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A NEW COMPUTATIONAL TECHNIQUE FOR FOURIER TRANSFORMS BY USING THE DIFFERENTIAL TRANSFORMATION METHOD

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Abstract. The aim of this study is to calculate the well-known Fourier Transforms of functions in a different way. Through our procedure, the Fourier Transform of functions is obtained by considering the Differential Transform Method (DTM) easily without resorting to complex integration.

1. Introduction

The Differential Transformation Method is one of the powerful methods that can be easily applied to a lot of linear and nonlinear problems and is able to reduce the amount of computational work. It is the method based on the Taylor series. DTM is a well-known and strong semi numerical-analytical technique that does not necessitate a lengthy integration phase. Therefore, very precise and efficient results are easily obtained. The idea of a Differential Transformation Method was suggested in the topic of electrical circuit analysis to the solution of linear and nonlinear (IVP) by Zhou [**17**]. They used the DTM to improve the fins [**15**][**16**], the Differential Transformation technique is employed in [**11**] for applying a free vibration of continuous systems. This technique is used in [**8**] [**10**] for solving initial value and higher order- initial-value problems. Ebaid also used this method to solve harmonic oscillator [**5**]. A method depending on the DTM was applied by Abbasov et al. [**2**] to get approximate solutions of equations connected to engineering issues, and the numerical results were found to be in good agreement with the analytical answers. This method doesn't require much time when it is applied to the computer.

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Integral Transforms are another important concept in mathematics, which can be applied in the solutions of linear problems. With these transformations, we can solve linear ordinary and partial differential equations, calculate some improper integrals. It is important to know reciprocal of functions under the integral transformation. In some cases, it may be necessary to calculate very complex integrals. For this reason, different methods have been used to calculate the Integral Transformations of functions. Some of these are Homotopy Perturbation, Adomian Decomposition Method, Variational Iteration method and DTM [**1**], [**4**], [**6**]. Inspired by these studies, we calculated the Fourier Transform of functions using DTM.

2. The basic definitions and theorems

2.1. DTM.

Definition 2.1. The Differential Transform of function *h(t)* is represented (seen [**3**], [**7**])as:

$$H(k) = \frac{1}{k!}[\frac{d^k h(t)}{dt^k}]_{t=0} \tag{2.1}$$

and inverse transformation is

$$h(t) = \sum_{k=0}^{\infty} H(k) t^k \tag{2.2}$$

Theorem 2.1. [**3**][**13**] *If* $h(t) = g(t) \mp w(t)$ *then* $H(k) = G(k) \mp W(k)$.

Theorem 2.2. [**3**][**13**] *If* $h(t) = \mu g(t)$ *then* $H(k) = \mu G(k)$, *where* μ *is a constant.*

Theorem 2.3. [**3**][**13**] *If* $h(t) = \frac{d^m g(t)}{dt^m}$ *then* $H(k) = \frac{(k+m)!}{k!} G(k+m)$.

Theorem 2.4. [**3**][**13**] *If* $h(t) = t^m$ *then* $H(k) = \delta(k-m)$, *where*

$$\delta(k-m) = \begin{cases} 1, \text{if } k = m \\ 0, \text{if } k \neq m \end{cases} .$$

Theorem 2.5. [**3**][**13**] *If* $h(t) = g(t)w(t)$ *then* $H(k) = \sum_{r=0}^{k} G(r)W(k-r)$.

Theorem 2.6. [**3**] *If* $h(t) = g_1(t) g_2(t) \cdots g_{m-1}(t) g_m(t)$ *then*

$$H(k) = \sum_{r_{m-1}=0}^{k} \sum_{r_{m-2}=0}^{r_{m-1}} \cdots \sum_{r_2=0}^{r_3} \sum_{r_1=0}^{r_2} G_1(r_1) G_2(r_2 - r_1) G_m(k - r_{m-1})$$

.

2.2. Fourier Transform. In order to begin our discussion on the new computational method for Fourier transform of functions, we define Fourier transform of $f(t)$ in non-unitary angular frequency form[**12**]

$$\mathcal{F}(f(t)) = \hat{f}(w) = \int_{-\infty}^{\infty} f(t) e^{-iwt} dt \tag{2.3}$$

and it's inverse transform:

$$f(t) = \mathcal{F}^{-1}(\hat{f}(w)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(w)e^{iwt}dw \tag{2.4}$$

Improper integral (2.3) is convergent to the function of complex variable w when

$$\int_{-\infty}^{\infty} |f(t)|dt < \infty \tag{2.5}$$

condition is satisfied. However, in real-life problems like signal processing and many others, we face up with trigonometric, polynomial and exponential functions which is non-integrable on the interval $(-\infty, \infty)$.

Fourier Transforms of trigonometric ,polynomial and exponential functions are defined with Dirac's δ-function, which is not a classical function. In the coming section 2.3 we will introduce to you one of the most useful and at the same time most rigorous function Dirac-delta function.

2.3. Dirac delta distribution. This function plays important role in Fourier analysis, probability theory and many other real applications of mathematics. Dirac δ-function has the following properties[**12**]

- $\delta(w) = 0$ for $w \neq 0$
- $\delta(0) = \infty$
- $\int_{-\infty}^{\infty} \delta(w)dw = 1$

As you can see it is not a type of classical function, it's a generalized function that can assign meaningful value inside the integral. Some proper functions can approximate δ-function in the limiting process. One of them is the family of Gaussian curves which is :

$$g(w, a) = \frac{1}{\sqrt{2\pi a}}e^{\frac{-w^2}{2a}}, a > 0 \tag{2.6}$$

As $a \to 0^+$ family of $g(w, a)$ functions satisfies exact the same properties of δ function. That means

$$\delta(w) = \lim_{a\to 0^+} g(w, a) \tag{2.7}$$

The Fourier transform of δ-function and it's inverse is given by[**12**]

$$\mathcal{F}(\delta(w)) = \int_{-\infty}^{\infty} \delta(w)e^{-iwt}dw = 1 \tag{2.8}$$

and

$$\mathcal{F}^{-1}(1) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 1.e^{iwt}dt = \delta(w) \tag{2.9}$$

δ-function is an even function that $\delta(-w) = \delta(w)$, and instead expression of delta function (2.9), it is more common to express in the form :

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iwt}dt = \delta(w) \tag{2.10}$$

From Fourier integral theorem :

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(w)e^{iwt}dw = \frac{1}{2\pi}\int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} f(y)e^{-iwy}dy)e^{iwt}dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)(\int_{-\infty}^{\infty} e^{-i(y-t)w}dw)dy. \tag{2.11}$$

From (2.10) and (2.11) $f(t)$ can be written equivalently as follows

$$f(t) = \int_{-\infty}^{\infty} f(y)\delta(y-t)dy \tag{2.12}$$

This is also known as convolution with a delta function. We can see here convolution with δ-function gives function itself.
Some other important properties of δ-function are as following [**14**]:

- $\delta(-w) = \delta(w)$
- $w\delta(w) = 0$
- $\delta(aw) = \frac{1}{|a|}\delta(w), a \neq 0$
- $\delta(w^2 - a^2) = \frac{1}{2|a|}(\delta(w-a) + \delta(w+a)), a \neq 0$
- $f(0) = \int_{-\infty}^{\infty} f(t)\delta(w)dt$

As we mentioned δ-function is meaningful with the function combined inside the integral and so does its derivatives. For example,

$$\int_{-\infty}^{\infty} f(y)\delta'(y-t)dy$$

$$= [\lim_{a\to-\infty, b\to\infty} f(y)\delta(y-t)]_{y=a}^{y=b} - \int_{-\infty}^{\infty} f'(y)\delta(y-t)dy = -f'(t)$$

And more generally,

$$\int_{-\infty}^{\infty} f(y)\delta^{(n)}(y-t)dy = (-1)^n f^{(n)}(t) \tag{2.13}$$

From the expression

$$\delta(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iwt}dt,$$

We get Fourier transform of $f(x) = 1$:

$$\mathcal{F}(1) = 2\pi\delta(w)$$

The Fourier transform of $f^{(n)}(t)$ can be constructed by partial integration:

$$\mathcal{F}(f^{(n)}(t)) = (iw)^n \mathcal{F}(f(t)). \tag{2.14}$$

And by the duality property we get :

$$\mathcal{F}(t^n f(t)) = i^n \frac{d^n}{dw^n}\mathcal{F}(f(t)) \tag{2.15}$$

We note here that this derivative formula holds on absolute integrable functions. But in the generalized Fourier transform of distributions, this idea was extended and putting $f(t) = 1$ in (2.15) gives us:

$$\mathcal{F}(t^n) = 2\pi i^n \delta^{(n)}(w)$$

.

Functions	**The fourier transforms**
1	$\mathcal{F}(1) = 2\pi\delta(w)$
t^n	$\mathcal{F}(t^n) = 2\pi i^n \delta^{(n)}(w)$
e^{at}	$\mathcal{F}(e^{at}) = 2\pi\delta(w + ia)$

Table1:The Fourier Transformations of some functions.

3. Results through the use of DTM

THEOREM 3.1. *If $f(t)$ is an analytic function and $w \in C$, and we consider the linear initial value problem as follow*

$$y' - iwy = if(t), y(0) = 0 \tag{3.1}$$

then the Fourier transform of $f(t)$ is

$$\mathcal{F}(f(t)) = [\frac{e^{-iwt}}{i} \sum_{k=0}^{\infty} Y(k)t^k]_{t=-\infty}^{t=\infty} \tag{3.2}$$

where $Y(k)$ is differential transform of $y(t)$.

PROOF. $y' - iwy = if(t)$, we can write this ODE as follow

$$(ye^{-iwt})' = if(t)e^{-iwt}. \tag{3.3}$$

By integrating both sides with respect to t from $-\infty$ to ∞ we obtain the relation between previous equation and Fourier transform as follow

$$[ye^{-iwt}]_{t=-\infty}^{t=\infty} = i\int_{-\infty}^{\infty} f(t)e^{-iwt}dt \tag{3.4}$$

that means :

$$\mathcal{F}(f(t)) = [\frac{e^{-iwt}}{i} y]_{t=-\infty}^{t=\infty} \tag{3.5}$$

In order to find the Fourier transform of $f(t)$, we construct the DTM form of equation (3.1) as

$$Y(k+1) = \frac{iF(k) + iwY(k)}{k+1}, Y(0) = 0 \tag{3.6}$$

where $F(k)$, $Y(k)$ are differential transform of functions $f(t)$ and $y(t)$ respectively. By using the inverse differential transform (2.2), we get

$$y = \sum_{k=0}^{\infty} Y(k)t^k \tag{3.7}$$

By substituting y into equation (3.5) we obtain

$$\mathcal{F}(f(t)) = [\frac{e^{-iwt}}{i}\sum_{k=0}^{\infty} Y(k)t^k]_{t=-\infty}^{t=\infty} \tag{3.8}$$

□

Now we will apply the differential transform method to find the Fourier transform of some functions:

EXAMPLE 3.1. Let $f(t) = 1$ in Theorem 3.1, then we obtain $Y(k+1) = \frac{i\delta(k)+iwY(k)}{k+1}$,Where $\delta(k)$ is the Kronecker delta and given as: $\delta(k) = \begin{cases} 1, \text{if } k = 0 \\ 0, \text{if } k \neq 0 \end{cases}$
, $Y(0) = 0$, Now we will find some of $Y(k)$

$$Y(1) = i, Y(2) = \frac{i^2 w}{2!}, Y(3) = \frac{i^3 w^2}{3!}, Y(4) = \frac{i^4 w^3}{4!}, Y(5) = \frac{i^5 w^4}{5!}, \cdots \tag{3.9}$$

Thus from (3.9) and (3.7), we obtain that

$$y(t) = \sum_{k=0}^{\infty} Y(k)t^k = it + \frac{i^2 w}{2!}t^2 + \frac{i^3 w^2}{3!}t^3 + \frac{i^4 w^3}{4!}t^4 + \frac{i^5 w^4}{5!}t^5 + \cdots \tag{3.10}$$

Equation (3.10) can be written equivalently as follows

$$y(t) = \sum_{k=0}^{\infty} Y(k)t^k = \frac{1}{w}(e^{iwt} - 1) \tag{3.11}$$

Finally, by setting equation (3.11) in (3.8) we obtain the Fourier transform of 1

$$\mathcal{F}(1) = [\frac{e^{-iwt}}{i}(\frac{1}{w}(e^{iwt} - 1))]_{t=-\infty}^{t=\infty} = [\frac{1}{iw}(1 - e^{-iwt})]_{t=-\infty}^{t=\infty} = \int_{-\infty}^{\infty} e^{-iwt}dt. \tag{3.12}$$

By equation (2.10), we obtain $\mathcal{F}(1) = 2\pi\delta(w)$

EXAMPLE 3.2. Let $f(t) = e^{at}$ in Theorem 3.1, then we obtain $Y(k+1) = \frac{i\frac{a^k}{k!} + iwY(k)}{k+1}$, $Y(0) = 0$
Now we will find some of $Y(k)$

$$\begin{aligned} Y(1) = i, Y(2) = \frac{ia + i^2 w}{2!}, Y(3) = \frac{ia^2 + i^2 wa + i^3 w^2}{3!}, \\ Y(4) = \frac{ia^3 + i^2 wa^2 + i^3 w^2 a + i^4 w^3}{4!}, \end{aligned} \tag{3.13}$$

$$Y(5) = \frac{ia^4 + i^2 wa^3 + i^3 w^2 a^2 + i^4 w^3 a + i^5 w^4}{5!},$$

$$Y(6) = \frac{ia^5 + i^2 wa^4 + i^3 w^2 a^3 + i^4 w^3 a^2 + i^5 w^4 a + i^6 w^5}{6!}, \cdots$$

Thus from (3.13) and (3.7), we obtain that

$$(3.14)\quad y(t)=\sum_{k=0}^{\infty}Y(k)t^k = it+\frac{ia+i^2w}{2!}t^2+\frac{ia^2+i^2wa+i^3w^2}{3!}t^3$$
$$+\frac{ia^3+i^2wa^2+i^3w^2a+i^4w^3}{4!}t^4$$
$$+\frac{ia^4+i^2wa^3+i^3w^2a^2+i^4w^3a+i^5w^4}{5!}t^5$$
$$+\frac{ia^5+i^2wa^4+i^3w^2a^3+i^4w^3a^2+i^5w^4a+i^6w^5}{6!}t^6+\cdots$$
$$=\frac{i}{a-iw}((e^{at}-1)-iwt-i^2w^2\frac{t^2}{2!}-i^3w^3\frac{t^3}{3!}-i^4w^4\frac{t^4}{4!}-\cdots)$$

Equation (3.14) can be written equivalently as follows

$$(3.15)\qquad y(t)=\sum_{k=0}^{\infty}Y(k)t^k=\frac{i}{a-iw}(e^{at}-e^{iwt})$$

Finally, by setting equation (3.15) in (3.8) we obtain the Fourier transform of e^{at}

$$\mathcal{F}(e^{at})=[\frac{e^{-iwt}}{i}(\frac{i}{a-iw}(e^{at}-e^{iwt}))]_{t=-\infty}^{t=\infty}=[\frac{1}{-i(ia+w)}(e^{-i(ia+w)t}-1)]_{t=-\infty}^{t=\infty}$$

$$(3.16)\qquad =\int_{-\infty}^{\infty}e^{-i(ia+w)t}dt.$$

By equation (2.10), we obtain $\mathcal{F}(e^{at})=2\pi\delta(ia+w)$

Example 3.3. Let $f(t)=t^n$ in Theorem 3.1, then we obtain $Y(k+1)=\frac{i\delta(k-n)+iwY(k)}{k+1}$, $Y(0)=0$
Now we will find some of $Y(k)$

(3.17)

$$Y(1)=Y(2)=Y(3)=\cdots=Y(n)=0, Y(n+1)=\frac{in!}{(n+1)!}, Y(n+2)=\frac{i^2wn!}{(n+2)!},$$
$$Y(n+3)=\frac{i^3w^2n!}{(n+3)!},\cdots,Y(n+m)=\frac{i^mw^{m-1}n!}{(n+m)!},\cdots$$

Thus from (3.17) and (3.7), we obtain that

$$(3.18)\quad y(t)=\sum_{k=0}^{\infty}Y(k)t^k=\frac{in!}{(n+1)!}t^{n+1}+\frac{i^2wn!}{(n+2)!}t^{n+2}+\frac{i^3w^2n!}{(n+3)!}t^{n+3}+\cdots$$

Equation (3.18) can be written equivalently as follows

$$y(t)=\sum_{k=0}^{\infty}Y(k)t^k=\frac{in!}{(iw)^{n+1}}(e^{iwt}-1-\frac{iwt}{1}-\frac{(iwt)^2}{2!}-\cdots-\frac{(iwt)^n}{n!}) \tag{3.19}$$

Finally, by setting equation (3.19) in (3.8) we obtain the Fourier transform of t^n

$$\mathcal{F}(t^n)=[\frac{e^{-iwt}}{i}(\frac{in!}{(iw)^{n+1}}(e^{iwt}-1-\frac{iwt}{1}-\frac{(iwt)^2}{2!}-\cdots-\frac{(iwt)^n}{n!}))]_{t=-\infty}^{t=\infty} \tag{3.20}$$
$$=[\tfrac{2\pi}{(-i)^n}(\tfrac{(-i)^n}{2\pi}(-1)e^{-iwt}(\tfrac{t^n}{iw}+\tfrac{nt^{n-1}}{(iw)^2}+\cdots+\tfrac{n!t}{(iw)^n}+\tfrac{n!}{(iw)^{n+1}}))]_{t=-\infty}^{t=\infty}=2\pi i^n\delta^{(n)}(w).$$

EXAMPLE 3.4. Let $f(t)=rect(t)=\begin{cases}1, \text{if } -\frac{1}{2}\leqslant t\leqslant\frac{1}{2}\\ 0, otherwise\end{cases}$ in Theorem 3.1 where $rect(t)$ is Rectangular function, then we obtain $Y(k+1)=\frac{i\delta(k)+iwY(k)}{k+1}, -\frac{1}{2}\leqslant t\leqslant\frac{1}{2}$, $Y(0)=0$
Now we will find some of $Y(k)$

$$Y(1)=i, Y(2)=\frac{i^2w}{2!}, Y(3)=\frac{i^3w^2}{3!}, Y(4)=\frac{i^4w^3}{4!}, Y(5)=\frac{i^5w^4}{5!},\cdots \tag{3.21}$$

Thus from (3.21) and (3.7), we obtain that

$$y(t)=\sum_{k=0}^{\infty}Y(k)t^k=it+\frac{i^2w}{2!}t^2+\frac{i^3w^2}{3!}t^3+\frac{i^4w^3}{4!}t^4+\frac{i^5w^4}{5!}t^5+\cdots \tag{3.22}$$

Equation (3.22) can be written equivalently as follows

$$y(t)=\sum_{k=0}^{\infty}Y(k)t^k=\frac{1}{w}(e^{iwt}-1) \tag{3.23}$$

Finally, by setting equation (3.23) in (3.8) we obtain the Fourier transform of $rect(t)$

$$\mathcal{F}(rect(t))=[\frac{e^{-iwt}}{i}(\frac{1}{w}(e^{iwt}-1))]_{t=-\infty}^{t=\infty}=[\frac{1}{iw}(1-e^{-iwt})]_{t=-\frac{1}{2}}^{t=\frac{1}{2}}=\frac{\sin(\frac{w}{2})}{\frac{w}{2}}, \tag{3.24}$$

therefore, we finally have,

$$\mathcal{F}(rect(t))=sinc(\frac{w}{2\pi}).$$

The definition of *sinc* function in[**9**]

4. Conclusion

We applied the Differential Transform Method (DTM) for calculating of the well-known Fourier transforms of functions in a different way. Furthermore, contrary to literature our presented method provides a well-known and strong analytic technique that does not necessitate a lengthy integration phase.

References

1. S. Abbasbandy, *Applications of He's homotopy perturbation method for Laplace transform.* Chaos Solitons Fractals., **30**(2006), 1206–1212.
2. A. Abbasov and A. R. Bahadir, *The investigation of the transient regimes in the nonlinear systems by the generalized classical method.* Math. Prob. Eng., **5**(2005), 503–519.
3. A. Arikoglu and I. Ozkol, *Solution of difference equations by using differential transform method.* Applied Mathematics and Computation., **174**(2006), 1216–1228.
4. E. Babolian, J. Biazar, and A.R. Vahidi, *A new computational method for Laplace transforms by decomposition method.* Math. Comput., **150**(2004), 841–846.
5. A. E. H. Ebaid, *Approximate periodic solutions for the non-linear relativistic harmonic oscillator via differential transformation method.* Communications in Nonlinear Science and Numerical Simulation., **15**(7)(2010), 1921–1927.
6. M. Gubes, *A new calculation technique for the Laplace and Sumudu transforms by means of the variational iteration method.* Mathematical Sciences., **13**(1)(2019), 21–25.
7. M. Gubes, *Different Computational Approach for Sumudu Integral Transform by Using Differential Transform Method.* Universal Journal of Mathematics and Applications., **1**(4)(2018), 233–238.
8. I. A. H. Hassan, *Differential transformation technique for solving higher-order initial value problems.* Applied Mathematics and Computation., **154**(2)(2004), 299–311.
9. A. Issa, N. Qatanani, and A. Daraghmeh, *Approximation Techniques for Solving Linear Systems of Volterra Integro-Differential Equations.* Journal of Applied Mathematics., (2020).
10. M. J. Jang, C. L. Chen, and Y.C. Liy, *On solving the initial value problems using the differential transformation method.* Appl. Math. Comput., **115**(2000), 145–160.
11. M. Malik and H.H. Dang, *Vibration analysis of continuous systems by differential transformation.* Applied Mathematics and Computation., **96**(1)(1998), 17–26.
12. B. Osgood, *The Fourier transform and its applications.* Lecture notes for EE., **261**(2009), 20.
13. M. Thongmoon and S. Pusjuso, *The numerical solutions of differential transform method and the Laplace transform method for a system of differential equations.* Nonlinear Analysis: Hybrid Systems., **4**(3)(2010), 425–431.
14. N. Wheeler, *Simplified production of Dirac delta function identities.* Reed College., (1997).
15. L.T. Yu and C.K. Chen, *The solution of the Blasius equation by the differential transformation method.* Math. Comput. Modelling., **28**(1)(1998), 101–111.
16. L.T. Yu and C.K. Chen, *Application of taylor transformation to optimize rectangular fins with variable thermal parameters.* Appl. Math. Model., **22**(1998), 11–21.
17. J. K. Zhou, *Differential transformation and its applications for electrical circuits.* Huazhong University Press, Wuhan, China., 1986.

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