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STABILITY ANALYSIS OF A MODIFIED LOTKA-VOLTERRA EQUATIONS MODELLING COVID-19 TRANSMISSION DYNAMICS

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Abstract. In this paper, we study the propagation of covid 19 in a population. We present this phenomen as a predator-prey problem. We consider two interacting populations and we propose a modified Lotka-Volterra system modeling the COVID-19 transmission dynamics and provide a study of the stability in the neighborhood of equilibrium points.

1. Introduction

Mathematical modeling is considered to be a very powerful theoretical tool for understanding several phenomens in different fields such as physics, chemistry, biology, ecology and in particular in medicine where in several situations they allow prevention, diagnosis, and treatment. The mathematician Vito Volterra and the chemist and statistician Alfred James Lotka [**7, 10**] were the first researchers who have studied population dynamics of prey-pedator problem and modeled it by a system of ordinary equations

$$
\begin{cases}
\overline{x} = ax - bxy, \\
\overline{y} = -cy + dxy, \\
a, b, c, d \text{ are positive numbers.} \\
x: \text{ number of prey} \\
y: \text{ number of predators} \\
\overline{x}, \overline{y}: \text{ rates of the prey and predators at an instant } t.
\end{cases}
\tag{1.1}
$$

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although the problem was discussed by Lotka in 1920 and by Volterra in 1926 but their conclusions were similar, that is the interraction of the two species would give rise to periodic oscillation in their poulation [**1**]. Prey-Predators modeling has proven to be effective in understanding and solving many complex problems in different fields: chemical reaction, astronomy, economic and evolutionary game theory [**3, 9, 4**]. Later on, the system (1.1) was modified to model more complexe problems such as modeling the structure of marine phage populations, to capture the essential physics of the dusty plasma problem [**6, 8**]. In this work, we aim to model the Covid 19 transmission by casting it into a problem of population dynamics. Let us consider some population which due to its lifestyle, we assume that all people are carriers of corona verrus, but there are respectively, healthy carriers who have no complications and carriers with complications. People from this population are classified in two compartments: a compartment A which includes the first category of individuals and compartment B of the second category. Let $x(t)$ denotes the number of people in compartment A at the time t and $y(t)$ the number of people in compartiment B at the same time t. We assume that the contact of a person having no complications with a person carrying the verus and who has complications causes contamination. In other words, complications of covid 19 are transmitted by the meeting of persons from compartment A with those of B, so people in compartment B are seen as predators for the preys which are in A. In absence of predators, the growth of the prey population $x(t)$ is proportional to its size that is

$$\frac{dx}{dt} = ax, \quad a > 0.$$

In absence of prey, the predators population will decrease proportionally to its seize, that is

$$\frac{dy}{dt} = -by, \quad b > 0.$$

When people of compartiment A meet those of compartment B, a decline in the prey population and a growth in the predators population will occur, each at a rate proportional to the frequency of encounters between individuals of the two compartments ($-\alpha xy$ for the prey; $\quad + \alpha xy \quad$ for predators $\alpha > 0$). If β denote the rate of natural mortality rates, then due to the natural death, the number of individual in A and B will respectively by βx and βy. On the other hand, people in compartment B are either recoverble from complication by treatment or they die. The recovery rate from complication, the death rate due to complications are respectively denoted by η and λ, consequently the number of individuals in B decrease by $(\eta + \lambda)y$ and population in A increase by ηy. We seek to propose a mathematical model which take in account the interaction between population in compartments A, B and allowed us to study the stabilty in the in the neighbourhood of equilibrium points.

2. The main result

The model is expressed as a system of coupled equations resembling the original Lotka-Volterra equations by writing

$$\begin{cases} \overline{x} = ax - \alpha xy - \beta x + \eta y, \\ \overline{y} = -by + \alpha xy - \beta y - (\eta + \lambda)y, \\ a, b, \beta, \eta, \lambda, \alpha \ \text{ are positive numbers}, \\ a > \beta. \end{cases} \tag{2.1}$$

That is

$$\begin{cases} \overline{x} = (a - \beta)x - \alpha xy + \eta y, \\ \overline{y} = -(b + \beta + \eta + \lambda)y + \alpha xy, \\ a, b, \beta, \eta, \lambda, \alpha \ \text{ are positive numbers}, \\ a > \beta. \end{cases} \tag{2.2}$$

The modified Lotka-Volterra system comprises two nonlinear ordinary differential equations. The classical methods for the calculation of equilibrium points for x and y separately, by setting

$$\overline{x} = 0 \quad \text{and} \quad \overline{y} = 0$$

give

$$\begin{cases} 1) \ \ x^* = 0 \ \text{ and } \ y^* = 0, \\ 2) \ \ x^* = \dfrac{b + \beta + \eta + \lambda}{\alpha} \ \text{ and } \ y^* = \dfrac{(a - \beta)(b + \beta + \eta + \lambda)}{\alpha(b + \beta + \lambda)}, \end{cases} \tag{2.3}$$

where x^*, y^* denote respectively equilibrium points for system (2.2).

To get information on the local stability of the equilibrium point, we proceed to the linearization of the system (2.2) at neighborhood of the equilibrium point. Let us consider the Jacobian matrix:

$$M = \begin{bmatrix} (a - \beta) - \alpha y & -\alpha x + \eta \\ \alpha y & \alpha x - (b + \beta + \eta + \lambda) \end{bmatrix}$$

replacing x, y by x * and y * we have

$$M\left(x^*, y^*\right) = \begin{bmatrix} (a - \beta) - \alpha y^* & -\alpha x^* + \eta \\ \alpha y^* & \alpha x^* - (b + \beta + \eta + \lambda) \end{bmatrix}$$

Set $M^* = M\left(x^*, y^*\right)$. If μ is an eigenvalue of M^* it satisfies

$$\mu^2 - tr(M^*)\mu + det(M^*) = 0$$

and

$$\Delta = (tr(M^*))^2 - 4det(M^*)$$

The first case where $(x^* = 0, y^* = 0)$ is to be eliminated because it means that both populations are extinc and this can't be seen in real life. Indeed, in this case the matrix

$$M^* = M(0,0) = \begin{bmatrix} (a-\beta) & \eta \\ 0 & -(b+\beta+\eta+\lambda) \end{bmatrix}$$

$$\Delta = (a-b-2\beta-\eta-\lambda)^2 + 4(a-\beta)(b+\beta+\eta+\lambda) > 0$$

with

$$det(M^*) = -(a-\beta)(b+\beta+\eta+\lambda) < 0$$

which means that $(0,0)$ is an unstable saddle point, and explaine the inability for a population to reach a population of zero. For the second case,

$$M(x^*, y^*) = \begin{bmatrix} \dfrac{-(a-\beta)\eta}{b+\beta+\lambda} & -(b+\beta+\lambda) \\ \dfrac{(a-\beta)(b+\beta+\eta+\lambda)}{b+\beta+\lambda} & 0 \end{bmatrix}$$

$$\mu = \frac{1}{2}\left[\frac{-(a-\beta)\eta}{b+\beta+\lambda} \pm \sqrt{\left(\frac{(a-\beta)^2\eta^2}{(b+\beta+\lambda)^2}\right) - 4(a-\beta)(b+\beta+\lambda+\eta)}\right]$$

$$\Delta = \frac{(a-\beta)^2\eta^2}{(b+\beta+\lambda)^2} - 4\big(a-\beta\big)\big(b+\beta+\lambda+\eta\big).$$

$$tr(M^*) = -\frac{(a-\beta)\eta}{\beta+\lambda+b} < 0 \tag{2.4}$$

and

$$det(M^*) = (a-\beta)(\beta+\lambda+b) > 0. \tag{2.5}$$

According to the values of Δ we will discuss three cases: $\Delta = 0; \Delta < 0; \Delta > 0$

(1) If $a = 4\dfrac{(b+\beta+\lambda+\eta)(b+\beta+\lambda)^2}{\eta^2} + \beta$, by (2.4) we have a double negative eigenvalues and therefore (x^*, y^*) is a degenerate stable node.

(2) If $a < 4\dfrac{(b+\beta+\lambda+\eta)(b+\beta+\lambda)^2}{\eta^2} + \beta$, and using (2.4) we deduce that we have a stable focus and trajectories spiral as they approach of the equilibrium point which is stable.

(3) If $a > 4\dfrac{(b+\beta+\lambda+\eta)(b+\beta+\lambda)^2}{\eta^2} + \beta$, and using (2.4) and (2.5) we deduce that the two eigenvalues are negative, it is a stable node or sink. When $t \to +\infty$ all trajectories tend towards the equilibrium point which is therefore stable.

Our model was tested for some values, we present here our results.

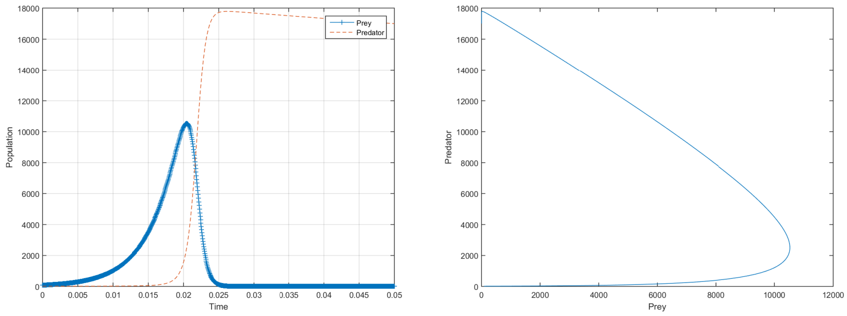


FIGURE 1. Example solution to the modified Lotka-Volterra equations, showing simulated behaviour of a predator-prey relationship applicable to dusty plasma populations. Parameters used were: $a = 250.3547; b = 2.; \alpha = 0.1; \beta = 0.01; \lambda = 0.02; \eta = 0.4$ and the starting point for calculation was $(x^*, y^*) = (24.30, 299.67)$.

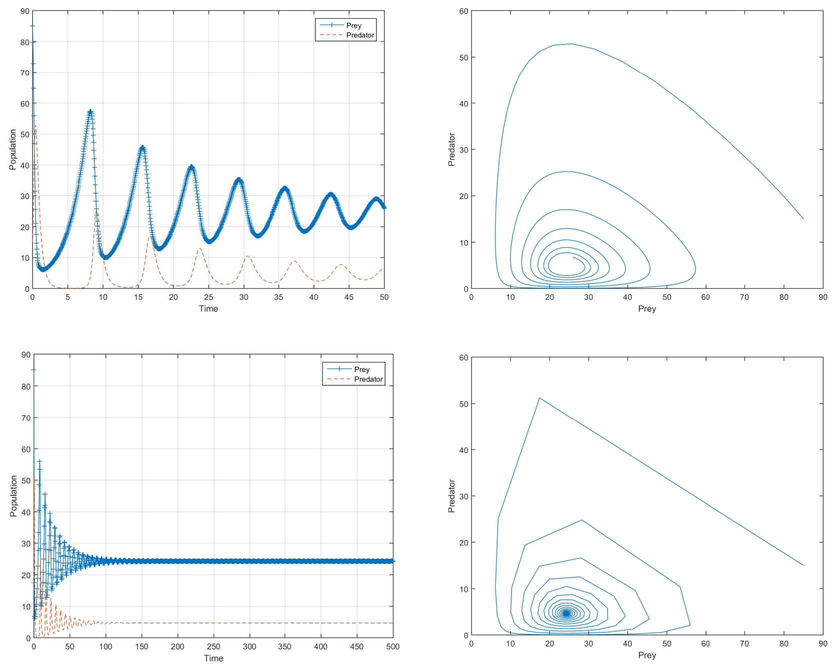
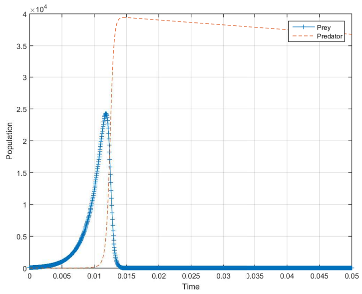


FIGURE 2. Example solution to the modified Lotka-Volterra equations, showing simulated behaviour of a predator-prey relationship applicable to dusty plasma populations. Parameters used were: $a = 0.4; b = 2.; \alpha = 0.1; \beta = 0.01; \lambda = 0.02; \eta = 0.4$ and the starting point for calculation was $(x^*, y^*) = (24.30, 0.4668)$.



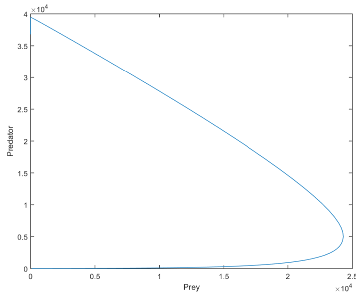


FIGURE 3. Example solution to the modified Lotka-Volterra equations, showing simulated behaviour of a predator-prey relationship applicable to dusty plasma populations. Parameters used were: $a = 500.; b = 2.; \alpha = 0.1; \beta = 0.01; \lambda = 0.02; \eta = 0.4$ and the starting point for calculation was $(x^*, y^*) = (24.30, 598.5102)$.

3. Conclusion

In this work we have modeled the propagation of covid 19 in a population. We have modified the model of Lotka-Voltera for population dynamic to study the stabilty of our proposed model in the neighborhood of the equilibrium point. We have seen that the cases of local stability are determined by the two condition $det(M^*) > 0$ and $tr(M^*) < 0$. Under these conditions any trajectory resulting from an initial condition taken in the neighborhood of equilibrium point will return to the equilibrium point (node or stable focus). All our results are based on the assumption that the rate of population growth in Compartiment A is upper than the rate of natural death in the same compartiment. If for some reasans this condition is droped so that $a < \beta$, in this case the two eigenvalues have opposite signs, we will have a saddle point. When $t \to +\infty$ the population of predators tends to ∞ while the other population tends to x^*. The equilibrium point is unstable. Let us recal that because $det(M^*) \neq 0$ we must have $a \neq \beta$.

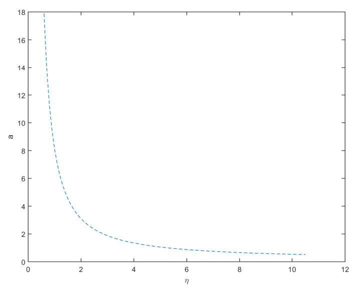


FIGURE 4. The range of behaviours of solutions to function of a par η where $b = 0.88; \beta = 0.1;$ and $\lambda = 0.02$.

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