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A SPECIAL ONE-POINT CONNECTIFICATION THAT CHARACTERIZES T_0 SPACES

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Abstract. This article shows in particular that every T_0 space X has some T_0 "special" one-point connectification X_∞ such that X is a closed subspace of X_∞; moreover, having such a one-point connectification characterizes T_0 spaces. As an application, it is also shown that i) our one-point connectification of every given topological n-manifold is a space more general than, but "close to" a topological n-manifold with boundary and ii) there exist "arbitrarily many" connected (compact second countable) T_0 spaces with a disconnected derived set, although every connected T_1 space has a connected derived set.

A topological space (or a topology) is assumed to possess a property if and only if a corresponding declaration is made. For our purposes, by a *one-point connectification* of a given topological space we mean precisely a connected space including (set-theoretically) the given space with exactly one additional element; thus no additional requirement such as denseness is built-in, although the relaxation of denseness is not always standard.

We indicate a one-point connectification result for T_0 spaces that, to some extent, formally resembles the classical Alexandroff one-point compactification result for (noncompact) locally compact Hausdorff spaces:

Theorem 1. *Let X be a topological space. Then X is T_0 if and only if there is some T_0 one-point connectification X_∞ of X such that X is a closed subspace of X_∞.*

Proof. The "if" part follows from the fact that T_0-ness is hereditary.

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To prove the converse, let $\mathscr{T}$ be the given topology of X. Let the set $X_\infty := X \cup \{\infty\}$ receive the "naive" topology

$$\mathscr{T}_\infty := \{\varnothing\} \cup \{G \cup \{\infty\} \mid G \in \mathscr{T}\}.$$

Then, since $\{\infty\} \in \mathscr{T}_\infty$, the collection $\mathscr{T}_\infty$ is in particular a connected T_0 topology of X_∞, and X is a closed subspace of X_∞. This completes the proof. □

Remark 1. In Theorem 1, the topology $\mathscr{T}_\infty$ of X_∞ is by construction never T_1 even if X is Hausdorff. And X is not $\mathscr{T}_\infty$-dense in X_∞ as X is by construction closed-$\mathscr{T}_\infty$, although $\mathscr{T}_\infty$ is separable. □

Due to Theorem 1, we may fix some terminology:

Definition 1. Let X be a T_0 space; let X_∞ be the specific topological space obtained from X in Theorem 1. Then X_∞ is called the *special one-point connectification* of X. Here the uniqueness is certainly understood in the up-to-homeomorphism sense.

In general, a special one-point connectification is by definition precisely a topological space X_∞ such that there is some T_0 space X for which X_∞ is the special one-point connectification of X. □

Nonexample 1. Let $X := \{0, 1\}$ be the Sierpiński space receiving the topology $\{\varnothing, \{0\}, \{0, 1\}\}$. Since X is T_0, we may by Theorem 1 consider the special one-point connectification X_∞ of X. And we evidently have

$$\mathscr{T}_\infty = \{\varnothing, \{\infty\}, \{0, \infty\}, \{0, 1, \infty\}\}.$$

Thus a connected compact T_0 space can still admit its special one-point connectification, which is not possible in the standard context (e.g. [**1**], [**2**], or [**3**]), where a one-point connectification is understood in a more stringent sense. □

Remark 2. If X is a (noncompact) locally compact separable metrizable space, and if X is connectible in the sense of [**1**], then the Alexandroff one-point compactification of X is not the special one-point connectification of X even though it is a one-point connectification (in the sense of [**1**]) of X by Proposition 13 in [**1**]; for, the special one-point connectification of X is not Hausdorff. □

Given any T_0 space X, the special one-point connectification of X preserves many nice properties of X; we indicate in particular two relevant ones:

Proposition 1. *i) If X is a locally compact T_0 space, then the special one-point connectification X_∞ of X is a locally compact T_0 space. ii) If X is a compact T_0 space, then X_∞ is a compact T_0 space.*

Proof. Let X be a T_0 space, so that the special one-point connectification X_∞ of X exists by Theorem 1.

For clarity and completeness, simple justifications of the statements are articulated.

To prove i), fix any point $x \in X_\infty$. If $x = \infty$, then, since ∞ lies in every nonempty open subset of X_∞, the set $\{\infty\}$ is also compact in X_∞. It then suffices to consider elements of X.

Let $x \in X$. Suppose X is locally compact, and choose (in X) some neighborhood G of x and some compact superset K of G. Since $G \cup \{\infty\}$ is open in X_∞, and since the set $K \cup \{\infty\}$ is compact in X_∞ by inspecting the topology of X_∞, the space X_∞ is locally compact.

For ii), let $\mathscr{C}$ be an open cover of X_∞. Then $\{V \smallsetminus \{\infty\} \mid V \in \mathscr{C}\}$ is an open cover of X and hence admits by assumption a finite subcover $\{V_1 \smallsetminus \{\infty\}, \ldots, V_n \smallsetminus \{\infty\}\}$. But since $\{V_1, \ldots, V_n\}$ is a finite subcover of $\mathscr{C}$, the proof is complete. □

Remark 3. It would be worth indicating that the special one-point connectifications may be useful in constructing counterexamples. □

By an n-manifold we always mean a topological n-manifold, i.e. a second-countable Hausdorff space where every point has some neighborhood homeomorphic to the Euclidean space $\mathbb{R}^n$. For our purposes, we introduce the following

Definition 2. Let $n \in \mathbb{N}$. A topological space X is called a *pseudo locally n-Euclidean space with boundary* if and only if X is a second countable T_0 space where every point has some neighborhood equinumerous to some subset of $\mathbb{H}^n_+ := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geqslant 0\}$ such that this subset less the topological boundary $\partial\mathbb{H}^n_+$ of $\mathbb{H}^n_+$ is open in $\mathbb{R}^n$.

We take the phrase "pseudo locally n-Euclidean with boundary" as a modifier. □

Thus every n-manifold (with boundary or not) is a pseudo locally n-Euclidean space with boundary.

An application of the previous results is the following.

Theorem 2. *If X is an* (*resp. compact*) *n-manifold, then the special one-point connectification X_∞ of X is a* (*resp. compact*) *locally compact connected T_0 space that is pseudo locally n-Euclidean with boundary.*

Proof. Since X is by definition Hausdorff and hence T_0, the special one-point connectification X_∞ of X exists as a T_0 space by Theorem 1. Since X is in addition locally compact from assumption, Proposition 1 implies also that X_∞ is locally compact. If X is compact, then X_∞ is compact by Proposition 1. The connectedness, T_0-ness, and second countableness of X_∞ are evident.

For the remaining desired property of X_∞, fix any $x \in X$ and choose by assumption some chart (G, φ) of X about x such that G is φ-homeomorphic to the Euclidean subspace $\mathbb{R}^n_{++} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i > 0 \text{ for all } 1 \leqslant i \leqslant n\}$. We are considering only those points of X as ∞ by construction lies in every nonempty open subset of X_∞. Let $G_\infty := G \cup \{\infty\}$, and define the map $\varphi_\infty : G_\infty \to \mathbb{R}^n_{++} \cup \{(0)^n_{i=1}\}$ by $\varphi_\infty|_G := \varphi$ with $\varphi_\infty(\infty) := (0)^n_{i=1}$. Then φ_∞ is a bijection. Since $(\mathbb{R}^n_{++} \cup \{(0)^n_{i=1}\}) \smallsetminus \partial\mathbb{H}^n_+ = \mathbb{R}^n_{++}$ is open in $\mathbb{R}^n$, we are through. □

We may now turn our attention to another application of special one-point connectification. Given any topological space X and any connected set A in X, it is a basic fact that the closure $\operatorname{cl}(A)$ of A is also a connected set in X. To what extent can one assert connectedness for the derived set $\operatorname{cl}'(A)$ of the connected set A in X?

An immediate observation, recorded here for completeness, is the following proposition:

PROPOSITION 2. *If X is a T_1 space, and if $A \subset X$ is connected with at least two elements, then* $\mathrm{cl}'(A)$ *is connected.*

PROOF. We claim that $A \subset \mathrm{cl}'(A)$. Indeed, if $x \in A$ is an isolated point of the subspace A, then, since $\{x\}$ is clopen in A from assumption, it follows that A is disconnected.

But then $\mathrm{cl}(A) = \mathrm{cl}'(A)$; the desired conclusion follows. □

Thus, in particular, every connected set with at least two elements in any given Hausdorff space has a connected derived set.

From Proposition 2 we immediately have

COROLLARY 1. *If X is a connected T_1 space, then* $\mathrm{cl}'(X)$ *is connected.*

PROOF. If X is the empty set or a singleton, we are through. Otherwise, the desired conclusion follows from Proposition 2. □

However, as will be shown, it is "easy" for a T_0 space to admit a disconnected derived set even though the space is connected (and compact second countable). We will indicate a general simple construction of such counterexamples via special one-point connectification.

We first prove a more fundamental result:

THEOREM 3. *If X is a T_0 space, then there is some connected separable T_0 space Y such that X is a subspace of Y and* $\mathrm{cl}'(Y) = X$.

PROOF. Denote by $\mathscr{T}_X$ the given topology of X; fix any point ∞ not contained in X. Since X is a T_0 space by assumption, let $Y := X \cup \{\infty\}$ be the special one-point connectification of X. Then Y is by Theorem 1 (for concreteness) a connected T_0 space, and $\mathscr{T}_X$ coincides with the subspace topology of X received from Y.

By inspecting the given topology $\mathscr{T}_Y$ of Y, it is seen that the only closed superset of $\{\infty\}$ in Y is Y itself; the space Y is therefore separable.

Moreover, by inspection of $\mathscr{T}_Y$, we have $X \subset \mathrm{cl}'(Y)$. But since $\{\infty\}$ is by construction a neighborhood of ∞, we obtain the desired equality $X = \mathrm{cl}'(Y)$. This completes the proof. □

The desired result is the following.

THEOREM 4. *There exists a compact second countable connected T_0 space with a disconnected derived set.*

PROOF. Fix any compact second countable disconnected T_0 space X, choices of which are certainly "abundant", e.g. the subspace $[0, 1] \cup [2, 3]$ of the Euclidean space $\mathbb{R}$; let there be given a point ∞ not contained in X.

If $Y := X \cup \{\infty\}$ is the special one-point connectification of X, then Y is by Proposition 1 compact. That Y is second countable is evident by inspecting the given topology of Y.

The desired conclusion then follows from Theorem 3. □

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