$

Considering the following table of operation:

$+$	μ	ν
μ	θ	δ
ν	δ	θ

Then $(\Gamma,+)$ is an abelian monoid on $\acute{\mathcal{O}}$ with identity element $0_\Gamma = \theta$. Now, considering the following operations:

μ	t	m	y
t	p	p	p
m	p	p	p
y	p	p	p

ν	t	m	y
t	p	p	p
m	p	p	p
y	p	p	p

In this case, $(M,+,\cdot)$ is a Γ-semiring on the weak near approximation space on $\mathcal{O}-\acute{\mathcal{O}}$, i.e., $(M,+,\cdot)$ is a Γ-nearness semiring. Now, we take $A=\{m,y\}\subseteq S$.

$$\begin{aligned} N_1(B)^* A &= \bigcup_{[x]_{\chi_i}\, \cap\, A\neq\varnothing} [x]_{\chi_i} \\ &= [p]_{\chi_1} \cup [v]_{\chi_1} \cup [p]_{\chi_2} \cup [x]_{\chi_2} \cup [t]_{\chi_3} \\ &= \{p,t,m,y,v,w\}. \end{aligned}$$

Hence, A is a subsemigorup of M and $A\Gamma M\Gamma A \subseteq N_r(B)^* A$. Therefore, A is a bi-ideal of Γ-nearness semiring M.

Theorem 3.1. *Let M be a Γ-nearness semiring and A be a non-empty subset of M. If M is commutative and $N_r(B)^*\left(N_r(B)^* A\right) = N_r(B)^* A$, then each quasi Γ-ideal of M is bi Γ-ideal of M.*

Proof. Let M be a commutative Γ-nearness semiring and A be a quasi Γ-ideal of M. Then, $A\Gamma M\Gamma A = (A\Gamma M\Gamma A)\cap(A\Gamma M\Gamma A) = A\Gamma(M\Gamma A)\cap(A\Gamma M)\Gamma A \subseteq M\Gamma(M\Gamma A)\cap(A\Gamma M)\Gamma M \subseteq (M\Gamma M)\Gamma A\cap A\Gamma(M\Gamma M)$. In this case, $(M\Gamma M)\Gamma A\cap A\Gamma(M\Gamma M) \subseteq (N_r(B)^* M)\Gamma(N_r(B)^* A)\cap(N_r(B)^* A)\Gamma(N_r(B)^* M)$ by Theorem 2.1.(i). We have that $(N_r(B)^* M)\Gamma(N_r(B)^* A)\cap(N_r(B)^* A)\Gamma(N_r(B)^* M) \subseteq N_r(B)^*(M\Gamma A)\cap N_r(B)^*(A\Gamma M)$ by Lemma 2.1.(ii). Afterwards, A is quasi Γ-ideal and each quasi nearness ideal of M is two sided nearness ideal of by Lemma 2.3, and so $N_r(B)^*(M\Gamma A)\cap N_r(B)^*(A\Gamma M) \subseteq N_r(B)^*\left(N_r(B)^* A\right)\cap N_r(B)^*\left(N_r(B)^* A\right) = N_r(B)^*\left(N_r(B)^* A\right) = N_r(B)^* A$ from hypothesis. Hence, $A\Gamma M\Gamma A\subseteq N_r(B)^* A$ and A is a bi Γ-ideal of M. □

Theorem 3.2. *Let M be a Γ-nearness semiring and A be a non-empty subset of M. Each right or left Γ-ideal of M is a bi Γ-ideal of S if $N_r(B)^*\left(N_r(B)^* A\right) = N_r(B)^* A$.*

Proof. Let A be left Γ-ideal of M. Then, we have that $M\Gamma A\subseteq N_r(B)^* A$. Thus $A\Gamma M\Gamma A = A\Gamma(M\Gamma A)\subseteq A\Gamma(N_r(B)^* A)$ by hypothesis. In this case, $A\Gamma(N_r(B)^* A) \subseteq \left(N_r(B)^* A\right)\Gamma\left(N_r(B)^* A\right) \subseteq N_r(B)^*(A\Gamma A)$ by Lemma 2.1.(ii). Then, $N_r(B)^*(A\Gamma A)\subseteq N_r(B)^*(N_r(B)^* A) = N_r(B)^* A$ by hypothesis. Hence, we get that $A\Gamma M\Gamma A\subseteq N_r(B)^* A$ and A is a bi Γ-ideal of M.

Similarly, we can easily show in case A is right nearness ideal. $\square$

LEMMA 3.1. *Let M be a Γ-nearness semiring and A be a non-empty subset of M. Every bi-nearness ideal of M is an bi upper-near ideal of M if* $N_r(B)^*\left(N_r(B)^*A\right) = N_r(B)^*A$.

PROOF. Let M be a Γ-nearness semiring and A is a bi Γ-ideal of M. $(N_r(B)^*A)\Gamma M\Gamma(N_r(B)^*A) \subseteq (N_r(B)^*A)\Gamma(N_r(B)^*M)\Gamma(N_r(B)^*A)$ by Theorem 2.1.(i). In this case, we have that $\left((N_r(B)^*A)\Gamma(N_r(B)^*M)\right)\Gamma(N_r(B)^*A) \subseteq N_r(B)^*(A\Gamma M\Gamma A)$ by Lemma2.1.(ii). Since A is a bi Γ-ideal of M, we see that $N_r(B)^*(A\Gamma M\Gamma A) \subseteq N_r(B)^*\left(N_r(B)^*A\right) = N_r(B)^*A$. Therefore, $(N_r(B)^*A)\Gamma M\Gamma(N_r(B)^*A) \subseteq N_r(B)^*A$ and A is a bi upper-near ideal of M. $\square$

COROLLARY 3.1. *Let M be a Γ-nearness semiring. If M is commutative and $N_r(B)^*\left(N_r(B)^*A\right) = N_r(B)^*A$, then each quasi nearness ideal is a bi upper-near ideal.*

PROOF. The proof is obvious from Theorem 3.1 and Lemma 3.1. $\square$

THEOREM 3.3. *Let M be a Γ-nearness semiring. If M is commutative and $N_r(B)^*\left(N_r(B)^*A\right) = N_r(B)^*A$, then the product of two quasi-nearness ideals of M is a bi nearness ideal of M.*

PROOF. Let A_1 and A_2 be quasi-nearness ideals of M. We show that $(A_1\Gamma A_2)\Gamma M\Gamma(A_1\Gamma A_2) \subseteq N_r(B)^*(A_1\Gamma A_2)$. In this case, $(A_1\Gamma A_2)\Gamma M\Gamma(A_1\Gamma A_2) \subseteq (A_1\Gamma A_2)\Gamma M\Gamma(M\Gamma A_2) \subseteq (A_1\Gamma A_2)\Gamma M\Gamma(N_r(B)^*A_2)$ from Lemma 2.3. Then we have that, $(A_1\Gamma A_2)\Gamma M\Gamma(N_r(B)^*A_2) \subseteq (A_1\Gamma N_r(B)^*A_2)\Gamma M\Gamma(N_r(B)^*A_2)$ from Theorem 2.1.(i). From here, $A_1\Gamma\left((N_r(B)^*A_2)\Gamma M\Gamma(N_r(B)^*A_2)\right) \subseteq A_1\Gamma(N_r(B)^*A_2)$ since each quasi-nearness ideal is a bi upper-near ideal by Corollary 3.1. Then, we get that $A_1\Gamma(N_r(B)^*A_2) \subseteq (N_r(B)^*A_1)\Gamma(N_r(B)^*A_2)$ by Theorem 2.1.(i). Afterwards, $(N_r(B)^*A_1)\Gamma(N_r(B)^*A_2) \subseteq N_r(B)^*(A_1\Gamma A_2)$ by Lemma 2.1.(ii). Hence, $(A_1\Gamma A_2)\Gamma M\Gamma(A_1\Gamma A_2) \subseteq N_r(B)^*(A_1\Gamma A_2)$ and the product of two Γ- quasi ideals of M is a bi Γ-ideal of M. $\square$

EXAMPLE 3.2. Let $\mathcal{O} = \{p, q, r, s, t, u, w, y, z\}$ be a set of sample objects where

$$p = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, q = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, r = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, s = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$
$$t = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, u = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, w = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, y = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, z = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

for universal set $U = \{\ [a_{ij}]_{2x1} \mid a_{ij} \in \mathbb{Z}_3\}$.

Also, $\acute{\mathcal{O}} = \{\theta, \alpha, \beta, \gamma, \mu, \eta, \nu, \sigma\}$ be a set of sample objects where

$$\theta = \begin{bmatrix} 0 & 0 \end{bmatrix}, \alpha = \begin{bmatrix} 1 & 0 \end{bmatrix}, \beta = \begin{bmatrix} 0 & 1 \end{bmatrix}, \gamma = \begin{bmatrix} 2 & 2 \end{bmatrix},$$
$$\mu = \begin{bmatrix} 0 & 2 \end{bmatrix}, \eta = \begin{bmatrix} 2 & 0 \end{bmatrix}, \sigma = \begin{bmatrix} 1 & 2 \end{bmatrix}, \delta = \begin{bmatrix} 2 & 1 \end{bmatrix}, \lambda = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

for universal set $\acute{U} = \{ [a_{ij}]_{1x2} \mid a_{ij} \in \mathbb{Z}_3\}, r = 1$, $B = \{\chi_1, \chi_2, \chi_3\} \subseteq \mathcal{F}$ be a set of probe functions and $M = \{q, r, s\} \subset \mathcal{O}, \Gamma = \{\eta, \sigma\} \subset \acute{\mathcal{O}}$. Also, we take $A = \{q, r\} \subset M$

Values of the probe functions for M and A:

$$\begin{aligned}
\chi_1 &: \mathcal{O} \to V_1 = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}, \\
\chi_2 &: \mathcal{O} \to V_2 = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}, \\
\chi_3 &: \mathcal{O} \to V_3 = \{\gamma_1, \gamma_2, \gamma_3\}
\end{aligned}$$

are presented in the following table:

	p	q	r	s	t	u	w	y	z
χ_1	γ_1	γ_2	γ_1	γ_3	γ_1	γ_3	γ_4	γ_2	γ_3
χ_2	γ_1	γ_2	γ_3	γ_1	γ_3	γ_4	γ_1	γ_2	γ_1
χ_3	γ_1	γ_2	γ_2	γ_2	γ_3	γ_1	γ_3	γ_1	γ_3

Also, values of the probe functions for Γ:

$$\begin{aligned}
\chi_1 &: \mathcal{O} \to V_1 = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}, \\
\chi_2 &: \mathcal{O} \to V_2 = \{\gamma_1, \gamma_2, \gamma_3\}, \\
\chi_3 &: \mathcal{O} \to V_3 = \{\gamma_1, \gamma_2, \gamma_3\}
\end{aligned}$$

are given in the following table:

	θ	α	β	γ	μ	η	σ	δ	λ
χ_1	γ_1	γ_2	γ_1	γ_3	γ_2	γ_1	γ_3	γ_2	γ_4
χ_2	γ_1	γ_2	γ_3	γ_1	γ_2	γ_1	γ_2	γ_3	γ_3
χ_3	γ_1	γ_2	γ_1	γ_1	γ_2	γ_3	γ_2	γ_3	γ_3

We find the near equivalence classes according to the indiscernibility relation $\sim_{B_r}$ of elements in $\mathcal{O}$ and $\acute{\mathcal{O}}$:

$$\begin{aligned}
[p]_{\chi_1} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_1(p) = \gamma_1\} = \{p, r, t\} \\
&= [r]_{\chi_1} = [t]_{\chi_1}, \\
[q]_{\chi_1} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_1(q) = \gamma_2\} = \{q, y\} \\
&= [y]_{\chi_1}, \\
[s]_{\chi_1} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_1(s) = \gamma_3\} = \{s, u, z\} \\
&= [u]_{\chi_1} = [z]_{\chi_1}, \\
[w]_{\chi_1} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_1(w) = \gamma_4\} = \{w\}.
\end{aligned}$$

Then, we get $\xi_{\chi_1} = \left\{ [p]_{\chi_1}, [q]_{\chi_1}, [s]_{\chi_1}, [w]_{\chi_1} \right\}$.

$$\begin{aligned}
[p]_{\chi_2} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_2(p) = \gamma_1\} = \{p, s, w, z\} \\
&= [s]_{\chi_2} = [w]_{\chi_2} = [w]_{\chi_2} = [z]_{\chi_2}\,, \\
[q]_{\chi_2} &= \{x \in \mathcal{O} \mid \chi_2(x) = \chi_2(q) = \gamma_2\} = \{q, y\} \\
&= [y]_{\chi_1}\,, \\
[r]_{\chi_2} &= \{x \in \mathcal{O} \mid \chi_2(x) = \chi_2(r) = \gamma_3\} = \{r, t\} \\
&= [t]_{\chi_2}\,, \\
[u]_{\chi_2} &= \{x \in \mathcal{O} \mid \chi_2(x) = \chi_2(u) = \gamma_4\} = \{u\}.
\end{aligned}$$

Then, we get $\xi_{\chi_2} = \left\{[p]_{\chi_2}, [q]_{\chi_2}, [r]_{\chi_2}, [u]_{\chi_2}\right\}$.

$$\begin{aligned}
[p]_{\chi_3} &= \{x \in \mathcal{O} \mid \chi_3(x) = \chi_3(p) = \gamma_1\} = \{p, u, y\} \\
&= [u]_{\chi_3} = [y]_{\chi_3}\,, \\
[q]_{\chi_3} &= \{x \in \mathcal{O} \mid \chi_3(x) = \chi_3(q) = \gamma_2\} = \{q, r, s\} \\
&= [q]_{\chi_3} = [s]_{\chi_3}\,, \\
[t]_{\chi_3} &= \{x \in \mathcal{O} \mid \chi_3(x) = \chi_3(t) = \gamma_3\} = \{t, w, z\} \\
&= [w]_{\chi_3} = [z]_{\chi_3}\,.
\end{aligned}$$

Then, we get $\xi_{\chi_3} = \left\{[p]_{\chi_3}, [q]_{\chi_3}, [t]_{\chi_3}\right\}$.

Therefore, for $r = 1$, a set of partitions of $\mathcal{O}$ is $N_1(B) = \{\xi_{\chi_1}, \xi_{\chi_2}, \xi_{\chi_3}\}$. Then, we can write

$$\begin{aligned}
N_1(B)^* M &= \bigcup_{[x]_{\chi_i} \cap M \neq \varnothing} [x]_{\chi_i} \\
&= [p]_{\chi_1} \cup [q]_{\chi_1} \cup [p]_{\chi_2} \cup [q]_{\chi_2} \cup [r]_{\chi_2} \cup [q]_{\chi_3} \\
&= \{p, q, r, s, t, u, w, y, z\}.
\end{aligned}$$

Considering the following table of operation:

$+$	q	r	s
q	w	s	t
r	s	y	u
s	t	u	z

In that case, $(M, +)$ is an abelian monoid on $\mathcal{O}$ with identity element $0_M = p$.

$$\begin{aligned}
[\theta]_{\chi_1} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_1(\theta) = \gamma_1\} = \{\theta, \beta, \eta\} \\
&= [\beta]_{\chi_1} = [\eta]_{\chi_1}, \\
[\alpha]_{\chi_1} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_1(\alpha) = \gamma_2\} = \{\alpha, \mu, \delta\} \\
&= [\mu]_{\chi_1} = [\delta]_{\chi_1}, \\
[\gamma]_{\chi_1} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_1(\gamma) = \gamma_3\} = \{\gamma, \sigma\} \\
&= [\sigma]_{\chi_1}, \\
[\lambda]_{\chi_1} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_1(\lambda) = \gamma_4\} = \{\lambda\}.
\end{aligned}$$

Then, we get $\xi_{\chi_1} = \left\{[\theta]_{\chi_1}, [\alpha]_{\chi_1}, [\gamma]_{\chi_1}, [\lambda]_{\chi_1}\right\}$.

$$\begin{aligned}
[\theta]_{\chi_2} &= \{x \in \mathcal{O} \mid \chi_1(x) = \chi_2(\theta) = \gamma_1\} = \{\theta, \gamma, \eta\} \\
&= [\gamma]_{\chi_2} = [\eta]_{\chi_2}, \\
[\alpha]_{\chi_2} &= \{x \in \mathcal{O} \mid \chi_2(x) = \chi_2(\alpha) = \gamma_2\} = \{\alpha, \mu, \sigma\} \\
&= [\mu]_{\chi_2} = [\sigma]_{\chi_2}, \\
[\beta]_{\chi_2} &= \{x \in \mathcal{O} \mid \chi_2(x) = \chi_2(\beta) = \gamma_3\} = \{\beta, \delta, \lambda\} \\
&= [\delta]_{\chi_2} = [\lambda]_{\chi_2}.
\end{aligned}$$

Then, we get $\xi_{\chi_2} = \left\{[\theta]_{\chi_2}, [\alpha]_{\chi_2}, [\beta]_{\chi_2}\right\}$.

$$\begin{aligned}
[\theta]_{\chi_3} &= \{x \in \mathcal{O} \mid \chi_3(x) = \chi_3(\theta) = \gamma_1\} = \{\theta, \beta, \gamma\} \\
&= [\beta]_{\chi_3} = [\gamma]_{\chi_3}, \\
[\alpha]_{\chi_3} &= \{x \in \mathcal{O} \mid \chi_3(x) = \chi_3(\alpha) = \gamma_2\} = \{\alpha, \mu, \sigma\} \\
&= [\mu]_{\chi_3} = [\sigma]_{\chi_3}, \\
[\eta]_{\chi_3} &= \{x \in \mathcal{O} \mid \chi_3(x) = \chi_3(\eta) = \gamma_3\} = \{\eta, \delta, \lambda\} \\
&= [\delta]_{\chi_3} = [\lambda]_{\chi_3}.
\end{aligned}$$

Then, we get $\xi_{\chi_3} = \left\{[\theta]_{\chi_3}, [\alpha]_{\chi_3}, [\eta]_{\chi_3}\right\}$.

Therefore, for $r = 1$, a set of partitions of $\acute{\mathcal{O}}$ is $N_1(B) = \{\xi_{\chi_1}, \xi_{\chi_2}, \xi_{\chi_3}\}$. Then, we can write

$$\begin{aligned}
N_1(B)^* \Gamma &= \bigcup_{[x]_{\chi_i} \cap \Gamma \neq \varnothing} [x]_{\chi_i} \\
&= [\theta]_{\chi_1} \cup [\gamma]_{\chi_1} \cup [\theta]_{\chi_2} \cup [\alpha]_{\chi_2} \cup [\alpha]_{\chi_3} \cup [\eta]_{\chi_3} \\
&= \{\theta, \alpha, \beta, \gamma, \mu, \eta, \sigma, \delta, \lambda\}.
\end{aligned}$$

Considering the following table of operation:

$+$	η	σ
η	α	μ
σ	μ	δ

Then $(\Gamma, +)$ is an abelian monoid on $\acute{\mathcal{O}}$ with identity element $0_\Gamma = \theta$. Now, considering the following operations:

η	q	r	s
q	w	w	q
r	y	y	r
s	z	z	s

σ	q	r	s
q	q	p	q
r	r	p	r
s	s	p	s

Therefore, $(M, +, \cdot)$ is a Γ-nearness semiring. Now, we find $N_1(B)^* A$

$$\begin{aligned} N_1(B)^* A &= \bigcup_{[x]_{\chi_i} \cap A \neq \varnothing} [x]_{\chi_i} \\ &= [p]_{\chi_1} \cup [q]_{\chi_1} \cup [q]_{\chi_2} \cup [r]_{\chi_2} \cup [q]_{\chi_3} \\ &= \{p, q, r, s, t, y\}. \end{aligned}$$

For $q \in A$, $\eta \in \Gamma$ and $s \in M$, $q\eta s\eta q = (q\eta s)\eta q = q\eta q = w$. In this case, $w \in A\Gamma M\Gamma A$, but $w \notin N_r(B)^* A$. Thus, A is not a bi Γ-ideal of M.

THEOREM 3.4. *Let M be a Γ-nearness semiring, T be a subset of M and A be a nearness ideal of M. If $N_r(B)^* (N_r(B)^* A) = N_r(B)^* A$, then the product AT is bi Γ-ideal of M.*

PROOF. It is easy to show that AM is subsemigroup of M. To show this, we prove $A\Gamma T + A\Gamma T \subseteq N_r(B)^* (A\Gamma T)$. Since A is nearness ideal and by Theorem 2.1.(i), $A\Gamma T + A\Gamma T = (A + A)\Gamma T \subseteq (N_r(B)^* A)\Gamma(N_r(B)^* T)$. Afterward, $(N_r(B)^* A)\Gamma(N_r(B)^* T) \subseteq N_r(B)^* (A\Gamma T)$ by Lemma 2.1.(ii).

Next, $(A\Gamma T)\Gamma M\Gamma(A\Gamma T) \subseteq (A\Gamma M)\Gamma M\Gamma(A\Gamma T)$ for T is a subset of M. Since A is an ideal, then, $(A\Gamma M)\Gamma M\Gamma(A\Gamma T) \subseteq (N_r(B)^* A)\Gamma M\Gamma(A\Gamma T)$. Similarly, $(N_r(B)^* A)\Gamma(M\Gamma A)\Gamma T \subseteq (N_r(B)^* A)\Gamma(N_r(B)^* A)\Gamma T$. In this case, $((N_r(B)^* A)\Gamma(N_r(B)^* A))\Gamma T \subseteq N_r(B)^* (A\Gamma A)\Gamma T$ by Lemma 2.1.(ii). After this, we have that $N_r(B)^* (A\Gamma A)\Gamma T \subseteq (N_r(B)^* (N_r(B)^* A))\Gamma T = (N_r(B)^* A)\Gamma T$ for A is a nearness ideal of M and from hypothesis. Thus, $(N_r(B)^* A)\Gamma T \subseteq (N_r(B)^* A)\Gamma$ $(N_r(B)^* T) \subseteq N_r(B)^* (A\Gamma T)$ by Lemma 2.1.(ii). Hence, the product AT is bi Γ-ideal of M.

□

THEOREM 3.5. *Let M be a Γ-nearness semiring. If M is commutative, then the product of two bi-nearness ideals of M is a bi-nearness ideal of M.*

PROOF. We prove the case two bi-nearness ideals A_1 and A_2 of M. First, we show that $A_1\Gamma A_2 + A_1\Gamma A_2 \subseteq N_r(B)^* (A_1\Gamma A_2)$. To prove this,

$$A_1\Gamma A_2 + A_1\Gamma A_2 \subseteq A_1\Gamma(A_2 + A_2) \subseteq (N_r(B)^* A_1)\Gamma(N_r(B)^* A_2)$$

by Theorem 2.1.(i) and the properties of bi-nearness ideals for A_2 is bi-nearness ideal. From here, $(N_r(B)^* A_1)\Gamma(N_r(B)^* A_2) \subseteq N_r(B)^*(A_1\Gamma A_2)$ by Lemma 2.1.(ii). Next,
$(A_1\Gamma A_2)\Gamma M\Gamma(A_1\Gamma A_2) = A_1\Gamma(A_2\Gamma M)\Gamma(A_1\Gamma A_2) = A_1\Gamma(M\Gamma A_2)\Gamma(A_1\Gamma A_2) = A_1\Gamma M\Gamma(A_2\Gamma A_1)\Gamma A_2 = A_1\Gamma M\Gamma(A_1\Gamma A_2)\Gamma A_2 = (A_1\Gamma M\Gamma A_1)\Gamma(A_2\Gamma A_2)$ since M is commutative. In this case, $(A_1\Gamma M\Gamma A_1)\Gamma(A_2\Gamma A_2) \subseteq (N_r(B)^* A_1)\Gamma(N_r(B)^* A_2)$ since A_1 and A_2 are bi-nearness ideals of M. Then, $(N_r(B)^* A_1)\Gamma(N_r(B)^* A_2) \subseteq N_r(B)^*(A_1\Gamma A_2)$ by Lemma 2.1.(ii). Hence, we have $(A_1\Gamma A_2)\Gamma M\Gamma(A_1\Gamma A_2) \subseteq N_r(B)^*(A_1\Gamma A_2)$. □

THEOREM 3.6. *Let M be a Γ-nearness semiring and $\{A_i | i \in I\}$ be set of Γ bi-nearness ideals of M with index set I. If $N_r(B)^*(\bigcap_{i\in I} A_i) = \bigcap_{i\in I} N_r(B)^* A_i$, then intersection of bi Γ-ideals of M is also a bi Γ-ideal.*

PROOF. Let $\bigcap_{i\in I} A_i = A$. Since A_i is Γ bi-nearness ideals of M for all $i \in I$, then we have that $A_i\Gamma M\Gamma A_i \subseteq N_r(B)^* A_i$ for all $i \in I$. Because $A \subseteq A_i$ for all $i \in I$, thus we get $A\Gamma M\Gamma A \subseteq A_i\Gamma M\Gamma A_i \subseteq N_r(B)^* A_i$ for all $i \in I$. Afterward, $A\Gamma M\Gamma A \subseteq \bigcap_{i\in I}(N_r(B)^* A_i) = N_r(B)^*(\bigcap_{i\in I} A_i) = N_r(B)^* A$ by hypothsesis. In this case, we get that $A\Gamma M\Gamma A \subseteq N_r(B)^* A$. □

4. Conclusions

As a recent study of Γ-nearness semirings, this paper presents some notion of bi-nearness ideals of Γ-nearness semirings which is a generalization of quasi-nearness ideals of Γ-nearness semirings. Besides that, it is explained that some of the concepts and definitions and an example is given with related to the subject. By these properties, further studies of nearness semirings will be more progressive.

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