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SEMI-DERIVATIONS ON HYPERRINGS

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Abstract. In this paper, we deal with Krasner hyperrings, a special type of hyperrings. We introduce and analyze the semi-derivation on prime hyperrings. Moreover, we prove that the semi-derivations of prime hyperrings are derivations or hypersemi-derivations of the form $f(r) = \lambda(r - g(r))$ for all $r \in R$, where λ is an element in the extended centroid of R.

1. Introduction

The study of derivations started in rings after Posner [**7**], who give remarkable results on derivations of prime rings. In the following years, the notion of a derivation was developed by many authors.

Semi-derivation concept in rings was given by J. Bergen in [**2**]. An additive mapping f of a ring R into R is called a semi-derivation if there exists a function $g : R \to R$ such that $f(rs) = f(r)g(s) + rf(s) = f(r)s + g(r)f(s)$ and $f(g(r)) = g(f(r))$ for all $r, s \in R$. In [**3**], Bresar obtained the structure of semi-derivations of prime rings.

The notion of Martindale rings of quotients, jointly with the notion of the extended centroid, play an important role in the study of prime rings satisfying a generalized polynomial identity, see [**12**]. There exist many results concerning the relationship between the quotient ring and the existence of certain specific types of the derivation of the ring.

Hyperstructures are known the algebraic structures equipped with at least one hyperoperation. The concept of multi-valued binary operations was introduced by Marty [**11**] and has been studied in the following decades by many mathematicians. The theory of hypergroups appears in [**4**]. Krasner [**10**] introduced the notion of

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hyperrings and hyperfields. A well-known type of a hyperring is called the Krasner hyperring. Krasner hyperring is an essential ring with approximately modified axioms in which addition is hyperoperation, while the multiplication is an operation. This type of hyperrings has been studied by a variety of authors. The book given in [**5**] is one of the sources for the applications of the theory of hyperstructures and the book given in [**6**] is a detailed study of hyperrings. The general structure of ordered Krasner hyperrings is studied in [**15**]. The bi-hyperideal, prime bi-hyperideal structure on such hyperrings can be found in [**13**]. The concept of inner hyperideal in ordered Krasner hyperrings is introduced and intra-regular ordered Krasner hyperrings are characterized according to the properties of these inner hyperideals in [**14**]. In [**20**], some basic concepts of hyperrings such as dense hyperideal, essential hyperideal, singular hyperideal have been defined and some results have been proven.

In [**1**], Asokkumar studied derivations in Krasner hyperrings and gives examples. Krasner hyperrings [**8**], ordered Krasner hyperrings [**17**], (ordered) hyper (near)-rings [**16**] are articles in which basic connections are established between hyperrings and derivations. In [**18**], m-k-hyperideals in semihyperrings were defined and some important properties of derivations and m-k-hyperideals are investigated. The Galois connection between ordered semihyperrings, which is a generalization of ordered semirings and semihyperrings, has been studied in detail and various interesting results have been obtained in [**9**].

In [**19**], the notation of extended centroid is applied to hyperring and we show that the extended centroid C of a hyperring is a hyperfield. Also, we show that if $a,\ b \in S$ such that $axb = bxa$ for all $x \in R$, then there exists $q \in C$ such that $qa = b$ where R is a prime hyperring and S is the central closure of R.

In this paper, we introduce semi-derivations in Krasner hyperrings and examine some of its basic properties.

2. Preliminaries

A mapping $\circ : H \times H \to P^*(H)$ is called a hyperoperation, where $P^*(H)$ is the set of non-empty subsets of H. An algebraic system $(H, \circ)$ is called a hypergroupoid.

Let A and B be two non-empty subsets of H and $h \in H$, we define

$$A \circ B = \bigcup_{\substack{a\in A\\ b\in B}} a \circ b\ ,\ A \circ h = A \circ \{h\}\ \text{ and }\ h \circ B = \{h\} \circ B.$$

A hyperoperation "$\circ$" is called associative if $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in H$, which means that

$$\bigcup_{u\in b\circ c} a \circ u = \bigcup_{v\in a\circ b} v \circ c.$$

A hypergroupoid with the associative hyperoperation is called a semihypergroup.

A hypergroupoid $(H, \circ)$ is a quasihypergroup, whenever $h \circ H = H = H \circ h$ for all $h \in H$. If $(H, \circ)$ is semihypergroup and quasihypergroup, then $(H, \circ)$ is called a hypergroup.

DEFINITION 2.1 ([**6**]). A Krasner hyperring is an algebraic structure $(R, +, \cdot)$ which satisfies the following axioms:
(1) $(R, +)$ is a canonical hypergroup, i.e.,
(i) $(r+s)+t = r+(s+t)$ for every $r, s, t \in R$,
(ii) $r+s = s+r$, for every $r, s \in R$,
(iii) For all $r \in R$, there exists $0 \in R$ such that $0+r = \{r\}$,
(iv) There exists an unique element denoted by $-r \in R$ for every $r \in R$ such that $0 \in r+(-r)$,
(v) $t \in r+s$ implies $s \in -r+t$ and $r \in t-s$, for every $r, s, t \in R$,
(2) $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element, i.e.,
(i) $(r \cdot s) \cdot t = r \cdot (s \cdot t)$, for every $r, s, t \in R$,
(ii) For all $r \in R$, $r \cdot 0 = 0 \cdot r = 0$,
(3) The multiplication is distributive with respect to the hyperoperation $+$, i.e., for every $r, s, t \in R$, $r \cdot (s+t) = r \cdot s + r \cdot t$ and $(r+s) \cdot t = r \cdot t + s \cdot t$.

Since $-A$ is defined by the set $\{-a \mid a \in A\}$, we have $-(-r) = r$ and $-(r+s) = -r-s$. In definition, for simplicity of notations we write sometimes rs instead of $r \cdot s$ and in (iii), $0+r = r$ instead of $0+r = \{r\}$.

If there exists an element $1 \in R$ such that $1a = a1 = a$ for every $a \in A$ in a hyperring R, then the element 1 is called the identity element of the hyperring R. The hyperring R is called a commutative hyperring, if $rs = sr$ for every $r, s \in R$. The center of R denoted by $Z(R) = \{r \in R \mid rs = sr \text{ for all } s \in R\}$. A hyperring R is called a hyperdomain if R does not have zero divisors. In other words, for $r, s \in R$ if $rs = 0$ then either $r = 0$ or $s = 0$. A Krasner hyperring is called a Krasner hyperfield, if $(R \backslash \{0\}, \cdot)$ is a group. Throughout this paper, by a hyperring we mean that Krasner hyperring.

Let R be a hyperring. A non-empty subset S of R is called a subhyperring of R, if $s_1 - s_2 \subseteq S$ and $s_1 s_2 \in S$ for all $s_1, s_2 \in S$.

A subhyperring I of a hyperring R is a left (resp. right) hyperideal of R if $ra \in I$ (resp. $ar \in I$) for all $r \in R$, $a \in I$. A hyperideal of R is both a left and a right hyperideal.

Let A and B be non-empty subsets of a hyperring R

$$A + B = \{x \mid x \in a + b \text{ for some } a \in A,\, b \in B\}$$

and

$$AB = \left\{ x \mid x \in \sum_{i=1}^{n} a_i bi,\, a_i \in A,\, b_i \in B,\, n \in \mathbb{Z}^+ \right\}.$$

A hyperring R is called a prime hyperring if $rRs = 0$ for all $r, s \in R$ implies $r = 0$ or $s = 0$. A hyperring R is said to be $2-$torsion free if $0 \in r+r$ for all $r \in R$ implies $r = 0$.

EXAMPLE 2.1. Let $R = \{0, x, y\}$ with the hyperoperation and the multiplication given in the following tables:

$+$	0	x	y
0	0	x	y
x	x	x	R
y	y	R	y

and

$\cdot$	0	x	y
0	0	0	0
x	0	x	y
y	0	y	x

Then, R is a prime and $2-$torsion free hyperring.

DEFINITION 2.2 ([**6**]). Let R_1 and R_2 be hyperrings. A mapping φ from R_1 into R_2 is said to be a strong homomorphism if for all $r, s \in R_1$,

$$\varphi(r + s) = \varphi(r) + \varphi(s),\ \varphi(rs) = \varphi(r)\varphi(s) \text{ and } \varphi(0) = 0.$$

If a strong homomorphism φ is one to one and onto, then φ is an isomorphism. If there exists isomorphism between hyperrings R_1 and R_2, we write $R_1 \cong R_2$.

DEFINITION 2.3 ([**1**]). Let R be a hyperring. A map $d : R \to R$ is said to be a derivation of R if d satisfies:
(i) $d(r + s) \subseteq d(r) + d(s)$
(ii) $d(rs) \in d(r)s + rd(s)$ for all $r, s \in R$.
If the map d is such that $d(r + s) = d(r) + d(s)$ and $d(rs) \in d(r)s + rd(s)$ for all $r, s \in R$, then d is called a strong derivation of R.

3. Semi-derivations On Hyperrings

DEFINITION 3.1. Let R be a hyperring. A map $f : R \to R$ is said to be a semi-derivation associated with a function $g : R \to R$ if for all $r, s \in R$,
(i) $f(r + s) \subseteq f(r) + f(s)$;
(ii) $f(rs) \in f(r)g(s) + rf(s) = f(r)s + g(r)f(s)$;
(iii) $f(g(r)) = g(f(r))$.

The hyperring R equipped with a semi-derivation f is called an $f-$semi-differential hyperring. If the map f satisfies the conditions $(ii), (iii)$ and $f(r+s) = f(r) + f(s)$ for all $r, s \in R$, then f is called a strong semi-derivation of R and the hyperring is called strongly $f-$semi-differential hyperring. Obviously, every derivation is a semi-derivation.

EXAMPLE 3.1. Let $R = \{r, s, t, w\}$ be a set with the hyperaddition $\oplus$ and the multiplication $\odot$ defined as follow:

$\oplus$	r	s	t	w
r	r	s	t	w
s	s	R	$\{s,t,w\}$	$\{s,t,w\}$
t	t	$\{s,t,w\}$	R	$\{s,t,w\}$
w	w	$\{s,t,w\}$	$\{s,t,w\}$	R

$\odot$	r	s	t	w
r	r	r	r	r
s	r	s	t	w
t	r	t	w	s
w	r	w	s	t

Then $(R, \oplus, \odot)$ is a Krasner hyperring ([**13**]). We define a map $f : R \to R$ by $f(r) = r, f(s) = t, f(t) = w, f(w) = s$. Here f is a strong semi-derivation of R, where $g : R \to R$ defined by $g(r) = r, g(s) = w, g(t) = s, g(w) = t$.

We define a map $f^* : R \to R$ by $f^*(r) = r,\ f^*(s) = w,\ f^*(t) = w,\ f^*(w) = s$. If $g : R \to R$ is identity map, we have f^* is a semi-derivation of R. Since $f^*(s + w) = \{s, w\} \subsetneq f^*(s) + f^*(w) = \{s, t, w\}$, f^* is not a strong semi-derivation of R.

EXAMPLE 3.2. Let $(R, +, \cdot)$ be a hyperring and $H(R) = \left\{\begin{pmatrix} r & s \\ 0 & 0 \end{pmatrix} \mid r, s \in R\right\}$. A hyperaddition $\oplus$ is defined on $H(R)$ by

$$\begin{pmatrix} r_1 & s_1 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} r_2 & s_2 \\ 0 & 0 \end{pmatrix} = \left\{\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \mid x \in r_1 + r_2,\ y \in s_1 + s_2\right\}$$

for all $\begin{pmatrix} r_1 & s_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} r_2 & s_2 \\ 0 & 0 \end{pmatrix} \in H(R)$. Now, we define a multiplication $\otimes$ on $H(R)$ by

$$\begin{pmatrix} r_1 & s_1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} r_2 & s_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} s_1 r_2 & s_1 s_2 \\ 0 & 0 \end{pmatrix}$$

for all $\begin{pmatrix} r_1 & s_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} r_2 & s_2 \\ 0 & 0 \end{pmatrix} \in H(R)$. Clearly, $H(R)$ is a Krasner hyperring.

Let $f : H(R) \to H(R)$ defined by $f\left(\begin{pmatrix} r & s \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}$ and we define a function g on $H(R)$ by $g\left(\begin{pmatrix} r & s \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & s \\ 0 & 0 \end{pmatrix}$. We will show that f is a strong semi-derivation.

For all $\begin{pmatrix} r_1 & s_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} r_2 & s_2 \\ 0 & 0 \end{pmatrix} \in H(R)$, the set

$$f\left(\begin{pmatrix} r_1 & s_1 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} r_2 & s_2 \\ 0 & 0 \end{pmatrix}\right) = f\left(\left\{\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \mid x \in r_1 + r_2, y \in s_1 + s_2\right\}\right)$$
$$= \left\{\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \mid x \in r_1 + r_2\right\}$$

and

$$f\left(\begin{pmatrix} r_1 & s_1 \\ 0 & 0 \end{pmatrix}\right) \oplus f\left(\begin{pmatrix} r_2 & s_2 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} r_1 & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} r_2 & 0 \\ 0 & 0 \end{pmatrix}$$
$$= \left\{\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \mid x \in r_1 + r_2\right\}.$$

Also, we have

$$f\left(\begin{pmatrix} r_1 & s_1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} r_2 & s_2 \\ 0 & 0 \end{pmatrix}\right) = f\left(\begin{pmatrix} s_1 r_2 & s_1 s_2 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} s_1 r_2 & 0 \\ 0 & 0 \end{pmatrix}$$
$$= \left\{f\left(\begin{pmatrix} r_1 & s_1 \\ 0 & 0 \end{pmatrix}\right) \otimes g\left(\begin{pmatrix} r_2 & s_2 \\ 0 & 0 \end{pmatrix}\right)\right\}$$
$$\oplus \left\{\begin{pmatrix} r_1 & s_1 \\ 0 & 0 \end{pmatrix} \otimes f\left(\begin{pmatrix} r_2 & s_2 \\ 0 & 0 \end{pmatrix}\right)\right\}$$
$$= \left\{f\left(\begin{pmatrix} r_1 & s_1 \\ 0 & 0 \end{pmatrix}\right) \otimes \begin{pmatrix} r_2 & s_2 \\ 0 & 0 \end{pmatrix}\right\}$$
$$\oplus \left\{g\left(\begin{pmatrix} r_1 & s_1 \\ 0 & 0 \end{pmatrix}\right) \otimes f\left(\begin{pmatrix} r_2 & s_2 \\ 0 & 0 \end{pmatrix}\right)\right\}.$$

As the last condition, we can see that $f\left(g\left(\begin{pmatrix} r & s \\ 0 & 0 \end{pmatrix}\right)\right) = g\left(f\left(\begin{pmatrix} r & s \\ 0 & 0 \end{pmatrix}\right)\right)$ for all $\begin{pmatrix} r & s \\ 0 & 0 \end{pmatrix} \in H(R)$. Thus, f is a strong semi-derivation of $H(R)$.

Lemma 3.1. *Let R be a prime hyperring and $f : R \to R$ be a semi-derivation associated with a surjective function g and $r \in R$. If $rf(x) = 0$ (or $f(x)r = 0$) for all $x \in R$, then either $r = 0$ or $f = 0$.*

Proof. Let $s, t \in R$ and suppose $rf(x) = 0$ for all $x \in R$. Then, we get

$$\begin{aligned} 0 &= rf(st) \in r(f(s)g(t) + sf(t)) \\ &= rf(s)g(t) + rsf(t) = rsf(t). \end{aligned}$$

By the primeness of R, we have $r = 0$ or $f(t) = 0$ for all $t \in R$, i.e., $f = 0$. When $f(x)r = 0$, the proof is similar. □

Lemma 3.2. *Let R be a $2-$torsion free prime hyperring and f be a semi-derivation associated with a surjective function g. If $f^2 = 0$, then $f = 0$*

Proof. Let $f^2 = 0$. For any $r, s \in R$, we have

$$\begin{aligned} 0 &= f^2(rs) = f(f(rs)) \in f(f(r)g(s) + rf(s)) \subseteq f(f(r)g(s)) + f(rf(s)) \\ &\subseteq f^2(r)g^2(s) + f(r)f(g(s)) + f(r)g(f(s)) + rf^2(s) \\ &= f(r)f(g(s)) + f(r)f(g(s)). \end{aligned}$$

Since R is 2-torsion free hyperring, we get $f(r)f(g(s)) = 0$. Since g is a surjective function and R is prime, we get $f(r) = 0$ for all $r \in R$. Hence, we have $f = 0$. □

Lemma 3.3. *Let f_1 and f_2 be semi-derivations of a $2-$torsion free prime hyperring R associated with the surjective functions g_1 and g_2, respectively. If $f_1f_2 = 0$, then $f_1 = 0$ or $f_2 = 0$.*

Proof. Let $r, s \in R$. We have

$$\begin{aligned} 0 &= (f_1f_2)(rs) = f_1(f_2(rs)) \in f_1(f_2(r)g_2(s) + rf_2(s)) \\ &\subseteq f_1(f_2(r)g_2(s)) + f_1(rf_2(s)) \\ &\subseteq (f_1f_2)(r)(g_1g_2)(s) + f_2(r)(f_1g_2)(s) + f_1(r)(g_1f_2)(s) + r(f_1f_2)(s) \\ &= f_2(r)(f_1g_2)(s) + f_1(r)(g_1f_2)(s). \end{aligned}$$

Replacing r by $f_2(r)$, we obtain $0 \in f_2(f_2(r))(f_1g_2)(s) + (f_1f_2)(r)(g_1f_2)(s)$. From hypothesis, $0 = f_2^2(r)f_1(g_2(s))$. Since g_2 is a surjective function, we get $f_2^2(r)f_1(t) = 0$. Replacing t by tw, using the primeness of R, we have $f_2^2 = 0$ or $f_1 = 0$. If $f_2^2 = 0$, then we get $f_2 = 0$, by Lemma 3.2. Thus, we have $f_1 = 0$ or $f_2 = 0$. The proof of the lemma is completed. □

Lemma 3.4. *Let R be a prime hyperring and C be the extended centroid of R. Suppose that f_1 and f_2 are semi-derivations of R such that $f_1(r)sf_2(t) = f_2(r)sf_1(t)$ for all $r, s, t \in R$ and $f_1 \neq 0$. Then, there exists $\lambda \in C$ such that $f_2(r) = \lambda f_1(r)$ for all $r \in R$.*

Proof. Given $r \in R$, we have $f_1(r)sf_2(r) = f_2(r)sf_1(r)$ for all $s \in R$. If $f_1(r) \neq 0$, then we have $f_2(r) = \lambda(r)f_1(r)$ for some $\lambda(r)$ in C, by [**19**, Theorem 3.6]. If $r, t \in R$ such that $f_1(r) \neq 0$ and $f_1(t) \neq 0$, then we can write

$$\begin{aligned} 0 &\in \lambda(t)f_1(r)sf_1(t) - \lambda(t)f_1(r)sf_1(t) = \lambda(t)f_1(r)sf_1(t) - f_1(r)sf_2(t) \\ &= \lambda(t)f_1(r)sf_1(t) - f_2(r)sf_1(t) = \lambda(t)f_1(r)sf_1(t) - \lambda(r)f_1(r)sf_1(t). \end{aligned}$$

Thus, we get $0 \in (\lambda(t) - \lambda(r))f_1(r)sf_1(t)$. Hence, there exists $\alpha \in \lambda(t) - \lambda(r)$ such that $0 = \alpha f_1(r)sf_1(t)$. Since R is a prime hyperring, we conclude by Lemma 3.1 that $\alpha = 0$. It means $\lambda(r) = \lambda(t)$ for all $r, t \in R$. For $f_1(r) \neq 0$, the proof is completed. Let $f_1(r) = 0$. Since $f_1 \neq 0$ and R is prime, we get $f_2(r) = 0$ for all $r \in R$. Therefore, we obtain $f_2(r) = \lambda f_1(r)$ for all $r \in R$. □

Remark 3.1. Let R be a hyperring. For all $r, s \in R$, we will denote the commutator sets $[r, s] = rs - sr$ and $(r, s) = rs + sr$. The following conditions hold: for all $r, s, t \in R$,

(*i*) $[r + s, t] = [r, t] + [s, t]$,
(*ii*) $[rs, t] \subseteq [r, t]s + r[s, t] = r[s, t] + [r, t]s$,
(*iii*) $(r + s, t) = (r, t) + (s, t)$,

(iv) $(rs,t) \subseteq (r,t)s + r[s,t] = r(s,t) - [r,t]s$.

PROOF. (i) For all $r, s, t \in R$,

$$\begin{aligned}[r+s,t] &= (r+s)t - t(r+s) = rt + st - tr - ts \\ &= rt - tr + st - ts = [r,t] + [s,t].\end{aligned}$$

(ii) Let $r, s, t \in R$. Then, we have

$$\begin{aligned}[rs,t] &= (rs)t - t(rs) = rst - trs + \{0\} \\ &\subseteq rts - trs + rst - rts \\ &= (rt - tr)s + r(st - ts) \\ &= [r,t]s + r[s,t].\end{aligned}$$

Similarly, one can see that $[rs,t] \subseteq r[s,t] + [r,t]s$. Therefore, we get $[rs,t] \subseteq [r,t]s + r[s,t] = r[s,t] + [r,t]s$.

The properties (iii) and (iv) are similar with the proof (i) and (ii), respectively. So, we omit them. □

THEOREM 3.1. *Let R be a $2-$torsion free prime hyperring, f be a non-zero semi-derivation of R associated with surjective function g and $r \in R$. If $[r, f(x)] = 0$ for all $x \in R$, then $r \in Z(R)$.*

PROOF. Suppose that $r \notin Z(R)$. We have

$$\begin{aligned}0 &= [r, f(xf(y))] \subseteq [r, f(x)f(y) + g(x)f^2(y)] \\ &= [r, f(x)f(y)] + [r, g(x)f^2(y)] \\ &\subseteq [r, f(x)]f(y) + f(x)[r, f(y)] + [r, g(x)]f^2(y) + g(x)[r, f^2(y)]\end{aligned}$$

by Remark 3.1. Since g is surjective, replacing $g(x)$ by z and $f(y) = t$, we have $0 \in [r,z]f(t)$ for all $z, t \in R$. This implies that $f(t) = 0$ or $0 \in [r,z]$. Since we assume $r \notin Z(R)$, we have $f(t) = 0$ for all $t \in R$. This leads a contradiction and so we obtain the desired result. □

THEOREM 3.2. *Let R be a $2-$torsion free prime hyperring, f be a non-zero semi-derivation of R associated with surjective function g and $r \in R$. If $(f(R), r) = 0$ then $f((R, r)) = 0$.*

PROOF. Firstly, we will show that $f(r) = 0$. If $r = 0$, then $f(r) = 0$. Hence, we suppose $r \neq 0$. From our hypothesis, $(f(x), r) = 0$ for all $x \in R$. Then, we get

$$\begin{aligned}0 &= (f(xr), r) \subseteq (f(x)g(r) + xf(r), r) \\ &= (f(x)g(r), r) + (xf(r), r) \\ &\subseteq f(x)[g(r), r] + (f(x), r)g(r) + x(f(r), r) - [x, r]f(r)\end{aligned}$$

and

$$0 \in [x, r]f(r) \text{ for all } x \in R.$$

Replacing x by xy, we get $0 \in [x,r]yf(r)$ for all $x,y \in R$. From this relation, we obtain $0 = sRf(r)$ for some $s \in [x,r]$. The primeness of R implies that $r \in Z(R)$ or $f(r) = 0$. We assume $r \in Z(R)$. Then, we get

$$0 = (f(r), r) = rf(r) + rf(r).$$

Since R is $2-$torsion free, $0 = rf(r)$. Since we assumed that $r \neq 0$, we get $f(r) = 0$, by Lemma 3.1. Thus, for all $x \in R$,

$$\begin{aligned} f((x,r)) &= f(xr + rx) \subseteq f(xr) + f(rx) \\ &\subseteq (f(x), r) + (g(x), f(r)) = \{0\} \end{aligned}$$

Therefore, we get $f((R,r)) = 0$. □

DEFINITION 3.2. Let R be a hyperring and $P^*(R)$ be the set of non-empty subsets of R. A map $f : R \to P^*(R)$ is said to be a hypersemi-derivation associated with function $g : R \to R$ if it satisfies: for all $r,s \in R$,

(i) $f(r+s) \subseteq f(r) + f(s)$;

(ii) $f(rs) \subseteq f(r)g(s) + rf(s) = f(r)s + g(r)f(s)$;

(iii) $f(g(r)) = g(f(r))$.

If the map f satisfies $f(r+s) = f(r) + f(s)$ for all $r,s \in R$ and the conditions $(ii), (iii)$ then f is called a strong hypersemi-derivation of R.

EXAMPLE 3.3. Consider the Krasner hyperring given in Example 2. We define $f : R \to P^*(R)$ by $f(r) = r$, $f(s) = \{s,t\}$, $f(t) = \{t,w\}$, $f(w) = \{s,w\}$ and $g : R \to R$ by $g(r) = r$, $g(s) = t$, $g(t) = w$, $g(w) = s$. Then, f is a hypersemi-derivation associated with function g.

EXAMPLE 3.4. Let R be a hyperring. We define a map $f : R \to P^*(R)$ by $f(r) = r - g(r)$ for each $r \in R$, where g is a strong endomorphism. Clearly, the map f is well defined. Let $x \in f(r+s)$. Then, $x \in f(t)$ for some $t \in r+s$ and so $x \in t - g(t) \subseteq r + s - g(r+s) = r - g(r) + s - g(s) = f(r) + f(s)$. Thus, $f(r+s) \subseteq f(r)+f(s)$ for all $r,s \in R$. Now, $f(rs) = rs - g(rs) = rs - g(r)g(s) + 0 \subseteq rs - rg(s) + rg(s) - g(r)g(s) = r(s - g(s)) + (r - g(r))g(s) = rf(s) + f(r)g(s)$. Then, we get $f(rs) \subseteq f(r)g(s) + rf(s)$. Similarly, we can see that $f(rs) \subseteq f(r)s + g(r)f(s)$ for all $r,s \in R$. If $x \in f(g(r))$, then $x \in g(r) - g(g(r)) = g(r - g(r)) = g(f(r))$. Also, if $x \in g(f(r))$, then $x \in g(r - g(r)) = g(r) - g(g(r)) = f(g(r))$. Hence, for all $r \in R$ we have $f(g(r)) = g(f(r))$. Therefore, f is a hypersemi-derivation.

In the following theorem, we give the structure of a semi-derivation in a prime hyperring.

THEOREM 3.3. *Let R be a prime hyperring and f be a semi-derivation of R associated with strong endomorphism $g : R \to R$. Then one of the following holds:*

(i) *f is an ordinary derivation,*

(ii) *f is a hypersemi-derivation and there exists a $\lambda \in C$ such that $f(r) = \lambda(r - g(r))$ for all $r \in R$, where C is the extended centroid of R.*

PROOF. From definition of semi-derivation, we can write

$$0 \in f(r)(s - g(s)) - (r - g(r))f(s) \text{ for all } r,s \in R.$$

In particular, $0 \in f(r)(st - g(st)) - (r - g(r))f(st)$ for all $r, s, t \in R$. Here, $f(r)(st - g(st)) \subseteq f(r)st - f(r)g(s)g(t) \subseteq f(r)(s - g(s))g(t) + f(r)s(t - g(t))$ and $(r - g(r))f(st) \subseteq (r - g(r))f(s)g(t) + (r - g(r))sf(t)$ for all $r, s, t \in R$. Thus, we have

$$\begin{aligned} 0 &\in f(r)(s - g(s))g(t) + f(r)s(t - g(t)) \\ &\quad -(r - g(r))f(s)g(t) - (r - g(r))sf(t) \\ &= [f(r)(s - g(s)) - (r - g(r))f(s)]g(t) \\ &\quad +f(r)s(t - g(t)) - (r - g(r))sf(t) \end{aligned}$$

and so we get $0 \in f(r)s(t - g(t)) - (r - g(r))sf(t)$. If $g = 1$, the identity map on R, f is a derivation. Hence, we suppose that the set $1 - g \neq 0$. As a result, we obtain that $f(r)s(1 - g)(t) = (1 - g)(r)sf(t)$ for all $r, s, t \in R$. The desired result is obtained by Lemma 3.4. □

4. Conclusions

In this paper, we study the concept of semi-derivations on prime hyperrings and present some result in this respect. Defining the semi-derivations in hyperrings and examining its general structure is the main subject of this article. We have shown that if f is a semi-derivation on prime hyperring R, then f is derivation or f is a hypersemi-derivation of the form $f(r) = \lambda(r - g(r))$ for all $r \in R$, where λ is an element in the extended centroid of R. This work is a pioneer in studies on structures such as ideals, homomorphisms of a hyperring with the help of semi-derivation notation. Also, the generalized semi-derivation concept can be studied on hyperrings. We hope that this paper would offer further study of the semi-derivations on hyperrings.

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