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REVERSE SOMBOR INDEX

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Abstract. The recently conceived Sombor index is defined as the sum over all edges uv of the terms $\sqrt{\deg(u)^2+\deg(v)^2}$, where deg denotes the degree of the respective vertex. Following Kulli's concept of δ-Sombor index, and Ediz's reverse Zagreb indices, we introduce a new vertex-degree-based graph invariant, the reverse Sombor index (RSO). Its is equal to the sum of the terms $\sqrt{\big[\Delta-\deg(u)+1\big]^2+\big[\Delta-\deg(v)+1\big]^2}$ where Δ is the greatest vertex degree. The basic mathematical properties of RSO are established.

1. Introduction

In this paper we are concerned with simple graph, that is, finite graphs without directed, weighted, or multiple edges, and without self loops. Let G be such a graph, with vertex set $\mathbf{V}(G)$ and edge set $\mathbf{E}(G)$. An edge of G, connecting the vertices u and v will be denoted by uv.

The degree (= number of first neighbors) of the vertex $u \in \mathbf{V}(G)$ will be denoted by $\deg(u)$.

The *Sombor index* is a recently invented vertex-degree-based topological index [**9**]. It is defined as

$$SO = SO(G) = \sum_{uv\in\mathbf{E}(G)} \sqrt{\deg(u)^2+\deg(v)^2}\,.$$

This graph invariant immediately attracted much attention, and a remarkably large number studies on this topic were published (e.g., see [**1–5**, **10**, **17**, **19**, **20**, **22**, **23**]).

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In the same time, various modifications of the Sombor index were proposed, such as the modified Sombor index [**11**, **15**]

$$\sum_{uv\in\mathbf{E}(G)} \frac{1}{\sqrt{\deg(u)^2+\deg(v)^2}}$$

the Banhatti–Sombor and second Banhatti–Sombor indices [**12**, **13**]

$$\sum_{uv\in\mathbf{E}(G)} \left[\left(\frac{1}{\deg(u)}\right)^2+\left(\frac{1}{\deg(v)}\right)^2\right]^{1/2} \text{ and } \sum_{uv\in\mathbf{E}(G)} \left[\left(\frac{1}{\deg(u)}\right)^2+\left(\frac{1}{\deg(v)}\right)^2\right]^{-1/2}$$

the δ-Sombor index [**14**]

$$\sum_{uv\in\mathbf{E}(G)} \sqrt{\big[\deg(u)-\delta-1\big]^2+\big[\deg(v)-\delta-1\big]^2} \tag{1.1}$$

where δ is the smallest vertex degree of the graph G, and many other [**9**, **16**, **18**, **21**]. To each of these graph invariants, a "reduced" invariant can be associated, in which for every vertex x, $\deg(x)$ is replaced by $\deg x - 1$.

Of the other vertex-degree-based graph invariants, needed in the subsequent part of this paper, we mention the first and second Zagreb indices [**24**, **25**]

$$M_1(G)=\sum_{uv\in\mathbf{E}(G)}\big[\deg(u)+\deg(v)\big]=\sum_{u\in\mathbf{V}(G)}\deg(u)^2$$

and

$$M_2(G)=\sum_{uv\in\mathbf{E}(G)}\deg(u)\,\deg(v)\,,$$

the forgotten index [**24**, **25**]

$$F(G)=\sum_{uv\in\mathbf{E}(G)}\big[\deg(u)^2+\deg(v)^2\big]=\sum_{u\in\mathbf{V}(G)}\deg(u)^3$$

and the"reverse Zagreb" indices, put forward by Ediz [**6**–**8**]

$$RM_1^{\alpha}(G) = \sum_{u\in\mathbf{V}(G)} c(u)^2 \tag{1.2}$$

$$RM_1^{\beta}(G) = \sum_{uv\in\mathbf{E}(G)} \big[c(u)+c(v)\big] \tag{1.3}$$

$$RM_2(G) = \sum_{uv\in\mathbf{E}(G)} c(u)\,c(v) \tag{1.4}$$

where $c(x)=\Delta-\deg(x)+1$ for any vertex $x\in\mathbf{V}(G)$, and where Δ is the greatest vertex degree of the graph G. Recall that in contrast to the first Zagreb index, the reverse first Zagreb indices RM_1^{α} and RM_2^{β} do not coincide.

Motivated by Kulli's δ-Sombor index, Eq. (1.1), and Ediz's reverse Zagreb indices, Eqs. (1.2)–(1.4), we now introduce a "*reverse Sombor index*", defined as

$$
\begin{aligned}
RSO = RSO(G) &= \sum_{uv \in \mathbf{E}(G)} \sqrt{c(u)^2 + c(v)^2} \\
&\equiv \sum_{uv \in \mathbf{E}(G)} \sqrt{\big[\Delta - \deg(u) + 1\big]^2 + \big[\Delta - \deg(v) + 1\big]^2}\,. \qquad (1.5)
\end{aligned}
$$

In the subsequent section we establish a few basic mathematical properties of RSO. Its chemical applications will be reported elsewhere.

2. Mathematical properties of reverse Sombor index

In order to avoid trivialities, from now on we assume that the graph G is connected and that $|\mathbf{V}(G)| > 2$. Then $\Delta \geqslant 2$.

We begin with finding the reverse Sombor index for some standard graphs. After that, we obtain bounds on this index in terms of other known graph parameters and topological indices.

Remark 2.1. For the path graph $P_n, n \geqslant 3$, $RSO(P_n) = 2\sqrt{5} + (n-3)\sqrt{2}$.

Remark 2.2. Let G be a graph of size $|\mathbf{E}(G)| = m$. If G is regular, then $RSO(G) = \sqrt{2}\,m$. Otherwise, $2\sqrt{2}\,m \leqslant RSO(G) < \sqrt{2}\,\Delta\, m$.

Proof. Suppose that G is k-regular. Then, for each $v \in \mathbf{V}(G)$, $\Delta = \deg(v) = k$ so that $c(v) = k - k + 1 = 1$. Thus, $RSO(G) = \sum_{uv \in \mathbf{E}(G)} \sqrt{1^2 + 1^2} = m\sqrt{2}$.

Suppose that G is not regular. Then, $\Delta \geqslant 2$, so that for any $v \in \mathbf{V}(G)$,

$$c(v) = \Delta - \deg(v) + 1 \geqslant 3 - \deg(v) \geqslant 3 - 1 = 2\,.$$

Also, $\deg(v) \geqslant 1$, so that $c(v) = \Delta - \deg(v) + 1 \leqslant \Delta$. Thus,

$$RSO(G) = \sum_{uv \in \mathbf{E}(G)} \sqrt{c(u)^2 + c(v)^2} \geqslant m\sqrt{2^2 + 2^2} = 2\sqrt{2}\,m\,.$$

For the upper bound, we have

$$RSO(G) = \sum_{uv \in \mathbf{E}(G)} \sqrt{c(u)^2 + c(v)^2} < m\sqrt{\Delta^2 + \Delta^2} = \sqrt{2}\,\Delta\, m\,.$$

The bound is strict since G is assumed to be non-regular. □

Theorem 2.1. *For any graph G of size $m \geqslant 1$,*

$$RSO(G) \leqslant \sqrt{m\big[2(\Delta+1)^2\, m + F(G) - 2(\Delta+1) M_1(G)\big]}\,.$$

PROOF. Using the Cauchy–Schwarz inequality, we have

$$\left[\sum_{uv\in\mathbf{E}(G)} \sqrt{c(u)^2+c(v)^2}\right]^2 \leqslant \sum_{uv\in\mathbf{E}(G)} (1) \sum_{uv\in\mathbf{E}(G)} \big[c(u)^2+c(v)^2\big] = m \sum_{uv\in\mathbf{E}(G)} \big[c(u)^2+c(v)^2\big]$$

Here,

$$\sum_{uv\in\mathbf{E}(G)} \big[c(u)^2+c(v)^2\big] = \sum_{uv\in\mathbf{E}(G)} \Big[(\Delta-\deg(u)+1)^2+(\Delta-\deg(v)+1)^2\Big]$$

$$= \sum_{uv\in\mathbf{E}(G)} \Big[2(\Delta+1)^2+\deg(u)^2+\deg(v)^2-2(\Delta+1)\big[\deg(u)+\deg(v)\big]\Big]$$

$$= 2(\Delta+1)^2\, m + F(G) - 2(\Delta+1)M_1(G)\,.$$

Thus,

$$RSO^2 = \left(\sum_{uv\in\mathbf{E}(G)} \sqrt{c(u)^2+c(v)^2}\right)^2 \leqslant m\big[2(\Delta+1)^2\, m+F(G)-2(\Delta+1)M_1(G)\big]\,.$$

□

THEOREM 2.2. *For any graph G, $RSO(G) \geqslant \frac{1}{\sqrt{2}}\, RM_1^{\beta}(G)$, with equality if G is regular.*

PROOF. For a concave function $g(x)$, by the Jensen inequality,

$$g\left(\frac{1}{n}\sum x_i\right) \geqslant \frac{1}{n}\sum g(x_i)$$

with equality for a strict concave function if $x_1 = x_2 = \cdots = x_n$. Choosing $g(x) = \sqrt{x}$, we get

$$\sqrt{\frac{c(u)^2+c(v)^2}{2}} \geqslant \frac{c(u)+c(v)}{2}$$

which implies

$$\sum_{uv\in\mathbf{E}(G)} \sqrt{c(u)^2+c(v)^2} \geqslant \frac{1}{\sqrt{2}} \sum_{uv\in\mathbf{E}(G)} \big[c(u)+c(v)\big]$$

i.e.,

$$RSO(G) \geqslant \frac{1}{\sqrt{2}}\, RM_1^{\beta}(G)\,.$$

In particular, if G is regular, then the result follows from the fact that $c(u) = c(v)$ for all $u, v \in \mathbf{V}(G)$. □

THEOREM 2.3. *For any graph G, $RSO(G) \leqslant \sqrt{2}\Big[RM_1^{\beta}(G) - \sqrt{RM_2(G)}\Big]$.*

PROOF. As discussed in [**18**], for $1 \leqslant x \leqslant y$, the function

$$f(x,y) = (x + y - \sqrt{xy}) - \sqrt{\frac{x^2 + y^2}{2}}$$

is decreasing for each y, so that $f(x,y) \geqslant f(y,y) = 0$. Therefore,

$$x + y - \sqrt{xy} \geqslant \sqrt{\frac{x^2 + y^2}{2}}\,.$$

Replacing x and y with $c(u)$ and $c(v)$, we have

$$c(u) + c(v) - \sqrt{c(u)\,c(v)} \geqslant \sqrt{\frac{c(u)^2 + c(v)^2}{2}}$$

from which

$$\begin{aligned} \frac{1}{\sqrt{2}} \sum_{uv \in \mathbf{E}(G)} \sqrt{c(u)^2 + c(v)^2} &\leqslant \sum_{uv \in \mathbf{E}(G)} \big[c(u) + c(v)\big] - \sum_{uv \in \mathbf{E}(G)} \sqrt{c(u)\,c(v)} \\ &\leqslant \sum_{uv \in \mathbf{E}(G)} \big[c(u) + c(v)\big] - \sqrt{\sum_{uv \in \mathbf{E}(G)} c(u)\,c(v)}\,. \end{aligned}$$

Theorem 2.3 follows now form Eqs. (1.3), (1.4), and (1.5). □

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