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HESITANT FUZZY RIGHT BI-QUASI IDEALS OF $\Gamma-$SEMIGROUPS

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Abstract. In this paper, we introduce hesitant fuzzy right bi-quasi ideals of a Γ-semigroup as a generalization of bi-ideals of a semigroup and study the properties of hesitant fuzzy right bi-quasi ideals. Regular $\Gamma-$semigroup is characterized in terms of hesitant fuzzy right bi-quasi ideals of a $\Gamma-$semigroup.

1. Introduction

Semigroup is a simple algebraic structure and has many applications in mathematical analysis, formal language and automata theory. As a generalization of semigroups, Sen and Saha[**21**] introduced the notion of Gamma semigroups. Quasi ideals of a semigroup were introduced by Steinfeld[**22**]. Quasi ideals are generalization of left(right) ideals, bi-ideals are generalization of quasi ideals. Murali Krishna Rao[**17, 18**] introduced fuzzy bi-quasi ideals in $\Gamma-$semigroups, further studied fuzzy bi-interior ideals of semigroups. Handling uncertainties in all fields is a very important task. As number of uncertainties prevail in many real world problems such as Decision making, Control Engineering, Artificial Intelligence, Robotics and many. To handle such, some of the mathematical tools such as randomness, rough set and fuzzy set were introduced. Fuzzy sets were proposed by Zadeh[**26**]. Rosenfeld[**20**] introduced the notion of fuzzy subgroup studied some of its properties. Kuroki[**9**] introduced fuzzy ideals in semigroup, and characterized several fuzzy left, fuzzy right and fuzzy bi-ideals of semigroups.

In 2009, Torra and Narukawa[**24**] and Torra[**23**] introduced the notion of hesitant fuzzy sets as a generalization of fuzzy sets which was proposed by Zadeh.

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Hesitant fuzzy sets play a remarkable role when there is ambiguity to decide the membership of an element, when different membership degree for an element was assigned by the decision makers. Hesitant fuzzy set allows the membership of an element of a set to be represented by several possible values, which can be a perfect tool for modeling peoples hesitancy. Recently many authors used these concepts in clustering analysis and decision making. Jun et.al.[**7**] studied hesitant fuzzy bi-ideals in semigroups and discussed relationships between hesitant fuzzy semigroups with frontier. Xia and Xu applied hesitant fuzzy set to decision making [**25**]. Kostaq Hila et al. studied a hesitant fuzzy set approach to ideal theory in $\Gamma-$semigroups [**1**].

2. Preliminaries

Some basic definitions needed in the next sections were given in this section.

DEFINITION 2.1. Let M and Γ be nonempty sets, then M is called a $\Gamma-$semigroup, if there exists a mapping $MX\Gamma XM \rightarrow M$ (images of l, α, m is denoted by $l\alpha m, l, m \in M, \alpha \in \Gamma$) such that it satisfies
i) $l\alpha m \in M$
ii) $l\alpha(m\beta n) = (l\alpha m)\beta n$, for all $l, m, n \in M, \alpha, \beta \in \Gamma$.

DEFINITION 2.2. Let M be a Γ-semigroup. A nonempty subset S of M is called a sub semigroup if $S\Gamma S \subseteq S$.

DEFINITION 2.3. Let M be a Γ-semigroup. A nonempty subset L of M is called left(right) ideal of M if $M\Gamma L \subseteq L(L\Gamma M \subseteq L)$. If it is both, a left and right ideal then it is called an ideal.

DEFINITION 2.4. Let M be a Γ-semigroup. A sub semigroup B of M is called a bi-ideal if $B\Gamma M\Gamma B \subseteq B$.

DEFINITION 2.5. Let M be a Γ-semigroup. A nonempty subset Q of M is called a quasi-ideal if $Q\Gamma M \cap M\Gamma Q \subseteq Q$.

DEFINITION 2.6. Let M be a nonempty set. A mapping $f : M \rightarrow [0, 1]$ is called a fuzzy subset of a Γ-semigroup M.

DEFINITION 2.7. Let f be a nonempty fuzzy subset of a $\Gamma-$semigroup M and $t \in [0, 1]$. Then the set $f_t = \{x \in M | f(x) \geqslant t$ is called the level subset of M with respect to f.

DEFINITION 2.8. A Γ-semigroup M is said to be regular if, for each $l \in M$ there exist $x \in M$ and $\alpha, \beta \in \Gamma$ such that $l = l\alpha x\beta l$.

3. Hesitant fuzzy ideals

DEFINITION 3.1. Let X be a reference set and let P[0,1] denote the power set of [0,1], then a mapping $h : X \rightarrow P[0, 1]$ is called a hesitant fuzzy set in X. The hesitant fuzzy empty[resp.whole]set, denoted by $h^0[resp.h^1]$, is a hesitant fuzzy set in X defined as, for each $x \in X$, $h^0 = \phi[resp.\ h^1(x) = [0, 1]$.
In this case, we will denote the set of all hesitant fuzzy sets in X as $HS(X)$.

DEFINITION 3.2. i) Let $h_1, h_2 \in HS(X)$. Then we say that h_1 is a subset of h_2 denoted by $h_1 \subset h_2$, if $h_1(x) \subset h_2(x)$, for each $x \in X$.
ii) we say that h_1 is equal to h_2, denoted by $h_1 = h_2$, if $h_1 \subset h_2$ and $h_2 \subset h_1$.

DEFINITION 3.3. Let $(X, \cdot)$ be a groupoid and let $h_1, h_2 \in HS(X)$. Then the hesitant fuzzy product of h_1 and h_2, is denoted by $h_1 \circ h_2$, is a hesitant fuzzy set in X defined by : for each $x \in X$,

$$(h_1 \circ h_2)(x) = \begin{cases} \widetilde{\bigcup}_{b\delta c=x}[h_1(b) \cap h_2(c)] \; if \; b\delta c = x \\ \phi \; otherwise. \end{cases}$$

DEFINITION 3.4. The hesitant characteristic function of a nonempty subset ω of $\Gamma-$semigroup M is defined as $\xi_\omega : M \to P([0,1])$ where,

$$\xi_w(x) = \begin{cases} [0,1], & if x \in \omega, \\ \phi, & if x \notin \omega. \end{cases}$$

DEFINITION 3.5. The hesitant fuzzy subset of a nonempty subset I of M is defined as $I : M \to P([0,1])/I(x) = [0,1]$.

DEFINITION 3.6. A hesitant fuzzy subset h of a Γ-semigroup M is said to be a hesitant fuzzy $\Gamma-$subsemigroup of M if

$$h(p\beta q) \supseteq h(p) \cap h(q) for \; all \; p, q \in M, \beta \in \Gamma.$$

DEFINITION 3.7. Let M be a $\Gamma-$semigroup. A hesitant fuzzy set h of M is said to be a hesitant fuzzy right(left) ideal of M if h is a hesitant fuzzy $\Gamma-$subsemigroup of M and $h(p\beta q) \supseteq h(p)(h(q))$ for all $p, q \in M$, $\beta \in \Gamma$.

DEFINITION 3.8. Let M be a Γ-semigroup. A hesitant fuzzy set h of M is said to be a hesitant fuzzy bi-ideal of M if it satisfies :
i) h is a fuzzy $\Gamma-$subsemigroup of M.
ii) $h(l\alpha m\beta n) \supseteq h(l) \cap h(n)$ for all $l, m, n \in$M, $\alpha, \beta \in \Gamma$.

DEFINITION 3.9. A hesitant fuzzy subset h of M is called a hesitant fuzzy left(right) bi-quasi ideal of M if $IohoIoh \subseteq h(hoIohoI \subseteq h)$.

DEFINITION 3.10. A hesitant fuzzy subset h of M is called a hesitant fuzzy bi-ideal of M if $hoIoh \subseteq h$.

DEFINITION 3.11. Let h be a hesitant fuzzy subset of a Γ-semigroup M and $t \in P([0,1])$. Then the set $h_t = \{x \in M/h(x) \supseteq t\}$ is called the t-cut of h.

THEOREM 3.1. *Let M be a* Γ*-semigroup, then for* $P, Q \subseteq M$, *we have*
i) $\xi_P \cap \xi_Q = \xi_{P\cap Q}$
ii) $\xi_P \circ \xi_Q = \xi_{PQ}$

PROOF. i) Let $x \in M$. $x \in P \cap Q$, implies $x \in P$ and $x \in Q$. Therefore

$$\begin{aligned}(\xi_P \cap \xi_Q)(x) &= \xi_P(x) \cap \xi_Q(x) \\ &= [0,1] \cap [0,1] = [0,1] = \xi_{P\cap Q}(x)\end{aligned}$$

If $x \notin P \cap Q$ then $x \notin P$ or $x \notin Q$. Therefore

$$\begin{aligned}(\xi_P \cap \xi_Q)(x) &= \xi_P(x) \cap \xi_Q(x) \\ &= \phi \cap \phi = \phi = \xi_{P\cap Q}(x)\end{aligned}$$

Hence $\xi_P \cap \xi_Q = \xi_{P\cap Q}$
ii)Let $a \in M$. Suppose $a \in P\Gamma Q$, then $a = x\alpha y$, $x \in P, y \in Q, \alpha \in \Gamma$.

$$\begin{aligned}(\xi_P \cap \xi_Q)(a) &= \bigcup_{a=x\alpha y} \{\xi_P(x) \cap \xi_Q(x)\} \\ &\supseteq \xi_P(x) \cap \xi_Q(x) = [0,1].\end{aligned}$$

Since $a \in P\Gamma Q$, we have $\xi_P \cap \xi_Q(x) = [0,1]$.
If $a \notin P\Gamma Q$, then $a \neq x\alpha y$ then

$$\begin{aligned}(\xi_P \cap \xi_Q)(a) &= \bigcup_{a=m\delta n} \{\xi_P(m) \cap \xi_Q(n)\} \\ &= \phi = \xi_{PQ}(a).\end{aligned}$$

Therefore $\xi_P \circ \xi_Q = \xi_{PQ}$. □

THEOREM 3.2. *Let ξ_ω be the hesitant characteristic function of the nonempty subset ω of a $\Gamma-$subsemigroup M. Then ω is a $\Gamma-$subsemigroup of M if and only if ξ_ω is a hesitant fuzzy $\Gamma-$subsemigroup of M.*

PROOF. Let ω be a $\Gamma-$subsemigroup of M. Then $m\alpha n \in \omega$ for $m, n \in M, \alpha \in \Gamma$.
If $m, n \in \omega$, then $\xi_\omega(m\alpha n) = [0,1] = \bigcup_{x=m\alpha n}\{\xi_\omega(m) \cap \xi_\omega(n).\}$
If $m \notin \omega$ or $n \notin \omega$ then $\xi_\omega(m\alpha n) = 0 = \bigcup_{x=m\alpha n}\{\xi_\omega(m) \cap \xi_\omega(n).\}$
Therefore if $m\alpha n \in \omega$, then $\xi_\omega(m\alpha n) \supseteq \bigcup_{x=m\alpha n}\{\xi_\omega(m) \cap \xi_\omega(n)$.
Conversely, suppose ξ_ω is a hesitant fuzzy $\Gamma-$subsemigroup of M.
Let $x, y \in \omega$, then $\xi_\omega(x) = \xi_\omega(y) = [0,1]$.
Thus $\xi_\omega(a\alpha b) \supseteq \bigcup_{x=a\alpha b}\{\xi_\omega(a) \cap \xi_\omega(b) = [0,1]$.
Therefore $a\alpha b \in \omega$. Hence ω is a $\Gamma-$subsemigroup of M. □

THEOREM 3.3. *A nonempty subset ω of a Γ-semigroup M is a bi-ideal of M if and only if the hesitant characteristic fuzzy set ξ_ω of ω is a hesitant fuzzy bi-ideal on M.*

PROOF. Let ω be a bi-ideal of M. Then ξ_ω is a hesitant fuzzy $\Gamma-$subsemigroup of M. $\xi_\omega(m) \cap \xi_\omega(n) = [0,1] = \xi_\omega(m\delta n)$.
If $m \notin \omega, n \notin \omega$, then $\xi_\omega(m) \cap \xi_\omega(n) = \phi = \xi_\omega(m\delta n)$.
Therefore ξ_ω is a hesitant fuzzy bi-ideal on M. Conversely suppose ξ_ω is a hesitant fuzzy bi-ideal on M. Then ω is a $\Gamma-$subsemigroup of M.

$$\begin{aligned}\xi_\omega(\Psi) &= \xi_\omega(a\alpha b\beta c) \\ &\supseteq \xi_\omega(a) \cap \xi_\omega(b) \cap \xi_\omega(c) \\ &\supseteq [0,1] \cap [0,1] \cap [0,1] \\ &\supseteq [0,1]\end{aligned}$$

□

THEOREM 3.4. *Let ω be a nonempty subset of a Γ-semigroup M and ξ_ω be the hesitant characteristic function of ω. Then ω is a bi-quasi ideal of M if and only if ξ_ω is a hesitant fuzzy bi-quasi ideal of M.*

PROOF. Let ω be a nonempty subset of a Γ-semigroup M and ξ_ω be the hesitant characteristic function of ω.
Suppose ω is a hesitant fuzzy bi-quasi ideal of M.
Therefore ξ_ω is a hesitant fuzzy $\Gamma-$subsemigroup of M.
Hence $\omega\Gamma M \cap \omega\Gamma M\Gamma\omega \subseteq \omega$. Then

$$\begin{aligned}\xi_\omega \circ \xi_M \cap \xi_\omega \circ \xi_M \circ \xi_\omega &= \xi_{\omega\Gamma M} \cap \xi_{\omega\Gamma M\Gamma\omega}\\ &= \xi_{\omega\Gamma M\cap\omega\Gamma M\Gamma\omega}\\ &\subseteq \xi_\omega\end{aligned}$$

Therefore ξ_ω is a hesitant fuzzy right bi-quasi ideal of M.
Similarly we can prove the other.
Conversely suppose ξ_ω is a a hesitant fuzzy bi-quasi ideal of M.
Then ω is a $\Gamma-$subsemigroup of M. Therefore

$$\begin{aligned}&\xi_\omega \circ \xi_M \cap \xi_\omega \circ \xi_M \circ \xi_\omega \subseteq \xi_\omega\\ &\Rightarrow \xi_{\omega\Gamma M} \cap \xi_{\omega\Gamma M\Gamma\omega} \subseteq \xi_\omega\\ &\Rightarrow \xi_{\omega\Gamma M} \cap \omega\Gamma M\Gamma\omega \subseteq \xi_\omega\end{aligned}$$

Therefore $\omega\Gamma M \cap \omega\Gamma M\Gamma\omega \subseteq \omega$.
Hence ω is a right bi-quasi ideal of M. □

THEOREM 3.5. *Let M be a Γ-semigroup and h be a non empty hesitant fuzzy subset of M. Then h is a hesitant fuzzy $\Gamma-$subsemigroup of M if and only if $hoh \subseteq h$.*

PROOF. Let $hoh \subseteq h$, then for $x, y \in M, \alpha \in \Gamma$
we have

$$h(r\delta s) \supseteq (hoh)(r\delta s) \supseteq \{h(r) \cap h(s)\}.$$

Therefore h is a hesitant fuzzy sub semigroup of M.
Conversely suppose h is a hesitant fuzzy $\Gamma-$subsemigroup of M and $a \in M$ then there exists $m, n \in M, \alpha \in M$ such that $a = m\alpha n$ then

$$\begin{aligned}&(hoh)(x) \neq \phi.\\ &\Rightarrow h(m\alpha n) \supseteq \{h(m) \cap h(n)\}.\\ &\Rightarrow h(m\alpha n) \supseteq \bigcup_{a=m\alpha n} \{h(m) \cap h(n)\}.\\ &\qquad = (hoh)(a)\end{aligned}$$

Hence $(hoh) \subseteq h$. □

THEOREM 3.6. *Let h be a nonempty hesitant fuzzy $\Gamma-$subsemi group of M. Then the following are equivalent:*
1) h is a hesitant fuzzy left(right) ideal of M.

2) $I \circ h \subseteq h$ $(h \circ I \subseteq h)$ where I is a hesitant fuzzy set and $I(x) = [0,1]$ for all $x \in M$.

THEOREM 3.7. *Let h be a non empty hesitant fuzzy subset of a $\Gamma-$semigroup M. Then h is a hesitant fuzzy $\Gamma-$subsemigroup of M if and only if h_s is a sub semigroup for every $s \in P([0,1])$, where $h_s = \{x \in M|\ h(x) \supseteq s\}$.*

PROOF. Let h be a hesitant fuzzy sub semigroup of M.
Let $s \in P([0,1])$, then for some $a \in M$ such that $h_a = s$.
So $a \in h_s \Rightarrow h_s \neq \phi$
Let $l, m \in h_s$. Then $h(l) \supseteq s, h(m) \supseteq s$. hence $\{h(l) \cap h(m)\} \supseteq s$.
Now for $\eta \in \Gamma$ $h(l\eta m) \supseteq \{h(l) \cap h(m)\} \supseteq s$.
Hence $l\eta m \in h_s$ for all $\eta \in \Gamma$ such that $h_s \Gamma h_s \subseteq h_s$.
Therefore h_s is a sub semigroup of M.
Conversely, suppose $h'_s s$ are sub semigroups of M for all $s \in P([0,1])$
Let $h(l) = s_1$ and $h(m) = s_2$ and $s_1 \supseteq s_2$, then $l, m \in h_{s_2}$.
Therefore $l\eta m \in h_{s_2}$
$\Rightarrow h(l\eta m) \supseteq s_2 = \{h(l) \cap h(m)\}$.
Therefore h is a hesitant fuzzy $\Gamma-$subsemigroup of M. □

THEOREM 3.8. *Let M be a $\Gamma-$semigroup and h be a non empty hesitant fuzzy subset of M. Then the following are equivalent:*
1) h is a hesitant fuzzy bi-ideal of M.
2) $h \circ h \subseteq h$ and $h \circ I \circ h \subseteq h$.

PROOF. Suppose h is a hesitant fuzzy bi-ideal of M. Let $a \in M$. then

$$\begin{aligned}(hoh)(a) &= \bigcup_{a=b\beta c} \{h(b) \cap h(c)\}.\\ &\subseteq \bigcup_{a=b\beta c} h(b\beta c)\\ &= h(a)\end{aligned}$$

Therefore $hoh \subseteq h$.

$$\begin{aligned}(h \circ I \circ h)(a) &= \bigcup_{a=l\alpha m} \{(h \circ I)(l) \cap h(m)\}\\ &= \bigcup_{a=l\alpha m} \left\{ \bigcup_{l=p\beta q} \{h(p) \cap I(q)\} \cap h(m) \right\}\\ &= \bigcup_{a=l\alpha m} \left\{ \bigcup_{l=p\beta q} \{h(p) \cap [0,1]\} \cap h(m) \right\}\end{aligned}$$

$$= \bigcup_{a=p\beta q\alpha m} \{h(p) \cap h(m)\}$$
$$\subseteq \bigcup_{a=p\beta q\alpha m} h(p\beta q\alpha m)$$
$$= h(a).$$

Hence $(h \circ I \circ h) \subseteq h$.
Conversely, suppose $h \circ h \subseteq h$ and $h \circ I \circ h \subseteq h$.
Let $h \circ h \subseteq h$. Then for $s, t \in M, \alpha \in \Gamma$.

$$h(s\alpha t) \supseteq h \circ h(s\alpha t) \supseteq h(s) \cap h(t)$$

Let $l, m, n \in M, \alpha, \beta \in \Gamma$ and $a = l\alpha m\beta n$.
Since $h \circ I \circ h \subseteq h$ we have,

$$\begin{aligned} h(l\alpha m\beta n) = h(a) &\supseteq (h \circ I \circ h)(a) \\ &= \bigcup_{a=l\alpha m\beta n} \{(hoI)(l\alpha m) \cap h(n)\} \\ &= \bigcup_{a=p\beta n} \left\{ \bigcup_{p=l\alpha m} \{(h)(l) \cap I(m)\} \cap h(n) \right\} \\ &= \bigcup_{a=p\beta n} \left\{ \bigcup_{p=l\alpha m} \{h(l) \cap [0,1]\} \cap h(n) \right\} \\ &\supseteq \bigcup_{a=l\alpha m\beta n} \{h(l) \cap h(n)\} \\ \Rightarrow h(l\alpha m\beta n) &\supseteq h(l) \cap h(n) \end{aligned}$$

Therefore $h(l\alpha m\beta n) \supseteq h(l) \cap h(n)$.
Hence h is a hesitant fuzzy bi-ideal of M. □

THEOREM 3.9. *Every hesitant fuzzy right ideal of a $\Gamma-$semigroup M is a hesitant fuzzy quasi-ideal of M.*

PROOF. Let hesitant fuzzy right ideal of M then

$$(hoI) \cap (Ioh) \subseteq hoI \subseteq h.$$

Therefore h is a hesitant fuzzy quasi-ideal of M. □

THEOREM 3.10. *Let r, l be right and left hesitant fuzzy ideal of M. Then $r \cap l$ is a hesitant fuzzy quasi-ideal of M.*

PROOF. Since $((r \cap l)oI \cap (Io(r \cap l)) \subseteq ((roI) \cap (Iol) \subseteq r \cap l$. Therefore $r \cap l$ is a hesitant fuzzy quasi-ideal of M. □

THEOREM 3.11. *Let M be a $\Gamma-$semigroup. Let r be a hesitant fuzzy right ideal and l be a hesitant fuzzy left ideal of M. Then $r \cup l$ is a hesitant fuzzy quasi-ideal of M.*

PROOF. Let r be a hesitant fuzzy right ideal and l be a hesitant fuzzy left ideal of M. Then $r(p\delta q) \supseteq r(p)$ and $l(p\delta q) \supseteq l(q)$.

$$\begin{aligned}((r \cup l) \circ I)(a) &= \bigcup_{a=p\delta q} [(r \cup l)(p) \cap I(q)] \\ &= \bigcup_{a=p\delta q} [\{(r(p) \cup l(p)\} \cap [0,1]] \\ &= \bigcup_{a=p\delta q} [\{(r(p) \cup l(p)\}] \\ &\subseteq \bigcup_{a=p\delta q} [\{(r(p\delta q) \cup l(p\delta q)\}] \\ &= r(a) \cup l(a) \\ &= (r \cup l)(a).\end{aligned}$$

Therefore $((r \cup l) \circ I)(a) \subseteq r \cup l(a)$.
Similarly it can be proved $(I \circ (r \cup l))(a) \subseteq r \cup l(a)$.
And $((r \cup l) \circ I) \cap (I \circ (r \cup l)) \subseteq (r \cup l) \cap (r \cup l) \subseteq (r \cup l)$
Therefore $r \cup l$ is a hesitant fuzzy quasi-ideal of M. □

THEOREM 3.12. *Any hesitant fuzzy quasi-ideal of M is a hesitant fuzzy bi-ideal of M.*

PROOF. Let h be a hesitant fuzzy quasi-ideal of M then

$$\begin{aligned} hoIoh &\subseteq hoIoI \subseteq hoI \\ hoIoh &\subseteq IoIoh \subseteq Ioh \\ \Rightarrow hoIoh &\subseteq hoI \cap Ioh \subseteq h.\end{aligned}$$

Therefore h is a hesitant fuzzy bi-ideal of M. □

THEOREM 3.13. *Let p, q be any two hesitant fuzzy quasi-ideals of M. Then $p\ o\ q$ is a hesitant fuzzy bi-ideal of M.*

PROOF. Since any hesitant fuzzy quasi-ideal is a hesitant fuzzy bi-ideal.
Therefore

$$\begin{aligned}(p\ o\ q)oMo(p\ o\ q) &= p\ o(q\ o(M\ o\ p)o\ q) \\ &\subseteq p\ o(q\ o\ M\ o\ M)o\ q \\ &\subseteq p\ o(q\ o\ M)o\ q \\ &\subseteq p\ o\ q\end{aligned}$$

Hence, $p\ o\ q$ is a hesitant fuzzy bi-ideal of M. □

THEOREM 3.14. *Let r and l be hesitant fuzzy right and left ideals of M. Then M is regular if and only if $r\ o\ l = r \cap l$.*

PROOF. Let M be a regular $\Gamma-$semigroup.
Let r be a hesitant fuzzy right ideal and l be a hesitant fuzzy left ideal of M.

Let $a \in M$

$$(r\ o\ l)(a) = \bigcup_{a=x\eta y} \{r(x) \cap l(y)\}$$
$$\subseteq \{r(x\eta y) \cap l(x\eta y)\}$$
$$\subseteq r(a) \cap l(a)$$
$$= (r \cap l)(a)$$

thus $r\ o\ l \subseteq r \cap l$.

Let $y \in M$. Since M is regular there exists $\alpha, \beta \in \Gamma, x \in M$ such that $y = y\alpha x\beta y$. And

$$(r\ o\ l)(y) = \bigcup_{y=b\alpha x\beta b} \{r(b\alpha x) \cap l(b)\}$$
$$= \bigcup_{y=p\beta b} \{ \bigcup_{p=b\alpha x} \{\{r(b) \cap r(x)\} \cap l(b)\}$$
$$\supseteq \bigcup_{y=b\alpha x\beta b} \{r(b) \cap l(b)\}$$
$$\supseteq \{r(y) \cap l(y)\}.$$

Since $y = y(\alpha x\beta)y$ and $\Gamma M\Gamma \subseteq \Gamma, \alpha x\beta \in \Gamma$ so $r\ o\ l \supseteq r \cap l$.
Thus $r\ o\ l = r \cap l$. □

Theorem 3.15. *A nonempty hesitant fuzzy subset h of a $\Gamma-$semigroup M is a hesitant fuzzy bi-ideal of M if and only if h_b is a bi-ideal of M for every $b \in P([0,1])$, where $h_b = \{x \in M|\ h(x) \supseteq b\}$.*

Proof. Let h be a hesitant fuzzy bi-ideal of M.
Then h be a hesitant fuzzy sub semigroup of M.
Therefore h_b is a sub semigroup of M for every $b \in P([0,1])$.
Let $b \in P([0,1])$, then for some $\alpha \in M$ such that $h_\alpha = b$.
So $\alpha \in h_b \Rightarrow h_b \neq \phi$
Let $x, z \in h_b$. Then $h(x) \supseteq b, h(z) \supseteq b$. hence $\{h(x) \cap h(z)\} \supseteq b$.
Now for $y \in M, \alpha, \beta \in \Gamma$ $h(x\alpha y\beta z) \supseteq \{h(x) \cap h(z)\} \supseteq b$. Hence $x\alpha y\beta z \in h_b$ such that $h_b\Gamma M\Gamma h_b \subseteq h_b$.
Therefore h_b is a bi-ideal of sub semigroup of M.
Conversely, suppose $h_b's$ are bi-ideals of M for all $b \in P([0,1])$
Then $h_b's$ are sub semigroups of M
Hence h be a hesitant fuzzy sub semigroup of M.
Let $x, y, z \in M, \alpha, \beta \in \Gamma$
Let $h(x) = b_1$ and $h(y) = b_2$ and $b_1 \supseteq b_2$, then $x, z \in h_{b_2}$.
Therefore $x\alpha y\beta z \in h_{b_2}$
$\Rightarrow h(x\alpha y\beta z) \supseteq b_2 = \{h(x) \cap h(z)\}$.
Therefore h is a hesitant fuzzy bi-deal of M. □

Theorem 3.16. *Let M be a $\Gamma-$semigroup, then every hesitant fuzzy right ideal h of M is a hesitant fuzzy bi-quasi ideal of M.*

PROOF. Let h be a hesitant fuzzy right ideal of M, and $x \in M$

$$\begin{aligned}(hoI)(x) &= \bigcup_{x=r\alpha s} \{h(r) \cap I(s)\}\\ &= \bigcup_{x=r\alpha s} \{h(r) \cap [0,1]\}\\ &\subseteq \bigcup_{x=r\alpha s} \{h(r)\}\\ &\subseteq \bigcup_{x=r\alpha s} \{h(r\alpha s)\}\\ &\subseteq \bigcup_{x=r\alpha s} \{h(x)\}\\ &\subseteq h(x)\end{aligned}$$

Therefore $(hoI)(x) \subseteq h(x)$.

$$\begin{aligned}(hoIoh)(x) &= \bigcup_{x=m\alpha n\beta p} \{(hoI)(m\alpha n) \cap h(p)\}\\ &\subseteq \bigcup_{x=m\alpha n\beta p} \{h(m\alpha n) \cap h(p)\}\\ &\subseteq \bigcup_{x=m\alpha n\beta p} h(x)\\ &\subseteq h(x).\end{aligned}$$

and

$$\begin{aligned}((hoI) \cap (hoIoh))(x) &= \{(hoI)(x) \cap (hoIoh)(x)\}\\ &\subseteq \{h(x) \cap h(x)\}\\ &= h(x).\end{aligned}$$

Therefore h is a hesitant fuzzy bi-quasi ideal of M. □

THEOREM 3.17. *Let h be a non empty hesitant fuzzy subset of a* $\Gamma-$*semigroup* M. *Then h is a hesitant fuzzy bi-quasi ideal of* M *if and only if the t-cut* h_t *of h is a bi-quasi ideal of* M *for every* $t \in P([0,1])$,*where* $h_t = \{x \in M|\ h(x) \supseteq t\}$.

PROOF. Let M be a $\Gamma-$semigroup and h be a nonempty hesitant fuzzy subset of M.
Suppose h be a hesitant fuzzy bi-quasi ideal of M, $h_t \neq \phi, t \in P([0,1])$.
and $n, p \in h_t, \alpha \in \Gamma$.
Let $x \in M\Gamma h_t \cap h_t\Gamma M\Gamma h_t$ then $x = m\alpha n = p\beta q\gamma r$ where $m, q \in M, n, p, r \in h_t, \alpha, \beta, \gamma \in \Gamma$. Then $(Ioh)(x) \supseteq t$ and $(hoIoh)(x) \supseteq t$
$\Rightarrow h(x) \supseteq (Ioh \cap hoIoh)(x) \supseteq t$.
Therefore $x \in h_t$. Hence h_t is a left bi-quasi ideal of M.
Similarly we can prove h_t is a right bi-quasi ideal of M.
Conversely, suppose h_t is a bi-quasi ideal of M for all $t \in P([0,1])$
Then h_t is a hesitant fuzzy sub semigroup of M.

Let $x, y \in M$, $h(x) = t_1$, $h(y) = t_2$ and $t_1 \supseteq t_2$, then $x, y \in h_{t_2}$ and $x\alpha y \in h_{t_2}$
$\Rightarrow h(x\alpha y) \supseteq t_2 = \{h(x) \cap h(y)\} \supseteq t_2$.
We have $M\Gamma h_a \cap h_a\Gamma M\Gamma h_a \subseteq h_t$, suppose $t \in P([0,1])$.
Then $M\Gamma h_t \cap h_t\Gamma M\Gamma h_t \subseteq h_t$.
Therefore $(Ioh \cap hoIoh)(x) \supseteq h$.
Hence h is a hesitant fuzzy left bi-quasi ideal of M.
Similarly we can prove h is a hesitant fuzzy right bi-quasi ideal of M. □

Theorem 3.18. *Let M be a regular $\Gamma-$semigroup and h be a hesitant fuzzy subset of M. Then h is a hesitant fuzzy bi-quasi ideal if $(Ioh \cap hoIoh)(x) \subseteq h$.*

Proof. Let h be a hesitant fuzzy bi-quasi ideal of a regular Γ-semigroup M. Then to prove that $(Ioh \cap hoIoh)(x) \subseteq h$.

$$\begin{aligned}(Ioh)(x) &= \bigcup_{x=a\alpha b\beta a} \{I(a\alpha b) \cap h(a)\}\\ &= \bigcup_{x=a\alpha b\beta a} \{[0,1] \cap h(a)\}\\ &\subseteq \bigcup_{x=a\alpha b\beta a} \{h(a)\}\\ &\subseteq \bigcup_{x=a\alpha b\beta a} \{h(a\alpha b\beta a)\}\\ &\subseteq \bigcup_{x=a\alpha b\beta a} \{h(x)\}\\ &= h(x).\end{aligned}$$

Therefore $(Ioh)(x) \subseteq h(x)$.

$$\begin{aligned}(hoIoh)(x) &= \bigcup_{x=a\alpha b\beta a} \{(h(a) \cap Ioh(b\beta a)\}\\ &= \bigcup_{x=a\alpha m} \{(h(a) \cap \{ \bigcup_{m=b\beta a} \{(I(b) \cap h(a))\}\}\\ &= \bigcup_{x=a\alpha m} \{(h(a) \cap \{ \bigcup_{m=b\beta a} \{([0,1] \cap h(a))\}\}\\ &\supseteq \bigcup_{x=a\alpha b\beta a} \{(h(a) \cap h(a))\}\\ &\supseteq h(x).\end{aligned}$$

and

$$\begin{aligned}((I \circ h) \cap (hoIoh))(x) &= \{(hoI)(x) \cap (hoIoh)(x)\}\\ &\subseteq \{h(x) \cap h(x)\}\\ &= h(x).\end{aligned}$$

Therefore h is a hesitant fuzzy bi-quasi ideal of M. □

THEOREM 3.19. *Let M be a $\Gamma-$semigroup, then every hesitant fuzzy right ideal of M is a hesitant fuzzy right bi-quasi ideal of M.*

PROOF. Let h be a hesitant fuzzy right ideal of a Γ-semigroup of M, and $a \in M$. Then

$$\begin{aligned}(hoI)(a) &= \bigcup_{a=x\alpha y} \{h(x) \cap I(y)\}\\ &= \bigcup_{a=x\alpha y} \{h(x) \cap [0,1]\}\\ &= \bigcup_{a=x\alpha y} \{h(x)\}\\ &\subseteq \bigcup_{a=x\alpha y} \{h(x\alpha y)\}\\ &\subseteq \bigcup_{a=x\alpha y} \{h(a)\}\\ &\subseteq h(a)\end{aligned}$$

Therefore $(hoI)(a) \subseteq h(a)$. And

$$\begin{aligned}(hoIoh)(a) &= \bigcup_{a=x\alpha y\beta z} \{(hoI(x\alpha y) \cap h(z)\}\\ &= \bigcup_{a=x\alpha y\beta z} \{\{ \bigcup_{p=x\alpha y} h(p) \cap I(p)\} \cap h(z)\}\\ &= \bigcup_{a=x\alpha y\beta z} \{h(p) \cap h(z)\}\\ &\subseteq \bigcup_{a=x\alpha y\beta z} \{(h(p) \cap h(z))\}\\ &\subseteq \{h(x\alpha y\beta z)\}\\ &\subseteq h(a).\end{aligned}$$

Now,

$$\begin{aligned}((hoI) \cap (hoIoh))(x) &= \{(hoI)(a) \cap (hoIoh)(a)\}\\ &\subseteq \{(hoI)(a) \cap h(a)\}\\ &\subseteq \{(h)(a) \cap h(a)\}\\ &\subseteq h(a).\end{aligned}$$

Therefore $(hoI) \cap (hoIoh) \subseteq h$.
Therefore h is a hesitant fuzzy right bi-quasi ideal of M. □

THEOREM 3.20. *Let M be a $\Gamma-$semigroup. Then M is regular if and only if $h = (h \circ I) \cap (h \circ I \circ h)$ for any hesitant fuzzy right bi-quasi ideal of M.*

PROOF. Let h be a hesitant fuzzy right bi-quasi ideal of M. Then

$$\begin{aligned} h \circ I(x) &= \bigcup_{a=a\alpha b\beta a} \{h(a) \cap I(b\beta a)\} \\ &= \bigcup_{a=a\alpha b\beta a} \{h(a) \cap [0,1]\} \\ &= h(x). \end{aligned}$$

$$\begin{aligned} h \circ I \circ h(x) &= \bigcup_{a=a\alpha b\beta a} \{h(a) \cap I \circ h(b\beta a)\} \\ &= \bigcup_{x=a\alpha p} \{h(a) \cap \bigcup_{p=b\beta a} \{I(b) \cap h(a)\} \\ &= \bigcup_{x=a\alpha p} \{h(a) \cap \bigcup_{p=b\beta a} \{[0,1] \cap h(a)\} \\ &= \bigcup_{x=a\alpha b\beta a} \{h(a) \cap h(a)\} \\ &\supseteq h(a\alpha b\beta a) \\ &= h(x). \end{aligned}$$

Therefore $(h \circ I) \cap (h \circ I \circ h) = h$.
Conversely suppose that $h = (h \circ I) \cap (h \circ I \circ h)$, for any hesitant fuzzy right bi-quasi ideal of M. Let L be a left bi-quasi ideal of M. Then ξ_L is a hesitant fuzzy left bi-quasi ideal of M. Therefore

$$\begin{aligned} \xi_L &= \xi_M \circ \xi_L \cap \xi_L \circ \xi_M \circ \xi_L \\ \xi_L &= \xi_{M\Gamma L} \cap \xi_{L\Gamma M\Gamma L} \\ \Rightarrow L &= M\Gamma L \cap L\Gamma M\Gamma L. \end{aligned}$$

Therefore M is a regular $\Gamma-$semigroup. □

4. Conclusion

In this paper we introduced the concepts of hesitant fuzzy $\Gamma-$subsemigroup, hesitant fuzzy ideals, hesitant fuzzy right bi-quasi ideals of $\Gamma-$semigroup and investigated some of the properties and relations between them. We characterized regular $\Gamma-$semigroup using hesitant fuzzy ideals.

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