BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Bull. Int. Math. Virtual Inst., **12**(2)(2022), 297-307
DOI: 10.7251/BIMVI2202297O

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

EIGENVALUE INTERVALS FOR ITERATIVE SYSTEMS OF THE SECOND-ORDER IMPULSIVE BOUNDARY VALUE PROBLEM

Dondu Oz and Ilkay Yaslan Karaca

ABSTRACT. The goal of this study is to find the eigenvalue intervals of the parameters $\lambda_1, \lambda_2, ..., \lambda_n$ for which positive solutions exist in the second-order impulsive boundary value problem's iterative systems. To arrive at our conclusions, we utilize the fixed point theorem. An example is also provided to show the application of the main results.

1. Introduction

Because it is substantially richer than the comparable theory of differential equations without impulsive effects, it is commonly accepted that the theory and applications of differential equations with impulsive effects are an important area of research. Impulsive differential equations can be used to express a variety of models, including population, ecology, biological systems, pharmacokinetics, biotechnology, and optimal control. Control theory, electronics, chemistry, mechanics, economics, medicine, electrical circuits, and population dynamics all benefit from impulsive differential equations. For an introduction to the general theory of impulsive differential equations, see references [**1, 2, 3, 12, 21, 22**], and for applications of impulsive differential equations, see references [**6, 15**].

In the literature, several researchers have examined at second-order impulsive boundary value problems; for a list, see [**5, 8, 10, 13, 14, 25, 26, 27**] in references.

2010 *Mathematics Subject Classification.* Primary 34B37; Secondary 34B18, 34B15.
Key words and phrases. Impulsive boundary value problems, positive solutions, fixed point theorems, iterative systems, eigenvalue interval, Green's function.
Communicated by Dusko Bogdanic.

In [**8**], Hu et al. studied the following nonlinear second-order impulsive differential equations:

$$\begin{cases} -u''(t) = h(t)f(t,u), \ t \in J' = [0,1]\backslash\{t_1, t_2, ..., t_m\}, \\ -\triangle u'|_{t=t_k} = I_k(u(t_k)), \\ \triangle u|_{t=t_k} = \bar{I}_k(u(t_k)), \quad k = 1,2,...,m, \\ \alpha u(0) - \beta u'(0) = 0, \\ \gamma u(1) + \delta u'(1) = 0, \end{cases}$$

where $\alpha, \beta, \gamma, \delta \geqslant 0$, $\rho = \beta\gamma + \alpha\gamma + \alpha\delta > 0$, $J = (0,1)$, $0 < t_1 < t_2 < ... < t_m < 1$, $J' = J\backslash\{t_1, t_2, ..., t_m\}$, $\bar{J} = [0,1]$, $J_0 = (0, t_1]$, $J_1 = (t_1, t_2]$, ..., $J_m = (t_m, 1)$, $f \in C(\bar{J} \times \mathbb{R}^+, \mathbb{R}^+)$, $I_k, \bar{I}_k \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\mathbb{R}^+ = [0, +\infty)$. The authors showed that the existence of one or two positive solutions are established by using the fixed point index theorem in cone.

In addition, some researchers have been interested in systems of the second-order impulsive boundary value problems; we recommend the reader to [**5, 13, 14**] for further information on these. Obtaining optimal eigenvalue intervals for the existence of positive solutions to iterative systems with nonlinear boundary value problems, on the other hand, has attracted interest due to the importance of both theory and applications. This line includes papers like [**4, 7, 9, 11, 16, 17, 18, 19, 20, 23**].

We consider the following iterative system of nonlinear second-order impulsive boundary value problem (IBVP) in this work, which is motivated by the above-mentioned result:

$$\begin{cases} y_i''(t) + \lambda_i p_i(t) g_i(y_{i+1}(t)) = 0, \ t \in J = [0,1], \ 1 \leqslant i \leqslant n, \\ y_{n+1}(t) = y_1(t), \\ \triangle y_i|_{t=t_k} = \lambda_i I_{ik}(y_{i+1}(t_k)), \ t \neq t_k, k = 1,2,...,p, \\ \triangle y_i'|_{t=t_k} = -\lambda_i J_{ik}(y_{i+1}(t_k)), \\ a y_i(0) - b y_i'(0) = 0, \\ c y_i(1) + d y_i'(1) = 0 \end{cases} \tag{1.1}$$

where $J = [0,1]$, $t \neq t_k$, $k = 1,2,...,p$ with $0 < t_1 < t_2 < ... < t_p < 1$. For $1 \leqslant i \leqslant n$, $\triangle y_i|_{t=t_k}$ and $\triangle y_i'|_{t=t_k}$ represent the jump of $y_i(t)$ and $y_i'(t)$ at $t = t_k$, i.e.,

$$\triangle y_i|_{t=t_k} = y_i(t_k^+) - y_i(t_k^-), \quad \triangle y_i'|_{t=t_k} = y_i'(t_k^+) - y_i'(t_k^-),$$

where $y_i(t_k^+)$, $y_i'(t_k^+)$ and $y_i(t_k^-)$, $y_i'(t_k^-)$ symbolize the right-hand limit and left-hand limit of $y_i(t)$ and $y_i'(t)$ at $t = t_k$, $k = 1,2,...,p$, respectively.

Throughout this paper, we suppose that the following conditions are provided.

- $(H1)$ $a, b, c, d \in [0, \infty)$ with $ac + ad + bc > 0$,
- $(H2)$ $g_i : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, for $1 \leqslant i \leqslant n$,
- $(H3)$ $p_i \in C([0,1], \mathbb{R}^+)$. On any closed subinterval of $[0,1]$, for $1 \leqslant i \leqslant n$, p_i does not vanish identically.

$(H4)$ $I_{ik} \in C(\mathbb{R}, \mathbb{R}^+)$ and $J_{ik} \in C(\mathbb{R}, \mathbb{R}^+)$ are bounded functions such that $[d+c(1-t_k)]J_{ik}(\tau) > cI_{ik}(\tau),\ t < t_k,\ k = 1, 2, ..., p$, for $1 \leqslant i \leqslant n$, where τ be any nonnegative number.

$(H5)$ Each of

$$g_i^0 = \lim_{y \to 0^+} \frac{g_i(y)}{y}, \quad I_{ik}^0 = \lim_{y \to 0^+} \frac{I_{ik}(y)}{y}, \quad J_{ik}^0 = \lim_{y \to 0^+} \frac{J_{ik}(y)}{y},$$

exists as positive real number.

The purpose of this research is to find the eigenvalue intervals of $\lambda_i, 1 \leqslant i \leqslant n$, for which the iterative system of nonlinear second-order IBVP (1.1) has positive solutions. The fixed point theorem is the primary tool used for this. According to the authors' knowledge, the iterative system of nonlinear second-order IBVP (1.1) has not been studied before. Therefore, this paper will contribute to the literature.

The following is the outline of this paper's main structure. In Section 2, we present a number of definitions and fundamental lemmas that are useful in understanding our main result. The eigenvalue intervals for which the IBVP (1.1) iterative system has positive solutions are found in Section 3. In Section 4, we demonstrate how our main results may be used through an example.

2. Preliminaries

We begin with some background definitions on Banach spaces in this part, and then introduce auxiliary lemmas that will actually be useful later.

Let $J' = J \backslash \{t_1, t_2, ..., t_n\}$. $C(J, \mathbb{R}^+)$ indicate the Banach space of all continuous mapping $y : J \to \mathbb{R}^+$ with the norm $\|y\| = \sup_{t \in J} |y(t)|$, $PC(J, \mathbb{R}^+) = \{y : J \to \mathbb{R}^+ : y \in C(J'),\ y(t_k^+)$ and $y(t_k^-)$ exist and $y(t_k^-) = y(t_k),\ k = 1, 2, ..., n\}$ is also a Banach space with norm $\|y\|_{PC} = \sup_{t \in J} |y(t)|$, and $PC^1(J, \mathbb{R}^+) = \{y \in PC(J, \mathbb{R}^+) : y' \in PC(J'),\ y'(t_k^+)$ and $y'(t_k^-)$ exist and $y'(t_k^-) = y'(t_k),\ k = 1, 2, ..., n\}$ is a real Banach space with norm $\|y\|_{PC^1} = \max\{\|y\|_{PC}, \|y'\|_{PC}\}$ where $\|y\|_{PC} = \sup_{t \in J} |y(t)|,\ \|y'\|_{PC} = \sup_{t \in J} |y'(t)|$. Let $\mathbb{B} = PC^1(J) \cap C^2(J')$. If a function $(y_1, ..., y_n) \in \mathbb{B}^n$ provides the iterative system of the IBVP (1.1), it is referred to as a solution of the iterative system of the IBVP (1.1).

We'll start with the situation of $i = 1$ in the iterative system of the IBVP (1.1). So, we will provide the solution y_1 of the IBVP (2.1). Then, since y_1 is known, we can find y_n. If we continue in this direction, we will have y_{n-1}, then y_{n-2}, and eventually y_2. As a consequence, the solution $(y_1, ..., y_n)$ for the IBVP (1.1) iterative system is found.

Let $h \in C[0, 1]$, then we consider the following IBVP:

$$\begin{cases} -y_1''(t) = h(t),\ t \in J = [0, 1], t \neq t_k, k = 1, 2, ..., p, \\ \triangle y_1|_{t=t_k} = \lambda_1 I_{1k}(y_2(t_k)), \\ \triangle y_1'|_{t=t_k} = -\lambda_1 J_{1k}(y_2(t_k)), \\ ay_1(0) - by_1'(0) = 0, \\ cy_1(1) + dy_1'(1) = 0. \end{cases} \tag{2.1}$$

The solutions of the corresponding homogeneous equation are denoted by θ and ϕ.

$$-y_1''(t) = 0, t \in [0,1], \tag{2.2}$$

under the initial conditions

$$\begin{cases} \theta(0) = b, & \theta'(0) = a, \\ \phi(1) = d, & \phi'(1) = -c. \end{cases} \tag{2.3}$$

Using the initial conditions (2.3), we can deduce from equation (2.2) for θ and ϕ the following equations:

$$\theta(t) = b + at, \quad \phi(t) = d + c(1-t). \tag{2.4}$$

Set

$$\rho := ad + ac + bc. \tag{2.5}$$

and

LEMMA 2.1. *Let (H1)-(H5) hold. If $y_1 \in \mathbb{B}$ is a solution of the equation*

$$y_1(t) = \int_0^1 G(t,s)h(s)ds + \sum_{k=1}^{p} W_{1k}(t,t_k), \tag{2.6}$$

where

$$G(t,s) = \frac{1}{\rho}\begin{cases} (b+as)[d+c(1-t)], & s \leqslant t, \\ (b+at)[d+c(1-s)], & t \leqslant s, \end{cases} \tag{2.7}$$

$$\begin{aligned} &W_{1k}(t,t_k) = \\ &\frac{1}{\rho}\begin{cases} (b+at)[-c\lambda_1 I_{1k}(y_2(t_k)) + (d+c(1-t_k))\lambda_1 J_{1k}(y_2(t_k))], & t < t_k, \\ (d+c(1-t))[a\lambda_1 I_{1k}(y_2(t_k)) + (b+at_k)J_{1k}(y_2(t_k))], & t_k \leqslant t, \end{cases} \end{aligned} \tag{2.8}$$

then y_1 is a solution of the IBVP (2.1).

PROOF. Let y_1 satisfies the integral equation (2.6), then we get

$$y_1(t) = \int_0^1 G(t,s)h(s)ds + \sum_{k=1}^{p} W_{1k}(t,t_k),$$

i.e.,

$$\begin{aligned} y_1(t) &= \frac{1}{\rho}\int_0^t (b+as)[d+c(1-t)]h(s)ds + \frac{1}{\rho}\int_t^1 (b+at)[d+c(1-s)]h(s)ds \\ &\quad + \frac{1}{\rho}\sum_{0<t_k<t} (d+c(1-t))[a\lambda_1 I_{1k}(y_2(t_k)) + (b+at_k)J_{1k}(y_2(t_k))] \\ &\quad + \frac{1}{\rho}\sum_{t<t_k<1} (b+at)[-c\lambda_1 I_{1k}(y_2(t_k)) + (d+c(1-t_k))\lambda_1 J_{1k}(y_2(t_k))], \end{aligned}$$

$$\begin{aligned} y_1'(t) &= \frac{1}{\rho}\int_0^t (-c)(b+as)h(s)ds + \frac{1}{\rho}\int_t^1 (a)[d+c(1-s)]h(s)ds \\ &+\frac{1}{\rho}\sum_{0<t_k<t}(-c)[a\lambda_1 I_{1k}(y_2(t_k)) + (b+at_k)J_{1k}(y_2(t_k))] \\ &+\frac{1}{\rho}\sum_{t<t_k<1}(a)[-c\lambda_1 I_{1k}(y_2(t_k)) + (d+c(1-t_k))\lambda_1 J_{1k}(y_2(t_k))]. \end{aligned}$$

Thus

$$\begin{aligned} y_1''(t) &= \frac{1}{\rho}(-ct-(d+c(1-t)))h(t) \\ &= -h(t), \end{aligned}$$

i.e.,

$$y_1''(t) + h(t) = 0.$$

Since

$$\begin{aligned} y_1(0) &= \frac{1}{\rho}\int_0^1 b[d+c(1-s)]h(s)ds \\ &+\frac{1}{\rho}\sum_{k=1}^{p} b[-c\lambda_1 I_{1k}(y_2(t_k)) + (d+c(1-t_k))\lambda_1 J_{1k}(y_2(t_k))] \end{aligned}$$

and

$$\begin{aligned} y_1'(0) &= \frac{1}{\rho}\int_0^1 (a)[d+c(1-s)]h(s)ds \\ &+\frac{1}{\rho}\sum_{k=1}^{p}(a)[-c\lambda_1 I_{1k}(y_2(t_k)) + (d+c(1-t_k))\lambda_1 J_{1k}(y_2(t_k))], \end{aligned}$$

we get

$$ay_1(0) - by_1'(0) = 0. \tag{2.9}$$

Since

$$\begin{aligned} y_1(1) &= \frac{1}{\rho}\int_0^1 (b+as)(c+d)h(s)ds \\ &+\frac{1}{\rho}\sum_{k=1}^{p}(c+d)[a\lambda_1 I_{1k}(y_2(t_k)) + (b+at_k)J_{1k}(y_2(t_k))] \end{aligned}$$

and

$$y_1'(1) = \frac{1}{\rho}\int_0^1 (-c)(b+as)h(s)ds + \frac{1}{\rho}\sum_{k=1}^{p}(-c)[a\lambda_1 I_{1k}(y_2(t_k)) + (b+at_k)J_{1k}(y_2(t_k))],$$

we get

$$cy_1(1) + dy_1^{'}(1) = 0. \tag{2.10}$$

From equations (2.9) and (2.10), we have that the conditions of the IBVP (2.1) are satisfied. □

Lemma 2.2. *Assume that (H1)-(H5) hold, then for any $t, s \in J$, we have*

$$0 \leqslant G(t,s) \leqslant G(s,s). \tag{2.11}$$

Proof. It is easily obtained from equation (2.7). □

We note that an n-tuple $(y_1(t), y_2(t), ..., y_n(t))$ is a solution of the iterative system of the IBVP (1.1) if and only if

$$\begin{aligned} y_1(t) =&\lambda_1 \int_0^1 G(t,s_1)p_1(s_1)g_1\Bigg(\lambda_2 \int_0^1 G(s_1,s_2)p_2(s_2)g_2\Bigg(\lambda_3 \int_0^1 G(s_2,s_3)p_3(s_3)g_3 ... \\ &g_{n-1}\Bigg(\lambda_n \int_0^1 G(s_{n-1},s_n)p_n(s_n)g_n(y_1(s_n))ds_n + \sum_{k=1}^{p} W_{nk}(s_{n-1},t_k)\Bigg)ds_{n-1} \\ &+ \sum_{k=1}^{p} W_{n-1,k}(s_{n-2},t_k)\Bigg)ds_{n-2} + ... + \sum_{k=1}^{p} W_{3k}(s_2,t_k)\Bigg)ds_2 \\ &+ \sum_{k=1}^{p} W_{2k}(s_1,t_k)\Bigg)ds_1 + \sum_{k=1}^{p} W_{1k}(t,t_k). \end{aligned}$$

$$y_i(t) =\lambda_i \int_0^1 G(t,s)p_i(s)g_i(y_{i+1}(s))ds + \sum_{k=1}^{p} W_{ik}(t,t_k),\ t \in J,$$

$y_{n+1}(t) = y_1(t)$.

and

$$W_{ik}(t,t_k) = \frac{1}{\rho}\begin{cases} (b+at)[-c\lambda_i I_{ik}(y_{i+1}(t_k)) + (d+c(1-t_k))\lambda_i J_{ik}(y_{i+1}(t_k))], & t < t_k, \\ (d+c(1-t))[a\lambda_i I_{ik}(y_{i+1}(t_k)) + (b+at_k)J_{ik}(y_{i+1}(t_k))], & t_k \leqslant t. \end{cases}$$

To identify the eigenvalue intervals for which the iterative system of the IBVP (1.1) has at least one positive solution, we will apply the following fixed point theorem [**24**].

THEOREM 2.1. [**24**] *Let E be a Banach space. Assume that Ω is an open bounded subset of E with $\theta \in \Omega$ and let $T : \bar{\Omega} \to E$ be a completely continuous operator such that*

$$\|Tu\| \leqslant \|u\|\,, \ \forall u \in \partial\Omega.$$

Then T has a fixed point in $\bar{\Omega}$.

3. Main results

In this section, we establish criteria to determine the eigenvalues for which the iterative system of the IBVP (1.1) has at least one positive solution.

Now, we define an integral operator $\mathbb{B} \to \mathbb{B}$, for $y_1 \in \mathbb{B}$, by

$$\begin{aligned} Ty_1(t) = \lambda_1 \int_0^1 & G(t,s_1)p_1(s_1)g_1\bigg(\lambda_2 \int_0^1 G(s_1,s_2)p_2(s_2)g_2\bigg(\lambda_3 \int_0^1 G(s_2,s_3)p_3(s_3)g_3 ... \\ & g_{n-1}\bigg(\lambda_n \int_0^1 G(s_{n-1},s_n)p_n(s_n)g_n(y_1(s_n))ds_n + \sum_{k=1}^{p} W_{nk}(s_{n-1},t_k)\bigg)ds_{n-1} \\ & + \sum_{k=1}^{p} W_{n-1,k}(s_{n-2},t_k)\bigg)ds_{n-2} + ... + \sum_{k=1}^{p} W_{3k}(s_2,t_k)\bigg)ds_2 \\ & + \sum_{k=1}^{p} W_{2k}(s_1,t_k)\bigg)ds_1 + \sum_{k=1}^{p} W_{1k}(t,t_k). \end{aligned} \tag{3.1}$$

The operator T is completely continuous by an application of the Arzela-Ascoli Theorem.

Let

$$N \ := \min_{1\leqslant i\leqslant n} \left\{ \left[\left(\int_0^1 G(s,s)p_i(s)ds + \frac{p}{\rho}(2a+b)(c+d) \right) \cdot \left(\max\{g_i^0, I_{ik}^0, J_{ik}^0\} \right) \right]^{-1} \right\}.$$

THEOREM 3.1. *Assume that conditions (H1)-(H5) are satisfied. Then, for each $\lambda_1, \lambda_2, ..., \lambda_n$ satisfying*

$$\lambda_i < N, \quad 1 \leqslant i \leqslant n, \tag{3.2}$$

there exists an n-tuple $(y_1, y_2, ..., y_n)$ satisfying (1.1) such that $y_i(t) > 0,\ 1 \leqslant i \leqslant n$, on J.

PROOF. Let $\lambda_r,\ 1 \leqslant r \leqslant n$, be as in (3.2). Now, let $\varepsilon > 0$ be chosen such that

$$\begin{aligned} \max_{1\leqslant r\leqslant n} \lambda_r \leqslant \min_{1\leqslant i\leqslant n} \bigg\{ \bigg[\bigg(\int_0^1 G(s,s)p_i(s)ds + \frac{p}{\rho}(2a+b)(c+d) \bigg) \\ \cdot \bigg(\max\{g_i^0+\varepsilon, I_{ik}^0+\varepsilon, J_{ik}^0+\varepsilon\} \bigg) \bigg]^{-1} \bigg\}. \end{aligned}$$

The fixed points of the completely continuous operator $T : \mathbb{B} \to \mathbb{B}$ defined by (3.1) are investigated. Based on the definitions of $g_i^0, I_{ik}^0, J_{ik}^0,\ 1 \leqslant i \leqslant n$, there is a $K_1 > 0$ such that, for each $1 \leqslant i \leqslant n$,

$$g_i(y) \leqslant (g_i^0 + \varepsilon)y,\ I_{ik}(y) \leqslant (I_{ik}^0 + \varepsilon)y,\ J_{ik}(y) \leqslant (J_{ik}^0 + \varepsilon)y,\ 0 < y < K_1.$$

Let $y_1 \in \mathbb{B}$ with $\|y_1\| = K_1$. We obtain from Lemma 2.2 and the choice of ε, for $0 \leqslant s_{n-1} \leqslant 1$,

$$\begin{aligned}
&\lambda_n \int_0^1 G(s_{n-1}, s_n) p_n(s_n) g_n(y_1(s_n)) ds_n + \sum_{k=1}^{p} W_{nk}(s_{n-1}, t_k) \\
&\leqslant \lambda_n \Bigg[\bigg(\int_0^1 G(s_n, s_n) p_n(s_n) ds_n + \frac{p}{\rho}(2a+b)(c+d) \bigg) \\
&\qquad \cdot \bigg(\max\{g_n^0 + \varepsilon, I_{nk}^0 + \varepsilon, J_{nk}^0 + \varepsilon\} \bigg) \Bigg] \|y_1\| \\
&\leqslant K_1.
\end{aligned}$$

It proceeds in the same way from Lemma 2.2, for $0 \leqslant s_{n-2} \leqslant 1$, that

$$\begin{aligned}
&\lambda_{n-1} \int_0^1 G(s_{n-2}, s_{n-1}) p_{n-1}(s_{n-1}) g_{n-1} \bigg(\lambda_n \int_0^1 G(s_{n-1}, s_n) p_n(s_n) g_n(y_1(s_n)) ds_n \\
&\qquad + \sum_{k=1}^{p} W_{nk}(s_{n-1}, t_k) \bigg) ds_{n-1} + \sum_{k=1}^{p} W_{n-1,k}(s_{n-2}, t_k) \\
&\leqslant \lambda_{n-1} \Bigg[\bigg(\int_0^1 G(s_{n-1}, s_{n-1}) p_{n-1}(s_{n-1}) ds_{n-1} + \frac{p}{\rho}(2a+b)(c+d) \bigg) \\
&\qquad \cdot \bigg(\max\{g_{n-1}^0 + \varepsilon, I_{n-1,k}^0 + \varepsilon, J_{n-1k}^0 + \varepsilon\} \bigg) \Bigg] \|y_1\| \\
&\leqslant \|y_1\| = K_1.
\end{aligned}$$

If we continue this bootstrapping argument, we get, for $0 \leqslant t \leqslant 1$,

$$\begin{aligned}
&\lambda_1 \int_0^1 G(t, s_1) p_1(s_1) g_1(\lambda_2 ...) ds_1 + \sum_{k=1}^{p} W_{1k}(t, t_k) \\
&\leqslant \lambda_1 \Bigg[\bigg(\int_0^1 G(s_1, s_1) p_1(s_1) ds_1 + \frac{p}{\rho}(2a+b)(c+d) \bigg) \\
&\qquad \cdot \bigg(\max\{g_1^0 + \varepsilon, I_{1k}^0 + \varepsilon, J_{1k}^0 + \varepsilon\} \bigg) \Bigg] K_1 \\
&\leqslant K_1 = \|y_1\|.
\end{aligned}$$

Thus, $\|Ty_1\| \leqslant K_1 = \|y_1\|$. If we established $\Omega_1 = \{y \in \mathbb{B} : \|y\| < K_1\}$, then

$$\|Ty_1\| \leqslant \|y_1\| \ \textit{for} \ y_1 \in \partial\Omega. \tag{3.3}$$

We can see that T has a fixed point $y_1 \in \bar{\Omega}$ by applying Theorem 2.1 to (3.3) . As a result, by setting $y_{n+1} = y_1$, we get a positive solution $(y_1, y_2, ..., y_n)$ of the iterative system of the IBVP (1.1) given iteratively by

$$y_r(t) = \lambda_r \int_0^1 G(t, s) p_r(s) g_r(y_{r+1}(s)) ds + \sum_{k=1}^{p} W_{rk}(t, t_k), \quad r = n, n-1, ..., 1.$$

□

4. An Example

Example 4.1 In the iterative system of the IBVP (1.1), suppose that $n = p = 3$, $p_i(t) = 1$ for $1 \leqslant i \leqslant 3$, $a = c = 6, b = d = 3$ i.e.,

$$\begin{cases} y_i''(t) + \lambda_i g_i(y_{i+1}(t)) = 0, \ t \in J = [0,1], t \neq t_k, 1 \leqslant i, k \leqslant 3, \\ \triangle y_i|_{t=t_k} = \lambda_i I_{ik}(y_{i+1}(t_k)), \\ \triangle y_i'|_{t=t_k} = -\lambda_i J_{ik}(y_{i+1}(t_k)), \\ 6y_i(0) - 3y_i'(0) = 0, \\ 6y_i(1) + 3y_i'(1) = 0. \end{cases} \tag{4.1}$$

where

$$g_1(y_2) = y_2(2 - e^{-y_2}),\ g_2(y_3) = y_3(4 - 3e^{-2y_3}),\ g_3(y_1) = y_1(3 - \tfrac{5}{2}e^{-3y_1}),$$

$$I_{1k}(y_2) = \frac{3y_2^2 + 4y_2}{4 + y_2},\ I_{2k}(y_3) = \frac{2y_3^3 + 4y_3}{5 + y_3},\ I_{3k}(y_1) = \frac{5y_1^2 + 2y_1}{5 + 2y_1},$$

$$J_{1k}(y_2) = \frac{6y_2^2 + 8y_2}{2 + y_2},\ J_{2k}(y_3) = \frac{4y_3^3 + 8y_3}{3 + y_3},\ J_{3k}(y_1) = \frac{10y_1^2 + 4y_1}{3 + y_1}.$$

It is clear that (H1)-(H5) has been satisfied. By simple calculation, we get $\rho = 72, \theta(t) = 3 + 6t, \phi(t) = 9 - 6t$ and

$$G(t,s) = \frac{1}{72}\begin{cases} (3+6s)(9-6t), & s \leqslant t, \\ (3+6t)(9-6s), & t \leqslant s. \end{cases}$$

We obtain

$$g_1^0 = 1,\ g_2^0 = 1,\ g_3^0 = \frac{1}{2},\ I_{1k}^0 = 1,\ I_{2k}^0 = \frac{4}{5},\ I_{3k}^0 = \frac{2}{5},\ J_{1k}^0 = 4,\ J_{2k}^0 = \frac{8}{3},\ J_{3k}^0 = \frac{4}{3},$$

and
$N = \min\{0.0410958904109589, 0.0616438356164384, 0.1232876712328767\}$.

Using Theorem 2.1, we obtain the optimal eigenvalue interval of

$$\lambda_i < 0.0410958904109589,\ i = 1, 2, 3,$$

which has a positive solution to the impulsive boundary value problem (4.1).

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Received by editors 24.4.2022; Revised version 9.5.2022; Available online 18.5.2022.

Dondu Oz, Department of Mathematics, Ege University, 35100 Bornova, Izmir, Turkey

Email address: dondu.oz@ege.edu.tr

Ilkay Yaslan Karaca, Department of Mathematics, Ege University, 35100 Bornova, Izmir, Turkey

Email address: ilkay.karaca@ege.edu.tr